

# OAK RIDGE NATIONAL LABORATORY

operated by

UNION CARBIDE CORPORATION

for the

U.S. ATOMIC ENERGY COMMISSION



ORNL-TM-919

65



## NOTE ON ELECTROSTATIC INSTABILITIES IN A PLASMA WITH ANISOTROPIC VELOCITY DISTRIBUTION

Laurence S. Hall  
Yaakov Shima

### NOTICE

This document contains information of a preliminary nature and was prepared primarily for internal use at the Oak Ridge National Laboratory. It is subject to revision or correction and therefore does not represent a final report. The information is not to be abstracted, reprinted or otherwise given public dissemination without the approval of the ORNL patent branch, Legal and Information Control Department.

#### LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

ORNL-TM-919

Contract No. W-7405-eng-26

THERMONUCLEAR DIVISION

NOTE ON ELECTROSTATIC INSTABILITIES IN A PLASMA  
WITH ANISOTROPIC VELOCITY DISTRIBUTION

Laurence S. Hall  
Lawrence Radiation Laboratory, Livermore, California

and

Yaakov Shima

SEPTEMBER 1964

OAK RIDGE NATIONAL LABORATORY  
Oak Ridge, Tennessee  
operated by  
UNION CARBIDE CORPORATION  
for the  
U. S. ATOMIC ENERGY COMMISSION



## ABSTRACT

The location of possibly unstable roots to the dispersion equation for the electrostatic waves in an infinite plasma situated in a constant magnetic field is carefully examined.



NOTE ON ELECTROSTATIC INSTABILITIES IN A PLASMA  
WITH ANISOTROPIC VELOCITY DISTRIBUTION\*

Laurence S. Hall  
Lawrence Radiation Laboratory, Livermore, California

and

Yaakov Shima  
Oak Ridge National Laboratory, Oak Ridge, Tennessee

In this note we wish to extend previous work<sup>1</sup> to delimit as much as possible the location of possibly unstable roots to the dispersion equation for the electrostatic waves in an infinite plasma situated in a constant magnetic field.

The dispersion equation may be written in the form

$$1 + \sum_j \omega_{pj}^2 \Omega_j^{-2} F_j = 0$$

where  $F_j$  is given in Ref. 1 for the bi-Maxwellian distribution function.

The theorems given in the previous article, which were based on an examination of  $\text{Im}(F_1)$ , are here extended to show that marginally unstable solutions ( $\text{Im}(\omega) = 0^+$ ) of the dispersion equation can occur only if both

$$\ell + \frac{1}{2} < \omega < \ell + 1 - \frac{T_{\parallel 1}}{T_{\perp 1}} \quad \text{and} \quad \omega < \frac{T_{\perp 1}}{T_{\parallel 1}} - 1 \quad (1)$$

where  $\ell$  is an integer.

Hence, in particular the system is stable if  $\frac{T_{\parallel 1}}{T_{\perp 1}} > \frac{1}{2}$ .

Following the same arguments as in Ref. 1, it is sufficient to show that  $\text{Im}(F)$  is non-negative in the regime of interest located in the first quadrant of the complex  $\omega$ -plane, where

$$F = n_{||}^2 F_C + n_{\perp}^2 F_B = \int_0^{\infty} dx (n_{||}^2 x + n_{\perp}^2 \sin x) \exp \left[ i\omega x - \mu x^2 - \lambda (1 - \cos x) \right]$$

and  $\text{Im}(F_B) \geq 0$ .

when  $\omega = l + \epsilon$  is real,  $0 \leq \epsilon < 1$ ,

$$\text{Im}(F_C) = A e^{-\lambda} \sum_{k=-\infty}^{\infty} g(k + \omega) I_k(\lambda)$$

where  $A = \frac{1}{2} \sqrt{\frac{\pi}{\mu}} \frac{1}{\lambda} \frac{T_{||}}{T_{\perp}} > 0$  and  $g(x) = x \exp\left(-\frac{x^2}{4\mu}\right)$ .

Thus, since  $I_{k-\ell}(\lambda) > I_{k+\ell}(\lambda) > 0$  for  $k, \ell > 0$ ,

$$\text{Im}(F_C) > A e^{-\lambda} \left\{ g(\epsilon) I_{\ell}(\lambda) + \sum_{k=1}^{\infty} \left[ g(k + \epsilon) - g(k - \epsilon) \right] I_{k+\ell}(\lambda) \right\}.$$

Notice that, when  $x > a > 0$ ,  $g(x) - g(x - a)$  is positive for  $x > x_a$ , and negative thereafter. Therefore, because  $I_k(\lambda)$  is a positive, decreasing function of  $k$ , there is an  $m \geq 0$  such that, for all  $k$ ,

$$\left[ g(k + \epsilon) - g(k - \epsilon) \right] \left[ I_{k+\ell}(\lambda) - I_{m+\ell}(\lambda) \right] \geq 0.$$

Therefore

$$\begin{aligned} \text{Im}(F_C) &> A e^{-\lambda} \left\{ g(\epsilon) I_{\ell}(\lambda) + \sum_{k=1}^{\infty} \left[ g(k + \epsilon) - g(k - \epsilon) \right] I_{m+\ell}(\lambda) \right\} \\ &> A e^{-\lambda} I_{m+\ell}(\lambda) \sum_{k=-\infty}^{\infty} g(k + \epsilon) \\ &= -2\mu A e^{-\lambda} I_{m+\ell}(\lambda) \frac{d}{d\omega} \left\{ \sum_{k=-\infty}^{\infty} \exp \left[ -\frac{(k + \omega)^2}{4\mu} \right] \right\} \end{aligned}$$

The infinite sum is just

$$e^{-\omega^2/4\mu} \theta_3 \left[ -i\omega/4\mu, e^{-\frac{1}{4}\mu} \right] = \sqrt{4\pi\mu} \theta_3 (\pi\omega, e^{-4\pi^2\mu})$$

where  $\theta_3 (z, q)$  is a theta-function.<sup>2</sup>

One can show<sup>3</sup>

$$\frac{d}{dz} \left\{ \log \theta_3 (z, q) \right\} = -B (z, q) \sin (2z)$$

where  $B (z, q)$  is positive if  $z, q > 0$ .

In our case,  $z = \pi\omega$ ,  $q = e^{-4\pi^2\mu}$  and  $\theta_3$  is positive. Therefore

$$\text{Im} (F_c) > D \sin (2\pi\omega), \quad D > 0 \quad (2)$$

which is non-negative whenever  $\ell \leq \omega \leq \ell + \frac{1}{2}$ .

Now, it was shown in Ref. 1 that  $\text{Im} (F_c) \geq 0$  when  $\omega = \ell + i\sigma$ , and by a completely analogous argument it is also easy to show that  $\text{Im} (F_c) \geq 0$  when  $\omega = \ell + \frac{1}{2} + i\sigma$ . Therefore, by Cauchy's theorem  $\text{Im} (F_c) \geq 0$  whenever  $\ell \leq \text{Re}(\omega) \leq \ell + \frac{1}{2}$ .

In order to complete the proof of (1), we write (for real  $\omega$ )

$$\text{Im} (F) = K \frac{T_{\perp}}{T_{\parallel}} \sum_{k=-\infty}^{\infty} \left[ \omega + k \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) \right] \exp \left[ -\frac{(\omega + k)^2}{4\mu} \right] I_k(\lambda)$$

and restrict our attention to the case  $\frac{T_{\parallel}}{T_{\perp}} < 1$ , since  $\text{Im} (F)$  is obviously positive otherwise. Let  $\left[ \underset{\sim}{x} \right]$  denote the largest integer

contained in  $x$ . Thus, if

$$n = \left[ \underset{\sim}{\omega / \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right)} \right],$$

then  $\omega = \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) (n + \delta)$ , where  $0 \leq \delta < 1$ . Hence,

$$\text{Im (F)} > K \left( \frac{T_{\perp}}{T_{\parallel}} - 1 \right) \sum_{\ell=1}^{\infty} \ell \left\{ I_{\ell-n}(\lambda) \exp \left\{ - \left[ \ell - n \frac{T_{\parallel}}{T_{\perp}} + \delta \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) \right]^2 / 4\mu \right\} \right. \\ \left. - I_{\ell+n}(\lambda) \exp \left\{ - \left[ \ell + n \frac{T_{\parallel}}{T_{\perp}} - \delta \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) \right]^2 / 4\mu \right\} \right\}$$

and so  $\text{Im (F)} > 0$  if  $n \frac{T_{\parallel}}{T_{\perp}} > \delta \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right)$ . Rewriting, this becomes

$$\left[ \omega / \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) \right] \gtrsim \omega, \text{ stable.} \quad (3)$$

Now if  $\omega = \ell + \epsilon$ , then for  $1 - \frac{T_{\parallel}}{T_{\perp}} < \epsilon < 1$

$$\left[ \omega / \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) \right] \gtrsim \left[ \ell / \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) \right] + 1 \gtrsim \ell + 1 > \omega$$

so that we have stability if  $\ell + \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) < \omega < \ell + 1$ .

Moreover,

$$0 \leq \left[ \omega / \left( \frac{T_{\perp}}{T_{\parallel}} - 1 \right) \right] \leq \left[ \omega / \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) \right] - \left[ \omega \right] < \left[ \omega / \left( 1 - \frac{T_{\parallel}}{T_{\perp}} \right) \right] + 1 - \omega$$

so that when  $\left[ \omega / \left( \frac{T_{\perp}}{T_{\parallel}} - 1 \right) \right] \geq 1$ , inequality (3) is satisfied, which

completes the proof of (1).

In summary, if we refer to Fig. 1, no roots to the dispersion equation can be found for  $\omega$  lying in the shaded portion of the complex  $\omega$ -plane. In addition, the condition of marginal instability  $[\text{Im}(\omega) = 0^+]$  is possible only if both  $\ell - \frac{1}{2} < \omega < \ell - \frac{T_{\parallel}}{T_{\perp}}$  and  $\omega < \frac{T_{\perp}}{T_{\parallel}} - 1$

It may be mentioned that these theorems which have been proved explicitly above for the case of bi-Maxwellian distribution of particle velocities,

$$f(v_{\parallel}, v_{\perp}) \sim \exp\left(-\frac{mv_{\parallel}^2}{2KT_{\parallel}} - \frac{mv_{\perp}^2}{2KT_{\perp}}\right)$$

also hold when the distribution of parallel velocities is Lorentzian, viz.

$$f(v_{\parallel}, v_{\perp}) \sim \left(1 + \frac{mv_{\parallel}^2}{2KT_{\parallel}}\right)^{-1} \exp\left(-\frac{mv_{\perp}^2}{2KT_{\perp}}\right)$$

\*Research sponsored by the U. S. Atomic Energy Commission under contract with the Union Carbide Corporation.

<sup>1</sup>L. S. Hall and W. Heckrotte, Phys. Rev. A 1474, Vol. 134 (1964).

<sup>2</sup>E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge Univ. Press (Cambridge, 1950), 4th ed., pp. 464 and 474.

<sup>3</sup>ibid., p. 489.

UNCLASSIFIED  
ORNL-DWG 64-7042

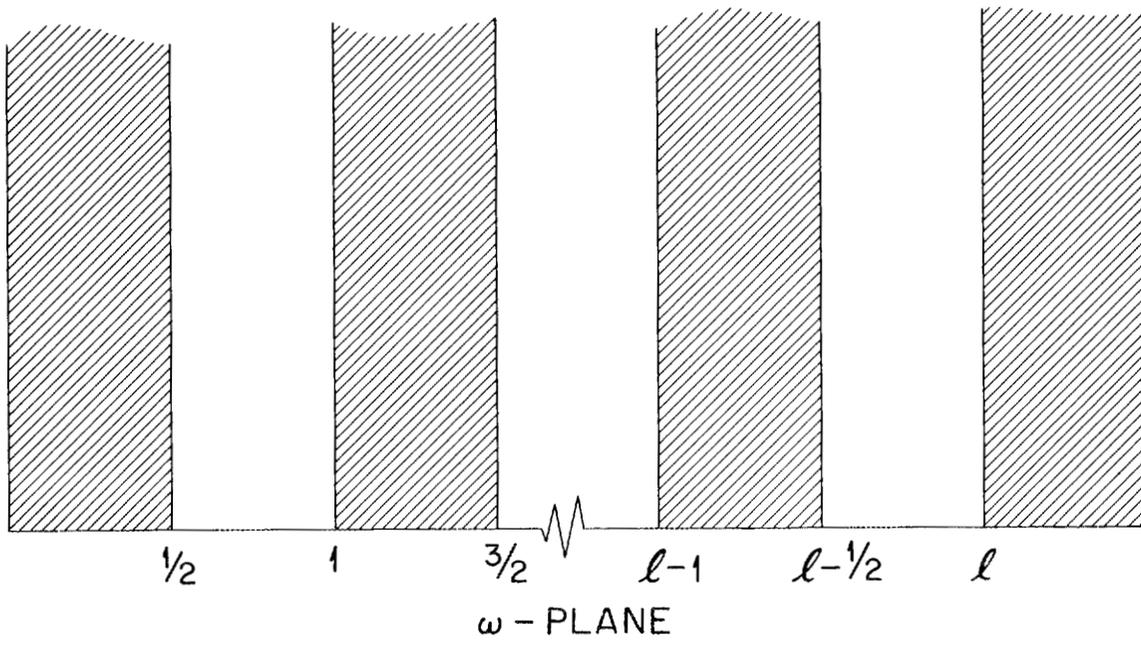


Fig. 1. Regions of Stability (Shaded) and Instability

## INTERNAL DISTRIBUTION

1. I. Alexeff
2. W. B. Ard
3. C. F. Barnett
4. C. O. Beasley
5. P. R. Bell
6. S. Cuperman
7. R. A. Dandl
8. R. A. Dory
9. J. L. Dunlap
10. H. O. Eason, Jr.
11. A. C. England
12. T. K. Fowler
13. W. F. Gauster
14. R. A. Gibbons
15. G. E. Guest
16. G. M. Haas
17. E. Guth
18. G. R. Haste
19. W. D. Jones
20. G. G. Kelley
21. C. E. Larson
22. N. H. Lazar
23. J. Lewin
24. J. N. Luton, Jr.
25. J. F. Lyon
26. J. R. McNally, Jr.
27. O. B. Morgan
28. R. V. Neidigh
29. J. Neufeld
30. C. E. Parker
31. H. Postma
32. J. F. Potts
33. M. Rankin
34. L. H. Reber
35. Y. Shima
36. E. D. Shipley
37. M. J. Skinner
38. A. H. Snell
39. W. L. Stirling
40. R. F. Stratton, Jr.
41. J. A. Swartout
42. A. M. Weinberg
43. T. A. Welton
44. O. C. Yonts
45. W. H. Bostick (consultant)
46. J. W. Flowers (consultant)
47. M. W. Garrett (consultant)
48. H. Grad (consultant)
49. E. G. Harris (consultant)
50. D. E. Harrison (consultant)
51. W. Heckrotte (consultant)
52. R. Hefferlin (consultant)
53. G. W. Hoffman (consultant)
54. V. W. Hughes (consultant)
55. E. W. McDaniel (consultant)
56. D. R. Montgomery (consultant)
57. C. E. Nielsen (consultant)
58. W. B. Pardo (consultant)
59. H. S. Robertson (consultant)
60. D. J. Rose (consultant)
61. L. P. Smith (consultant)
62. P. M. Stier (consultant)
63. J. D. Tillman (consultant)
- 64-65. Thermonuclear Division Library
- 66-67. Central Research Library
- 68-90. Laboratory Records Department
91. Laboratory Records - RC

## EXTERNAL DISTRIBUTION

- 92. I. B. Bernstein, Princeton University
- 93. S. C. Brown, Massachusetts Institute of Technology
- 94. M. B. Gottlieb, Princeton University
- 95-100. L. S. Hall, University of California, Lawrence  
Radiation Laboratory
- 101. T. Kammash, University of Michigan
- 102. D. W. Kerst, University of Wisconsin
- 103. A. C. Kolb, Naval Research Laboratory
- 104. J. A. Phillips, Los Alamos Scientific Laboratory
- 105. R. F. Post, University of California, Lawrence  
Radiation Laboratory
- 106. M. N. Rosenbluth, General Atomic, San Diego, California
- 107. A. Simon, General Atomic, San Diego, California
- 108. L. D. Smullin, Massachusetts Institute of Technology
- 109. L. Spitzer, Princeton University
- 110. J. L. Tuck, Los Alamos Scientific Laboratory
- 111. C. M. Van Atta, University of California, Lawrence  
Radiation Laboratory
- 112-113. Controlled Thermonuclear Research Branch, AEC, Washington
- 114-115. Research and Development Division, AEC, ORO
- 116-130. Division of Technical Information Extension, AEC, ORO

1

1

1

---

