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# REMARKS ON ESTIMATION PROBLEMS FOR THE PARAMETERS OF THE NEYMAN TYPE A DISTRIBUTION

K. O. Bowman
L. R. Shenton

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Errata

# REMARKS ON ESTIMATION PROBLEMS FOR THE PARAMETERS OF THE NEYMAN TYPE A DISTRIBUTION

K. O. Bowman and L. R. Shenton

The titles for Tables 2 and 3, pages 19 and 20, should be changed to read as follows:

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# REMARKS ON ESTIMATION PROBLEMS FOR THE PARAMETERS OF THE NEYMAN TYPE A DISTRIBUTION

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# REMARKS ON ESTIMATION PROBLEMS FOR THE PARAMETERS OF THE NEYMAN TYPE A DISTRIBUTION *

K. O. Bowman
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## Abstract

The Neyman Type A distribution defined on the non-negative integers depends on two parameters $\Theta_1$, $\Theta_2$, and is structurally the simplest of the Neyman contagious distributions. These have wide application in biology, ecology and other branches of natural science. At the present time very little is known about the distributional properties of the estimators of the parameters involved, and large sample theory is invoked with naive uncriticality. The study which is confined to the Type A distribution shows that moment estimators only have moments in a conditional sense, and that in their expansions higher order asymptotics may dominate the traditional lower order terms, in particular in two regions of the parameter space. The first region is that part of the parameter space in which the index of dispersion or contagion $\Theta_2$ is small; here the distribution becomes tangential to the Poisson distribution. The second region is that part of the parameter space contained by a parabolic arc sweeping away from the origin with an axis inclination of about 45°; within this region the distribution has high modality, the modes being spaced at intervals of $\Theta_2$ approximately.

In both these regions samples of more than 500 may be required to give the appearance of stability to the moment expansions of moment statistics and maximum likelihood statistics. As a rough assessment the estimators of $\Theta_1$ (the index parameter) require larger samples for stabilization than those for $\Theta_2$. Again the sample size is about the same for moment and maximum likelihood estimators. Moreover, in general, a larger sample is needed for a higher moment than for a lower moment.

Critical remarks are also offered on the meaning of asymptotic efficiency and its misleading undertones.

---

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# 1. INTRODUCTION

In several previous studies (see Bowman and Shenton 1965, 1966; Shenton and Myers, 1963) we have commented on the importance of second order terms in the biases and covariances of moment, mixed moment, and maximum likelihood estimators for the parameters of the negative binomial distribution and other over-dispersed distributions. We have also given a general method for the determination of the second order terms in the first two maximum likelihood estimator moments. Here we set out our findings for the Neyman Type A distribution (NTAd). A few seremdipic by-products which seem worth recording are given in the appendix.

To make the study self-contained would require the repetition of a good deal of material which is available elsewhere. We feel, however, that even a brief account of the importance of asymptotics in estimation theory and statistics is needed, and in such a rich field fresh examples abound. The mean value of a statistic, or the expected value of a random variable or vector, when it exists, is taken to be the population value as a first approximation. Similarly large sample results in statistical epidemiology stemming from stochastic models are asymptotically equivalent to those stemming from deterministic models. Again there are problems in pure and applied science for which asymptotics are free from the unimportant minutia of a complete solution. For this reason alone asymptotic results are valuable and may become the major objective in complicated structures - for example some asymptotic results are known for the otherwise hopeless problem of spatial distributions in self-avoiding random walks.

Returning to mean values of statistics, our aim is to call attention to some possible dangers in the overenthusiastic usage of O-term-ology. Is it good enough to assume that the equation

$$E\,t = \tau + o(1/n)$$

t being an arbitrary statistic, can be interpreted as $Et = \tau$ provided $n > N$, where the value of N is a subjective choice? The moments of many moment estimators for the parameters of a large class of modern compound distributions and mixtures of distributions are only available asymptotically. In many cases this also applies to maximum likelihood estimators, first order terms in biases and covariances being nonetheless aesthetically pleasing; however, the so-called high efficiency of these estimators <u>is in many cases a first-order property</u>, and in general they should not be exempt from further scrutiny merely because of the misleading terminology regarding their superiority.

We have shown elsewhere that there are techniques for evaluating higher order terms for moments of moments, and of course one can always fall back on the use of Fisher's k-statistics, or Fisher's k-statistics as modified by F. N. David (1949). These, however, are of little use in the application to maximum likelihood estimators, and for the latter formulae up to the $n^{-2}$ terms only are available at the moment; however the complication is such that they are only suitable for numerical work on a digital machine. Further development of these results by the methods already used seems somewhat remote.

## 2. MOMENTS OF MAXIMUM LIKELIHOOD ESTIMATORS

For a distribution depending on h parameters $\Theta_1, \Theta_2, \ldots \Theta_h$, for which the multivariate derivatives up to order five exist in some $\Theta$-interval, the second-order term in a covariance of any two maximum likelihood (m.l.) estimators are given by (Bowman & Shenton, 1965).

$$\mathrm{Cov}_2(\hat{\Theta}_a, \hat{\Theta}_b) = \Theta_1 + \Theta_3 + \Theta_4, \tag{1}$$

where

$$\Theta_1 = - L^{ab}$$

$$\Theta_3 = L^{a\beta} L^{b\alpha} L^{\gamma\delta} \{[\alpha\delta,\beta,\gamma] + [\beta\delta,\alpha,\gamma] + [\alpha\beta\gamma\delta]$$

$$+ 3[\alpha\beta,\gamma\beta] + 2[\alpha\beta\gamma,\delta] + \frac{1}{2}[\beta\gamma\delta,\alpha] + \frac{1}{2}[\alpha\gamma\delta,\beta]\}$$

$$\Theta_4 = = L^{a\beta} L^{b\alpha} L^{\gamma\varepsilon} L^{\delta\zeta} \{\frac{1}{2}[\alpha\gamma\zeta][\beta,\delta,\varepsilon] + \frac{1}{2}[\beta\gamma\zeta][\alpha,\delta,\varepsilon]$$

$$+ [\alpha\gamma\beta][\delta\varepsilon\zeta] + 5[\alpha\gamma\delta][\beta\varepsilon\zeta]/2 + [\alpha,\delta\varepsilon][\beta\zeta\gamma] + [\beta,\delta\varepsilon][\alpha\zeta\gamma]$$

$$+ 2[\alpha\beta\zeta][\alpha\gamma,\varepsilon] + 3[\alpha\zeta,\varepsilon][\beta\gamma\delta] + 3[\alpha\gamma\delta][\beta\zeta,\varepsilon]$$

$$+ \frac{1}{2}[\alpha,\beta\zeta][\gamma\delta\varepsilon] + \frac{1}{2}[\alpha\zeta,\beta][\gamma\delta\varepsilon] + [\alpha\delta,\varepsilon][\gamma\zeta,\beta] + [\beta\delta,\varepsilon][\gamma\zeta,\alpha]$$

$$+ [\alpha\delta,\beta][\gamma\zeta,\varepsilon] + [\beta\delta,\alpha][\gamma\zeta,\varepsilon] + [\alpha\delta,\gamma][\beta\varepsilon,\zeta]\}.$$

In this the Greek symbols run through the integers 1, 2, ..., h and there is an implied summation over matching symbols over all permutations. In addition, with probability function $p_r$, the terms in parenthesis are defined as

$$[\alpha,\beta,\gamma] = (p\Gamma_\alpha\Gamma_\beta\Gamma_\gamma) = \sum_r p_r \frac{\partial \log p_r}{\partial\Theta_\alpha}\frac{\partial \log p_r}{\partial\Theta_\beta}\frac{\partial \log p_r}{\partial\Theta_\gamma} ,$$

$$[\alpha\beta,\gamma] = (p\Gamma_{\alpha\beta}\Gamma_\gamma) = \sum_r p_r \frac{\partial^2 \log p_r}{\partial\Theta_\alpha\partial\Theta_\beta}\frac{\partial \log p_r}{\partial\Theta_\gamma} .$$

with obvious generalizations. Lastly,

$$L_{\alpha\beta} = -(p\Gamma_{\alpha\beta}) = (p\Gamma_\alpha\Gamma_\beta)$$

$$L_{\alpha\beta}L^{\beta\gamma} = \varepsilon_{\alpha\gamma} \quad \text{(Kronecker delta).}$$

Formulae for the biases of maximum likelihood estimators are more involved and have been described in Bowman & Shenton (1965) and Shenton & Bowman (1963). We have, using $E_1$ and $E_2$ for the $n^{-1}$ and $n^{-2}$ terms,

$$E_1\hat{\Theta}_a = L^{a\alpha}L^{\beta\gamma}B_2(\alpha,\beta,\gamma) \tag{2a}$$

$$\begin{aligned} E_2\hat{\Theta}_a = {} & -L^{a\alpha}L^{\beta\gamma}B_2(\alpha,\beta,\gamma) + L^{a\alpha}L^{\beta\delta}L^{\gamma\varepsilon}B_3(\alpha,\beta,\delta,\gamma,\varepsilon) \\ & + L^{a\alpha}L^{\beta\gamma}L^{\delta\zeta}L^{\varepsilon\eta}B_4(\alpha,\beta,\gamma,\delta,\zeta,\varepsilon,\eta) \\ & + L^{a\alpha}L^{\beta\gamma}L^{\delta\varepsilon}L^{\zeta\eta}L^{\theta i}B_5(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta,\theta,i) \end{aligned} \tag{2b}$$

where

$$2B_2 = -[\alpha\beta\gamma] - 2[\alpha\beta,\gamma],$$

$$\begin{aligned} 8B_3 = {} & [\alpha\beta\gamma\delta\varepsilon] + 4[\alpha\gamma,\beta\delta\varepsilon] + 8[\delta\varepsilon,\alpha\beta\gamma] + 4[\alpha\beta\gamma\varepsilon,\delta] \\ & + 4[\alpha\beta\gamma,\delta,\varepsilon] + 8[\alpha\beta,\gamma\delta,\varepsilon], \end{aligned}$$

$$
\begin{aligned}
4B_4 = {} & (2[\alpha\zeta\varepsilon\delta][\eta\beta,\gamma] + 2[\beta\varepsilon\delta\zeta][\alpha\gamma,\eta] + 4[\alpha\beta\varepsilon\delta][\eta\zeta,\gamma]) \\
& + ([\alpha\zeta\varepsilon\delta][\eta\gamma\beta] + 2[\alpha\beta\varepsilon\delta][\eta\gamma\zeta] + 2[\delta\beta\varepsilon\eta][\alpha\gamma\zeta]) \\
& + (2[\alpha\varepsilon\delta][\eta\beta,\zeta\gamma] + 4[\alpha\gamma\zeta][\delta\eta,\varepsilon\beta] + 4[\beta\varepsilon\delta][\alpha\gamma,\eta\zeta] \\
& + 2[\zeta\gamma\beta][\alpha\eta,\varepsilon\delta]) \\
& + (4[\alpha\zeta\varepsilon,\eta][\delta\beta,\gamma] + 4[\alpha\zeta\varepsilon,\gamma][\delta\beta,\eta] + 4[\delta\beta\varepsilon,\eta][\alpha\zeta,\gamma]) \\
& + (2[\alpha\beta\varepsilon,\eta][\gamma\delta\zeta] + 4[\delta\beta\varepsilon,\eta][\alpha\gamma\zeta] + 4[\alpha\beta\varepsilon,\zeta][\gamma\delta\eta] \\
& + 2[\delta\eta\varepsilon,\beta][\alpha\gamma\zeta]) \\
& + (4[\alpha\eta,\zeta\gamma][\varepsilon\delta,\beta] + 4[\varepsilon\delta,\zeta\gamma][\alpha\eta,\beta] + 4[\alpha\eta,\varepsilon\delta][\zeta\gamma,\beta]) \\
& + (4[\alpha\gamma\eta][\varepsilon\zeta,\beta,\delta] + 2[\gamma\delta\varepsilon][\alpha\beta,\zeta,\eta]) \\
& + 2[\alpha\beta\delta\varepsilon][\gamma,\zeta,\eta]/3,
\end{aligned}
$$

$$
\begin{aligned}
8B_5 = {} & [\alpha\delta\zeta]\ \{[\varepsilon\beta\gamma][\eta\theta i] + 2[\varepsilon\theta\gamma][\eta\beta i] + 4[\eta\beta\varepsilon][\theta\gamma i] + 8[\eta\beta\theta][\varepsilon\gamma i] \\
& + 2[\varepsilon\beta\gamma][\eta\theta,i] + 4[\varepsilon\theta\gamma][\eta\beta,i] + 2[\eta\theta i][\varepsilon\beta,\gamma] + 4[\eta\theta\gamma][\varepsilon\beta,i] \\
& + 8[\eta\beta\theta][\gamma\varepsilon,i] + 8[\eta\beta\theta][\gamma i,\varepsilon] + 8[\eta\beta\varepsilon][\gamma\theta,i] + 8[\gamma\theta\varepsilon][\eta\beta,i] \\
& + 4[\gamma\theta i][\eta\beta,\varepsilon] + 4[\eta\theta,i][\varepsilon\beta,\gamma] + 4[\varepsilon\beta,i][\eta\theta,\gamma] + 8[\eta\theta,\beta][i\gamma,\varepsilon] \\
& + 8[\eta\theta,\varepsilon][i\gamma,\beta] + 4[\beta\varepsilon\theta][\eta,\gamma,i]\} \\
& + [\delta\gamma i]\ \{8[\beta\eta\theta][\alpha\varepsilon,\zeta] + 4[\beta\eta\zeta][\alpha\varepsilon,\theta] + 8[\alpha\varepsilon,\theta][\beta\eta,\zeta] \\
& + 8[\alpha\varepsilon,\zeta][\beta\eta,\theta] + 8[\alpha\zeta,\beta][\eta\varepsilon,\theta]\} \quad .
\end{aligned}
$$

Evidently the main problem in the application of the formulae to a given distribution is the evaluation of the expectation of products of derivatives of the logarithm of the probability function (or derivatives of log. likelihood). Since these derivatives go as far as the fifth order, and since many discrete compound distributions have their probability functions defined recursively (the order of the recursion often variable), a structure suitable for machine use must be sought. The scheme we now illustrate for the Neyman Type A distribution (NTAd) has

wider application which we shall describe elsewhere.

## 3. NEYMAN TYPE A PROBABILITY FUNCTION AND A RECURSIVE SCHEME FOR ITS DERIVATIVES

### 3.1 The Distribution

The Neyman contagious distributions were introduced in 1939 and have proved to be of considerable interest in biology, ecology and other branches of natural science; the original paper has 70 references in a recent citation bibliography. The two-parameter type A distribution in particular has many applications and has been applied more frequently that the multi-parameter types B and C.

The distributions arose out of a study of the distribution of larvae in experimental plots. The larvae are hatched from eggs which occur in colonies (or masses) so that the presence of a single larva is less likely than under a Poisson law. This aspect was noticed by experimentalists who expected the Poisson structure to apply; they found too few occurrences of "no events" and too many of "one event."

A similar concept arises in statistical ecology where quadrat sampling is used to study time changes in local flora and fauna.

Now suppose the number (N) of colonies per quadrat is independently distributed according to the Poisson law with mean $\theta_1$; thus the probability of n colonies is

$$\Pr(N = n) = e^{-\theta_1} \theta_1^n/n! \qquad (n = 0,1,\ldots;\theta_1 > 0). \qquad (3)$$

If in addition the number (N') of members per colony is also distributed according to the Poisson law with mean $\theta_2$, then the distribution of the number of members per quadrat is the distribution of the random variate X, where

$$\Pr(X = x) = \sum_u \Pr(X = x \quad N = u) \Pr(N = u)$$

$$= \sum_o^\infty P_s \, q_x(s\theta_2)$$

and

$$q_x(\theta_2) = e^{-\theta_2} \theta_2^x / x!$$

$$P_s = e^{-\theta_1} \theta_1^s / s! \ .$$

Thus

$$\Pr(X = x)$$

$$= P_x$$

$$= \frac{e^{-\theta_1} \theta_2^x}{x!} \{ 0^x + \frac{\lambda 1^x}{1!} + \frac{\lambda^2 2^x}{2!} + \ldots . \} \tag{4}$$

where $\lambda = \theta_1 e^{-\theta_2}$ ,

$$\begin{cases} 0^x = 1, & x = 0 \\ \quad = 0, & x \neq 0. \end{cases}$$

There are two alternative forms for $P_x$ (see, for example Douglas (1965)), namely

$$P_{x+1} = \frac{\theta_1 \, \theta_2 \, e^{-\theta_2}}{x + 1} \sum_{s=o}^{x} \frac{\theta_2^s}{s!} P_{x-s} \quad (x = 0,1,\ldots) \tag{5}$$

$$P_o = e^{\theta_1 (e^{-\theta_2} - 1)} ,$$

and

$$P_x = \frac{P_o\ \theta_2^x}{x!}\ \mu_x' , \tag{6}$$

where $\mu_x'$ is the x-th. crude moment of a Poisson variate with mean $\lambda$.

Further properties concerning for example modality, tail-length, and approximations, are relegated to the appendix since they are not of paramount importance for our immediate concern.

3.2 Recursive Schemes for the Derivatives

By differentiating the probability generating function (pgf) with respect to $\theta_1$, $\theta_2$, and t, where

$$\begin{aligned} \text{pgf} &= G(t) \\ &= e^{\theta_1(e^{\theta_2(t-1)} - 1)} \qquad (\theta_1, \theta_2 > 0) \end{aligned} \tag{7}$$

we find (Shenton, 1949)

$$x_1 = \frac{\partial \log P_x}{\partial \theta_1} = -1 + \frac{(x+1)}{\theta_1\theta_2}\frac{P_{x+1}}{P_x} \tag{8.1}$$

$$x_2 = \frac{\partial \log P_x}{\partial \theta_2} = \frac{x}{\theta_2} + \theta_1(x_1 + 1). \tag{8.2}$$

It is convenient to extend the above notation; for example

$$x_{11} = \frac{\partial x_1}{\partial \theta_1}, \qquad x_{12} = \frac{\partial x_1}{\partial \theta_2} = \frac{\partial x_2}{\partial \theta_1},$$

$$x_{122} = x_{212} = x_{221} = \frac{\partial^2 x_1}{\partial \theta_2^2} \qquad \text{etc.}$$

Moreover we shall have occasion to increment the argument; thus

$$(x+1)_1 = \frac{\partial \log P_{x+1}}{\partial \theta_1}$$

and clearly the distributive law cannot be invoked. In passing, the reader should note that only when a subscript is used with the argument x does it imply a derivative.

We can now set up a correlated chain of relations encompassing derivatives of any desired order. One illustration will suffice to describe the development. Thus

$$
\begin{aligned}
x_{11} &= \frac{\partial x_1}{\partial \theta_1} \\
&= \frac{\partial}{\partial \theta_1} \left\{ \frac{(x+1)}{\theta_1 \theta_2} \frac{P_{x+1}}{P_x} \right\} \\
&= \frac{(x+1)}{\theta_2} \left\{ -\frac{1}{\theta_1^2} \frac{P_{x+1}}{P_x} + \frac{1}{\theta_1} \frac{1}{P_x} \frac{\partial P_{x+1}}{\partial \theta_1} - \frac{P_{x+1}}{\theta_1 P_x^2} \frac{\partial P_x}{\partial \theta_1} \right\} \\
&= -\frac{1}{\theta_1}(x_1 + 1) - \frac{(x+1)}{\theta_1 \theta_2} \frac{P_{x+1}}{P_x} \left\{ -1 + \frac{(x+1)}{\theta_1 \theta_2} \frac{P_{x+1}}{P_x} \right\} \\
&\quad + \frac{(x+1)}{\theta_1 \theta_2} \frac{P_{x+1}}{P_x} \left\{ -1 + \frac{(x+2)}{\theta_1 \theta_2} \frac{P_{x+2}}{P_{x+1}} \right\} \\
&= -\frac{(x_1 + 1)}{\theta_1} - (x_1 + 1)(x_1 - (x+1)_1).
\end{aligned}
$$

Hence

$$
x_{11} = x_1(x+1)_1 - x_1^2 - x_1\left(\frac{1}{\theta_1} + 1\right) + (x+1)_1 - \frac{1}{\theta_1}, \tag{8.3}
$$

so that this second derivative can be found from (8.1) and its incremented form. Similarly, omitting details, we find:

$$x_{111} = x_{11}((x + 1)_1 - 2x_1 - \frac{1}{\theta_1}) + (x + 1)_{11} - x_{11}$$

$$+ x_1(x + 1)_{11} + \frac{(x_1 + 1)}{\theta_1^2} \tag{8.4}$$

$$x_{1111} = x_{111}y_1 + 2x_{11}y_{11} + (x_1 + 1)y_{111} \tag{8.5}$$

$$x_{11111} = x_{1111}y_1 + 3x_{111}y_{11} + 3x_{11}y_{111} + (x_1 + 1)y_{1111}$$

$$(y_1 = (x + 1)_1 - x_1 - \frac{1}{\theta_1}, \; y_{11} = \frac{\partial y_1}{\partial \theta_1} \text{ etc.}) \tag{8.6}$$

$$x_2 = \frac{x}{\theta_2} - \theta_1(x_1 + 1) \tag{8.7}$$

$$x_{21} = - 1 - x_1 - \theta_1 x_{11} \tag{8.8}$$

$$x_{211} = - 2x_{11} - \theta_1 x_{111} \tag{8.9}$$

$$x_{2111} = - 3x_{111} - \theta_1 x_{1111} \tag{8.10}$$

$$x_{21111} = -4x_{1111} - \theta_1 x_{11111} \tag{8.11}$$

$$x_{22} = - \frac{x}{\theta_2^2} - \theta_1 x_{12} \tag{8.12}$$

$$x_{221} = - x_{12} - \theta_1 x_{112} \tag{8.13}$$

$$x_{2211} = -\ 2x_{112} - \theta_1 x_{1112} \tag{8.14}$$

$$x_{22111} = -\ 3x_{1112} - \theta_1 x_{11112} \tag{8.15}$$

$$x_{222} = \frac{2x}{\theta_2^3} - \theta_1 x_{122} \tag{8.16}$$

$$x_{2221} = -\ x_{122} - \theta_1 x_{1122} \tag{8.17}$$

$$x_{22211} = -\ 2x_{1122} - \theta_1 x_{11122} \tag{8.18}$$

$$x_{2222} = -\ \frac{6x}{\theta_2^4} - \theta_1 x_{1222} \tag{8.19}$$

$$x_{22221} = -\ x_{1222} - \theta_1 x_{11222} \tag{8.20}$$

$$x_{22222} = \frac{24x}{\theta_2^5} - \theta_1 x_{12222}\ . \tag{8.21}$$

The recursive nature of the scheme will be appreciated by tracing any one formula back to (8.1) and/or (8.2). For example there is the tree,

$$\begin{tikzcd}[row sep=small]
 & & & (8.1) & \\
 & (8.13) \arrow[r] & (8.8) \arrow[ur] \arrow[dr] & & \\
(8.17) \arrow[ur] \arrow[dr] & & & (8.3) \arrow[r] & (8.1) \\
 & (8.14) \arrow[r] \arrow[dr] & (8.9) \arrow[ur] \arrow[dr] & & \\
 & & & (8.4) \arrow[r] & (8.3) \arrow[r] & (8.1) \\
 & & (8.10) \arrow[ur] \arrow[dr] & & \\
 & & & (8.5) \arrow[r] & (8.4) \arrow[r] & (8.3) \arrow[r] & (8.1)
\end{tikzcd} \tag{9}$$

The relations in (8) with (5)-(6) can be used in (1) and (2) to derive expressions for the biases and covariances of $\hat{\theta}_1$, $\hat{\theta}_2$, the m.l. estimators of $\theta_1$, $\theta_2$. This scheme has been programmed and the

numerical results used to produce the graphical analysis exhibited in Figures 1-5 Before commenting on the results it is convenient to make some remarks about moment estimators for $\Theta_1$, $\Theta_2$.

## 4. MOMENT ESTIMATORS FOR $\Theta_1$, $\Theta_2$

### 4.1 Notation

These, in their simplest form, are

$$\Theta_1^* = \frac{m_1^2}{m_2 - m_1 - m_1^2} \tag{10a}$$

$$\Theta_2^* = \frac{m_2'}{m_1} - m_1 - 1 \tag{10b}$$

where $m_1 = \Sigma\, x_j/n$,

$m_s' = \Sigma\, x_j^s/n$, $(s = 2,3,\ldots)$

for a random sample $(x_1, x_2, \ldots, x_n)$. Evidently $\Theta_1^*$ can be infinite or negative if $m_2' \leq m_1 + m_1^2$, whereas $\Theta_2^*$ can lack definition in the unlikely circumstance $x_1 = x_2 = \ldots = x_n = 0$; it may also be negative. Actually the exact value of $E\Theta_2^*$ can be found. Ignoring the probability that the sample is concentrated at the origin, we can show (see, for example, Bowman and Shenton (1966)) that

$$E\Theta_2^* = -\Theta_1\Theta_2 - e^{-N} + \int_0^N e^{-x}\left(1 + \Theta_1 - \frac{x}{n}\right)\left\{\Theta_2 + \log\left(1 - \frac{x}{n\Theta_1}\right)\right\}dx \tag{11}$$

$$(N = n\Theta_1(1 - e^{-\Theta_2})).$$

This can be expanded as an asymptotic series in $n^{-1}$, yielding

$$E\Theta_2^* = \Theta_2 - e^{-N}(1 + \Theta_2 + \Theta_1\Theta_2) + \frac{\Lambda_1}{n} + \frac{\Lambda_2}{n^2} + \ldots \tag{12}$$

where

$$\Lambda_1 = -\left(\Theta_2 + \frac{1 + \Theta_1}{\Theta_1}\right)\left(1 - (N + 1)e^{-N}\right),$$

$$\Lambda_s = (-1)^s s!\left(\frac{1}{(s - 1)\Theta_1^{s-1}} - \frac{1 + \Theta_1}{s\Theta_1^s}\right)\left(1 - e^{-N}\sum_{r=o}^{s}\frac{N^r}{r!}\right)$$

$(s = 2,3,\ldots)$.

Otherwise the integral in (11) can be evaluated by quadrature to any desired accuracy. There is no corresponding formula for $E\Theta_1^*$ and indeed the higher moments of $\Theta_2^*$ become complicated.

## 4.2 Singularities and Sample Size

Since $\Theta_1^*$ and $\Theta_2^*$ are both of the form $f(m_1, m_2')$, asymptotic expansions for the moments can be found following the methods set out in Shenton and Myers (1965). In essence one merely expands $f(m_1, m_2')$ about $(\mu_1, \mu_2')$ and then forms the expected value of the power of f desired.

The singularity in the denominator of $\Theta_1^*$ (ignoring the possibility of a matching singularity in the numerator) is a warning that the asymptotic expansion of its moments may be unstable. In fact $E(\Theta_1^*)^s$ $(s = 1,2,\ldots)$ only exists conditional on $m_2' - m_1 - m_1^2 \neq 0$; if our interest goes to the stage of admissibility then the expectation is only valid if $m_2' - m_1 - m_1^2 > 0$. Now this condition is arbitrary in the sense that the inequality on the moments has to be quantified before we can reach an evaluation. Thus we can evaluate the expectation of $(\Theta_1^*)^s$

conditional on $m_2' - m_1 - m_1^2 > \varepsilon > 0$, so that we would be better advised to write the mean value as $E_\varepsilon(\Theta_1^*)^s$; this illustrates the important point that there is no unique value for the expectation for given $(\Theta_1, \Theta_2, n)$; nonetheless there may be an approximate value nearly independent of $\varepsilon$ if n is large enough. Put crudely, we can assign an approximate value to the Stieltjes integral by sharpening the tail-spread of the bivariate surface with a larger value of n; this has the effect of withdrawing the major part of the distribution from the singular curve in the $(m_1, m_2')$ plane. An intuitive approach to this idea is to arrange a sample size n so that $Et \geq c\sigma(t)$, where $t = m_2' - m_1 - m_1^2$; the value of $c > 3$ would preferably be chosen conservatively.

Now

$$Et = \Theta_1\Theta_2^2 - \frac{\Theta_1\Theta_2(1 + \Theta_2)}{n} \tag{13a}$$

and to order $n^{-1}$,

$$\text{Var}t \sim \frac{\Theta_1\Theta_2^2}{n}\left(2\Theta_1(1 + \Theta_2)^2 + 2 + 4\Theta_2 + \Theta_2^2\right). \tag{13b}$$

A few illustrations of the required sample size (approximately) with $c = 5$ are as follows:

| $\Theta_1$ | $\Theta_2$ | n (greater than) |
|---|---|---|
| 1 | 1 | 375 |
| 1 | 5 | 120 |
| 5 | 1 | 235 |
| 5 | 5 | 82 |
| 5 | 10 | 70 |
| 10 | 1 | 220 |
| 10 | 5 | 80 |
| 10 | 10 | 65 |
| 10 | 20 | 60 |

With these sample sizes we have for example, using the appropriate asymptotic expansions,

| | | | |
|---|---|---|---|
| $\Theta_1 = 1$ | $\Theta_2 = 1$ | $n = 375$ | $E\Theta_1^* \sim 1 + .0347 + .0010 + .0001$ |
| | | | $E\hat{\Theta}_1 \sim 1 + .0152 + .0007$ |
| $\Theta_1 = 5$ | $\Theta_2 = 1$ | $n = 1200$ | $E\Theta_1^* \sim 5 + .0442 + .0006$ |
| | | | $E\hat{\Theta}_1 \sim 5 + .0362 + .0005$ |
| $\Theta_1 = 5$ | $\Theta_2 = 5$ | $n = 400$ | $E\Theta_1^* \sim 5 + .0517 + .0005$ |
| | | | $E\hat{\Theta}_1 \sim 5 + .0498 + .0005$ |

Ignoring the importance of higher order terms we might take, with $\Theta_1 = \Theta_2 = 1$, $n = 50$, $E\Theta_1^* \sim 1$, whereas the subsequent terms are

$E\Theta_1^* \sim 1 + .2600 + .0568 + .0491 + .0006 + .0805$,

showing a tendency to increase for the higher order terms.

By contrast using $Et \geq c\sigma(t)$ for the denominator of $\Theta_2^*$, we have

$$Em_1 = \Theta_1\Theta_2 , \tag{14a}$$

$$\text{Var } m_1 = \frac{\Theta_1\Theta_2(1 + \Theta_2)}{n} , \tag{14b}$$

so that a "safe" sample size n is given by

$$n = \frac{c^2(1 + \Theta_2)}{\Theta_1\Theta_2} . \tag{15}$$

Simple calculations show that this sample size is very much smaller than the one required to control $E\Theta_1^*$.

In both cases a "safe" sample size for the higher moments could be found similarly, and one would expect the highest moment to demand the largest sample, although there could be exceptions to this.

### 4.3 Higher Order Terms in $E\Theta_1^*$

Methods for finding the higher order terms in the moments of moment estimators have been described in some detail in Shenton and Myers (1963). The glossary of terms in this paper has been programmed and applied to the bias of $\Theta_1^*$. The results are given in Table 1. For the parameter values used, that is $\Theta_1$, $\Theta_2$ ranging from 1 to 25, a sample size of a hundred is large enough to control the terms in the asymptotic series. However for small $\Theta_2$, there is a tendency for the $n^{-1}$ and $n^{-2}$ terms to be of about the same size, and this is particularly so if in addition $\Theta_1$ is small. It will be noticed that the sample sizes given in 4.2 for $E\Theta_1^*$ are in general very conservative.

BIAS OF $\theta_1^*$ $N^{-1}$ TO $N^{-5}$ TERMS WHEN N = 100 (MOMENTS ESTIMATOR)

| $\theta_1 \backslash \theta_2$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | .1300<br>$.1420^{-1}$<br>$.6134^{-2}$<br>$.4036^{-4}$<br>$.2517^{-2}$ | $.7000^{-1}$<br>$.2400^{-2}$<br>$.7930^{-3}$<br>$-.1661^{-3}$<br>$.1659^{-3}$ | $.5444^{-1}$<br>$.1131^{-2}$<br>$.3558^{-3}$<br>$-.7628^{-4}$<br>$.5230^{-4}$ | $.4750^{-1}$<br>$.7656^{-3}$<br>$.2299^{-3}$<br>$-.4645^{-4}$<br>$.2670^{-4}$ | $.4360^{-1}$<br>$.6096^{-3}$<br>$.1741^{-3}$<br>$-.3310^{-4}$<br>$.1710^{-4}$ | $.4111^{-1}$<br>$.5272^{-3}$<br>$.1434^{-3}$<br>$-.2588^{-4}$<br>$.1244^{-4}$ | $.3939^{-1}$<br>$.4775^{-3}$<br>$.1243^{-3}$<br>$-.2148^{-4}$<br>$.9785^{-5}$ | $.3812^{-1}$<br>$.4447^{-3}$<br>$.1113^{-3}$<br>$-.1854^{-4}$<br>$.8114^{-5}$ | $.3716^{-1}$<br>$.4217^{-3}$<br>$.1020^{-3}$<br>$-.1647^{-4}$<br>$.6980^{-5}$ |
| 2 | .2300<br>$.3090^{-1}$<br>$.9339^{-2}$<br>$.2413^{-2}$<br>$.1736^{-2}$ | .1300<br>$.7537^{-2}$<br>$.1215^{-2}$<br>$.5888^{-4}$<br>$.7223^{-4}$ | .1033<br>$.4263^{-2}$<br>$.5333^{-3}$<br>$.3246^{-5}$<br>$.2116^{-4}$ | $.9125^{-1}$<br>$.3145^{-2}$<br>$.3400^{-3}$<br>$-.2943^{-5}$<br>$.1072^{-4}$ | $.8440^{-1}$<br>$.2607^{-2}$<br>$.2555^{-3}$<br>$-.3910^{-5}$<br>$.6900^{-5}$ | $.8000^{-1}$<br>$.2296^{-2}$<br>$.2095^{-3}$<br>$-.3940^{-5}$<br>$.5056^{-5}$ | $.7694^{-1}$<br>$.2095^{-2}$<br>$.1811^{-3}$<br>$-.3776^{-5}$<br>$.4009^{-5}$ | $.7469^{-1}$<br>$.1955^{-2}$<br>$.1619^{-3}$<br>$-.3585^{-5}$<br>$.3347^{-5}$ | $.7296^{-1}$<br>$.1853^{-2}$<br>$.1481^{-3}$<br>$-.3408^{-5}$<br>$.2898^{-5}$ |
| 3 | .3300<br>$.4793^{-1}$<br>$.1347^{-1}$<br>$.4242^{-2}$<br>$.2246^{-2}$ | .1900<br>$.1270^{-1}$<br>$.1781^{-2}$<br>$.1971^{-3}$<br>$.7231^{-4}$ | .1522<br>$.7401^{-2}$<br>$.7743^{-3}$<br>$.4835^{-4}$<br>$.1833^{-4}$ | .1350<br>$.5527^{-2}$<br>$.4889^{-3}$<br>$.2077^{-4}$<br>$.8649^{-5}$ | .1252<br>$.4605^{-2}$<br>$.3646^{-3}$<br>$.1163^{-4}$<br>$.5352^{-5}$ | .1189<br>$.4065^{-2}$<br>$.2973^{-3}$<br>$.7557^{-5}$<br>$.3827^{-5}$ | .1145<br>$.3713^{-2}$<br>$.2558^{-3}$<br>$.5392^{-5}$<br>$.2984^{-5}$ | .1112<br>$.3466^{-2}$<br>$.2279^{-3}$<br>$.4097^{-5}$<br>$.2462^{-5}$ | .1088<br>$.3284^{-2}$<br>$.2079^{-3}$<br>$.3256^{-5}$<br>$.2112^{-5}$ |
| 4 | .4300<br>$.6505^{-1}$<br>$.1782^{-1}$<br>$.5978^{-2}$<br>$.2934^{-2}$ | .2500<br>$.1787^{-1}$<br>$.2383^{-2}$<br>$.3178^{-3}$<br>$.8640^{-4}$ | .2011<br>$.1054^{-1}$<br>$.1031^{-2}$<br>$.8609^{-4}$<br>$.1998^{-4}$ | .1787<br>$.7909^{-2}$<br>$.6474^{-3}$<br>$.4011^{-4}$<br>$.8869^{-5}$ | .1660<br>$.6604^{-2}$<br>$.4807^{-3}$<br>$.2409^{-4}$<br>$.5269^{-5}$ | .1578<br>$.5834^{-2}$<br>$.3906^{-3}$<br>$.1666^{-4}$<br>$.3661^{-5}$ | .1520<br>$.5331^{-2}$<br>$.3350^{-3}$<br>$.1259^{-4}$<br>$.2795^{-5}$ | .1478<br>$.4977^{-2}$<br>$.2978^{-3}$<br>$.1009^{-4}$<br>$.2269^{-5}$ | .1446<br>$.4715^{-2}$<br>$.2712^{-3}$<br>$.8421^{-5}$<br>$.1921^{-5}$ |
| 5 | .5300<br>$.8220^{-1}$<br>$.2226^{-1}$<br>$.7685^{-2}$<br>$.3672^{-2}$ | .3100<br>$.2304^{-1}$<br>$.3000^{-2}$<br>$.4322^{-3}$<br>$.1047^{-3}$ | .2500<br>$.1368^{-1}$<br>$.1294^{-2}$<br>$.1211^{-3}$<br>$.2303^{-4}$ | .2225<br>$.1029^{-1}$<br>$.8099^{-3}$<br>$.5779^{-4}$<br>$.9822^{-5}$ | .2068<br>$.8602^{-2}$<br>$.5995^{-3}$<br>$.3536^{-4}$<br>$.5663^{-5}$ | .1967<br>$.7604^{-2}$<br>$.4859^{-3}$<br>$.2485^{-4}$<br>$.3846^{-5}$ | .1896<br>$.6949^{-2}$<br>$.4161^{-3}$<br>$.1902^{-4}$<br>$.2884^{-5}$ | .1844<br>$.6488^{-2}$<br>$.3692^{-3}$<br>$.1541^{-4}$<br>$.2308^{-5}$ | .1804<br>$.6146^{-2}$<br>$.3358^{-3}$<br>$.1300^{-4}$<br>$.1932^{-5}$ |
| 6 | .6300<br>$.9937^{-1}$<br>$.2674^{-1}$<br>$.9378^{-2}$<br>$.4430^{-2}$ | .3700<br>$.2821^{-1}$<br>$.3624^{-2}$<br>$.5436^{-3}$<br>$.1247^{-3}$ | .2989<br>$.1682^{-1}$<br>$.1560^{-2}$<br>$.1548^{-3}$<br>$.2666^{-4}$ | .2662<br>$.1267^{-1}$<br>$.9742^{-3}$<br>$.7466^{-4}$<br>$.1109^{-4}$ | .2476<br>$.1060^{-1}$<br>$.7197^{-3}$<br>$.4607^{-4}$<br>$.6266^{-5}$ | .2356<br>$.9373^{-2}$<br>$.5824^{-3}$<br>$.3259^{-4}$<br>$.4186^{-5}$ | .2271<br>$.8567^{-2}$<br>$.4980^{-3}$<br>$.2508^{-4}$<br>$.3097^{-5}$ | .2209<br>$.7998^{-2}$<br>$.4414^{-3}$<br>$.2042^{-4}$<br>$.2451^{-5}$ | .2162<br>$.7577^{-2}$<br>$.4011^{-3}$<br>$.1730^{-4}$<br>$.2033^{-5}$ |
| 7 | .7300<br>.1165<br>$.3125^{-1}$<br>$.1106^{-1}$<br>$.5198^{-2}$ | .4300<br>$.3339^{-1}$<br>$.4251^{-2}$<br>$.6534^{-3}$<br>$.1456^{-3}$ | .3478<br>$.1996^{-1}$<br>$.1829^{-2}$<br>$.1877^{-3}$<br>$.3060^{-4}$ | .3100<br>$.1506^{-1}$<br>$.1140^{-2}$<br>$.9108^{-4}$<br>$.1252^{-4}$ | .2884<br>$.1260^{-1}$<br>$.8407^{-3}$<br>$.5645^{-4}$<br>$.6977^{-5}$ | .2744<br>$.1114^{-1}$<br>$.6795^{-3}$<br>$.4007^{-4}$<br>$.4607^{-5}$ | .2647<br>$.1018^{-1}$<br>$.5804^{-3}$<br>$.3093^{-4}$<br>$.3376^{-5}$ | .2575<br>$.9509^{-2}$<br>$.5141^{-3}$<br>$.2524^{-4}$<br>$.2650^{-5}$ | .2520<br>$.9009^{-2}$<br>$.4667^{-3}$<br>$.2143^{-4}$<br>$.2183^{-5}$ |
| 8 | .8300<br>.1337<br>$.3577^{-1}$<br>$.1275^{-1}$<br>$.5971^{-2}$ | .4900<br>$.3856^{-1}$<br>$.4882^{-2}$<br>$.7622^{-3}$<br>$.1669^{-3}$ | .3967<br>$.2310^{-1}$<br>$.2098^{-2}$<br>$.2202^{-3}$<br>$.3470^{-4}$ | .3537<br>$.1744^{-1}$<br>$.1306^{-2}$<br>$.1072^{-3}$<br>$.1405^{-4}$ | .3292<br>$.1460^{-1}$<br>$.9622^{-3}$<br>$.6663^{-4}$<br>$.7751^{-5}$ | .3133<br>$.1291^{-1}$<br>$.7770^{-3}$<br>$.4739^{-4}$<br>$.5076^{-5}$ | .3022<br>$.1180^{-1}$<br>$.6632^{-3}$<br>$.3665^{-4}$<br>$.3693^{-5}$ | .2941<br>$.1102^{-1}$<br>$.5870^{-3}$<br>$.2995^{-4}$<br>$.2882^{-5}$ | .2878<br>$.1044^{-1}$<br>$.5327^{-3}$<br>$.2546^{-4}$<br>$.2360^{-5}$ |
| 9 | .9300<br>.1509<br>$.4030^{-1}$<br>$.1443^{-1}$<br>$.6748^{-2}$ | .5500<br>$.4373^{-1}$<br>$.5513^{-2}$<br>$.8703^{-3}$<br>$.1886^{-3}$ | .4456<br>$.2624^{-1}$<br>$.2368^{-2}$<br>$.2524^{-3}$<br>$.3892^{-4}$ | .3975<br>$.1982^{-1}$<br>$.1472^{-2}$<br>$.1232^{-3}$<br>$.1563^{-4}$ | .3700<br>$.1660^{-1}$<br>$.1084^{-2}$<br>$.7668^{-4}$<br>$.8564^{-5}$ | .3522<br>$.1468^{-1}$<br>$.8747^{-3}$<br>$.5461^{-4}$<br>$.5575^{-5}$ | .3398<br>$.1342^{-1}$<br>$.7462^{-3}$<br>$.4228^{-4}$<br>$.4035^{-5}$ | .3306<br>$.1253^{-1}$<br>$.6601^{-3}$<br>$.3459^{-4}$<br>$.3134^{-5}$ | .3236<br>$.1187^{-1}$<br>$.5988^{-3}$<br>$.2942^{-4}$<br>$.2557^{-5}$ |
| 10 | $.1030^{+1}$<br>.1681<br>$.4484^{-1}$<br>$.1611^{-1}$<br>$.7526^{-2}$ | .6100<br>$.4891^{-1}$<br>$.6147^{-2}$<br>$.9780^{-3}$<br>$.2104^{-3}$ | .4944<br>$.2938^{-1}$<br>$.2638^{-2}$<br>$.2844^{-3}$<br>$.4320^{-4}$ | .4412<br>$.2221^{-1}$<br>$.1639^{-2}$<br>$.1390^{-3}$<br>$.1725^{-4}$ | .4108<br>$.1860^{-1}$<br>$.1206^{-2}$<br>$.8664^{-4}$<br>$.9405^{-5}$ | .3911<br>$.1645^{-1}$<br>$.9726^{-3}$<br>$.6176^{-4}$<br>$.6094^{-5}$ | .3773<br>$.1504^{-1}$<br>$.8293^{-3}$<br>$.4785^{-4}$<br>$.4393^{-5}$ | .3672<br>$.1404^{-1}$<br>$.7333^{-3}$<br>$.3917^{-4}$<br>$.3400^{-5}$ | .3594<br>$.1330^{-1}$<br>$.6650^{-3}$<br>$.3334^{-4}$<br>$.2765^{-5}$ |
| 11 | $.1130^{+1}$<br>.1853<br>$.4939^{-1}$<br>$.1778^{-1}$<br>$.8306^{-2}$ | .6700<br>$.5408^{-1}$<br>$.6781^{-2}$<br>$.1085^{-2}$<br>$.2323^{-3}$ | .5433<br>$.3252^{-1}$<br>$.2909^{-2}$<br>$.3162^{-3}$<br>$.4752^{-4}$ | .4850<br>$.2459^{-1}$<br>$.1806^{-2}$<br>$.1548^{-3}$<br>$.1890^{-4}$ | .4516<br>$.2059^{-1}$<br>$.1328^{-2}$<br>$.9653^{-4}$<br>$.1026^{-4}$ | .4300<br>$.1822^{-1}$<br>$.1071^{-2}$<br>$.6886^{-4}$<br>$.6627^{-5}$ | .4149<br>$.1666^{-1}$<br>$.9125^{-3}$<br>$.5337^{-4}$<br>$.4763^{-5}$ | .4037<br>$.1555^{-1}$<br>$.8066^{-3}$<br>$.4372^{-4}$<br>$.3676^{-5}$ | .3952<br>$.1473^{-1}$<br>$.7313^{-3}$<br>$.3722^{-4}$<br>$.2982^{-5}$ |
| 12 | $.1230^{+1}$<br>.2025<br>$.5393^{-1}$<br>$.1946^{-1}$<br>$.9087^{-2}$ | .7300<br>$.5926^{-1}$<br>$.7415^{-2}$<br>$.1193^{-2}$<br>$.2544^{-3}$ | .5922<br>$.3566^{-1}$<br>$.3180^{-2}$<br>$.3479^{-3}$<br>$.5188^{-4}$ | .5287<br>$.2697^{-1}$<br>$.1974^{-2}$<br>$.1705^{-3}$<br>$.2057^{-4}$ | .4924<br>$.2259^{-1}$<br>$.1451^{-2}$<br>$.1064^{-3}$<br>$.1114^{-4}$ | .4689<br>$.1999^{-1}$<br>$.1169^{-2}$<br>$.7592^{-4}$<br>$.7171^{-5}$ | .4524<br>$.1827^{-1}$<br>$.9958^{-3}$<br>$.5887^{-4}$<br>$.5141^{-5}$ | .4403<br>$.1706^{-1}$<br>$.8800^{-3}$<br>$.4823^{-4}$<br>$.3959^{-5}$ | .4310<br>$.1616^{-1}$<br>$.7976^{-3}$<br>$.4107^{-4}$<br>$.3206^{-5}$ |
| 13 | $.1330^{+1}$<br>.2197<br>$.5849^{-1}$<br>$.2114^{-1}$<br>$.9868^{-2}$ | .7900<br>$.6443^{-1}$<br>$.8050^{-2}$<br>$.1299^{-2}$<br>$.2765^{-3}$ | .6411<br>$.3880^{-1}$<br>$.3452^{-2}$<br>$.3796^{-3}$<br>$.5627^{-4}$ | .5725<br>$.2935^{-1}$<br>$.2141^{-2}$<br>$.1861^{-3}$<br>$.2225^{-4}$ | .5332<br>$.2459^{-1}$<br>$.1573^{-2}$<br>$.1162^{-3}$<br>$.1202^{-4}$ | .5078<br>$.2176^{-1}$<br>$.1267^{-2}$<br>$.8294^{-4}$<br>$.7722^{-5}$ | .4900<br>$.1989^{-1}$<br>$.1079^{-2}$<br>$.6434^{-4}$<br>$.5525^{-5}$ | .4769<br>$.1857^{-1}$<br>$.9535^{-3}$<br>$.5273^{-4}$<br>$.4248^{-5}$ | .4668<br>$.1760^{-1}$<br>$.8640^{-3}$<br>$.4491^{-4}$<br>$.3434^{-5}$ |

| $\theta_1 \backslash \theta_2$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|
| 1 | $.3640^{-1}$<br>$.4048^{-3}$<br>$.9497^{-4}$<br>$-.1494^{-4}$<br>$.6168^{-5}$ | $.3579^{-1}$<br>$.3919^{-3}$<br>$.8951^{-4}$<br>$-.1377^{-4}$<br>$.5562^{-5}$ | $.3528^{-1}$<br>$.3817^{-3}$<br>$.8515^{-4}$<br>$-.1284^{-4}$<br>$.5095^{-5}$ | $.3485^{-1}$<br>$.3735^{-3}$<br>$.8159^{-4}$<br>$-.1209^{-4}$<br>$.4724^{-5}$ | $.3449^{-1}$<br>$.3667^{-3}$<br>$.7862^{-4}$<br>$-.1148^{-4}$<br>$.4424^{-5}$ | $.3418^{-1}$<br>$.3611^{-3}$<br>$.7612^{-4}$<br>$-.1096^{-4}$<br>$.4176^{-5}$ | $.3391^{-1}$<br>$.3563^{-3}$<br>$.7398^{-4}$<br>$-.1052^{-4}$<br>$.3969^{-5}$ | $.3367^{-1}$<br>$.3522^{-3}$<br>$.7214^{-4}$<br>$-.1015^{-4}$<br>$.3793^{-5}$ |
| 2 | $.7160^{-1}$<br>$.1774^{-2}$<br>$.1378^{-3}$<br>$-.3254^{-5}$<br>$.2574^{-5}$ | $.7050^{-1}$<br>$.1713^{-2}$<br>$.1298^{-3}$<br>$-.3121^{-5}$<br>$.2332^{-5}$ | $.6958^{-1}$<br>$.1663^{-2}$<br>$.1234^{-3}$<br>$-.3007^{-5}$<br>$.2145^{-5}$ | $.6882^{-1}$<br>$.1622^{-2}$<br>$.1182^{-3}$<br>$-.2909^{-5}$<br>$.1996^{-5}$ | $.6816^{-1}$<br>$.1587^{-2}$<br>$.1139^{-3}$<br>$-.2824^{-5}$<br>$.1876^{-5}$ | $.6760^{-1}$<br>$.1558^{-2}$<br>$.1102^{-3}$<br>$-.2751^{-5}$<br>$.1776^{-5}$ | $.6711^{-1}$<br>$.1533^{-2}$<br>$.1071^{-3}$<br>$-.2686^{-5}$<br>$.1692^{-5}$ | $.6668^{-1}$<br>$.1511^{-2}$<br>$.1044^{-3}$<br>$-.2629^{-5}$<br>$.1621^{-5}$ |
| 3 | .1068<br>$.3144^{-2}$<br>$.1930^{-3}$<br>$.2676^{-5}$<br>$.1864^{-5}$ | .1052<br>$.3034^{-2}$<br>$.1814^{-3}$<br>$.2255^{-5}$<br>$.1679^{-5}$ | .1039<br>$.2944^{-2}$<br>$.1722^{-3}$<br>$.1939^{-5}$<br>$.1537^{-5}$ | .1028<br>$.2870^{-2}$<br>$.1647^{-3}$<br>$.1695^{-5}$<br>$.1425^{-5}$ | .1018<br>$.2808^{-2}$<br>$.1585^{-3}$<br>$.1501^{-5}$<br>$.1335^{-5}$ | .1010<br>$.2755^{-2}$<br>$.1532^{-3}$<br>$.1344^{-5}$<br>$.1260^{-5}$ | .1003<br>$.2709^{-2}$<br>$.1487^{-3}$<br>$.1215^{-5}$<br>$.1198^{-5}$ | $.9969^{-1}$<br>$.2670^{-2}$<br>$.1449^{-3}$<br>$.1106^{-5}$<br>$.1145^{-5}$ |
| 4 | .1420<br>$.4514^{-2}$<br>$.2513^{-3}$<br>$.7249^{-5}$<br>$.1678^{-5}$ | .1399<br>$.4354^{-2}$<br>$.2359^{-3}$<br>$.6386^{-5}$<br>$.1499^{-5}$ | .1382<br>$.4225^{-2}$<br>$.2236^{-3}$<br>$.5727^{-5}$<br>$.1362^{-5}$ | .1367<br>$.4118^{-2}$<br>$.2137^{-3}$<br>$.5210^{-5}$<br>$.1255^{-5}$ | .1355<br>$.4028^{-2}$<br>$.2054^{-3}$<br>$.4795^{-5}$<br>$.1169^{-5}$ | .1344<br>$.3952^{-2}$<br>$.1985^{-3}$<br>$.4455^{-5}$<br>$.1099^{-5}$ | .1335<br>$.3886^{-2}$<br>$.1925^{-3}$<br>$.4172^{-5}$<br>$.1040^{-5}$ | .1327<br>$.3828^{-2}$<br>$.1874^{-3}$<br>$.3934^{-5}$<br>$.9907^{-6}$ |
| 5 | .1772<br>$.5883^{-2}$<br>$.3108^{-3}$<br>$.1129^{-4}$<br>$.1670^{-5}$ | .1746<br>$.5675^{-2}$<br>$.2915^{-3}$<br>$.1003^{-4}$<br>$.1480^{-5}$ | .1725<br>$.5506^{-2}$<br>$.2761^{-3}$<br>$.9061^{-5}$<br>$.1336^{-5}$ | .1707<br>$.5367^{-2}$<br>$.2637^{-3}$<br>$.8299^{-5}$<br>$.1223^{-5}$ | .1692<br>$.5249^{-2}$<br>$.2533^{-3}$<br>$.7685^{-5}$<br>$.1133^{-5}$ | .1679<br>$.5149^{-2}$<br>$.2446^{-3}$<br>$.7181^{-5}$<br>$.1060^{-5}$ | .1667<br>$.5062^{-2}$<br>$.2372^{-3}$<br>$.6760^{-5}$<br>$.9994^{-6}$ | .1657<br>$.4987^{-2}$<br>$.2308^{-3}$<br>$.6404^{-5}$<br>$.9483^{-6}$ |
| 6 | .2124<br>$.7253^{-2}$<br>$.3709^{-3}$<br>$.1508^{-4}$<br>$.1744^{-5}$ | .2093<br>$.6996^{-2}$<br>$.3477^{-3}$<br>$.1343^{-4}$<br>$.1535^{-5}$ | .2068<br>$.6788^{-2}$<br>$.3292^{-3}$<br>$.1217^{-4}$<br>$.1377^{-5}$ | .2047<br>$.6615^{-2}$<br>$.3141^{-3}$<br>$.1118^{-4}$<br>$.1255^{-5}$ | .2029<br>$.6470^{-2}$<br>$.3017^{-3}$<br>$.1037^{-4}$<br>$.1157^{-5}$ | .2013<br>$.6346^{-2}$<br>$.2912^{-3}$<br>$.9715^{-5}$<br>$.1078^{-5}$ | .1999<br>$.6239^{-2}$<br>$.2823^{-3}$<br>$.9164^{-5}$<br>$.1012^{-5}$ | .1987<br>$.6146^{-2}$<br>$.2746^{-3}$<br>$.8698^{-5}$<br>$.9574^{-6}$ |
| 7 | .2476<br>$.8623^{-2}$<br>$.4314^{-3}$<br>$.1871^{-4}$<br>$.1861^{-5}$ | .2440<br>$.8317^{-2}$<br>$.4042^{-3}$<br>$.1670^{-4}$<br>$.1629^{-5}$ | .2411<br>$.8069^{-2}$<br>$.3825^{-3}$<br>$.1515^{-4}$<br>$.1455^{-5}$ | .2386<br>$.7863^{-2}$<br>$.3649^{-3}$<br>$.1393^{-4}$<br>$.1320^{-5}$ | .2365<br>$.7690^{-2}$<br>$.3503^{-3}$<br>$.1295^{-4}$<br>$.1213^{-5}$ | .2347<br>$.7543^{-2}$<br>$.3380^{-3}$<br>$.1214^{-4}$<br>$.1126^{-5}$ | .2331<br>$.7415^{-2}$<br>$.3276^{-3}$<br>$.1146^{-4}$<br>$.1054^{-5}$ | .2317<br>$.7305^{-2}$<br>$.3186^{-3}$<br>$.1089^{-4}$<br>$.9943^{-6}$ |
| 8 | .2828<br>$.9993^{-2}$<br>$.4922^{-3}$<br>$.2226^{-4}$<br>$.2003^{-5}$ | .2788<br>$.9638^{-2}$<br>$.4609^{-3}$<br>$.1988^{-4}$<br>$.1746^{-5}$ | .2754<br>$.9350^{-2}$<br>$.4360^{-3}$<br>$.1806^{-4}$<br>$.1554^{-5}$ | .2726<br>$.9111^{-2}$<br>$.4158^{-3}$<br>$.1662^{-4}$<br>$.1405^{-5}$ | .2702<br>$.8911^{-2}$<br>$.3991^{-3}$<br>$.1546^{-4}$<br>$.1287^{-5}$ | .2681<br>$.8740^{-2}$<br>$.3850^{-3}$<br>$.1450^{-4}$<br>$.1192^{-5}$ | .2663<br>$.8592^{-2}$<br>$.3731^{-3}$<br>$.1370^{-4}$<br>$.1113^{-5}$ | .2647<br>$.8463^{-2}$<br>$.3627^{-3}$<br>$.1302^{-4}$<br>$.1048^{-5}$ |
| 9 | .3180<br>$.1136^{-1}$<br>$.5530^{-3}$<br>$.2574^{-4}$<br>$.2162^{-5}$ | .3135<br>$.1096^{-1}$<br>$.5177^{-3}$<br>$.2301^{-4}$<br>$.1879^{-5}$ | .3097<br>$.1063^{-1}$<br>$.4897^{-3}$<br>$.2091^{-4}$<br>$.1667^{-5}$ | .3066<br>$.1036^{-1}$<br>$.4669^{-3}$<br>$.1925^{-4}$<br>$.1503^{-5}$ | .3039<br>$.1013^{-1}$<br>$.4480^{-3}$<br>$.1791^{-4}$<br>$.1374^{-5}$ | .3016<br>$.9937^{-2}$<br>$.4322^{-3}$<br>$.1681^{-4}$<br>$.1270^{-5}$ | .2995<br>$.9769^{-2}$<br>$.4186^{-3}$<br>$.1589^{-4}$<br>$.1184^{-5}$ | .2978<br>$.9622^{-2}$<br>$.4070^{-3}$<br>$.1511^{-4}$<br>$.1112^{-5}$ |
| 10 | .3532<br>$.1272^{-1}$<br>$.6140^{-3}$<br>$.2918^{-4}$<br>$.2332^{-5}$ | .3482<br>$.1228^{-1}$<br>$.5747^{-3}$<br>$.2610^{-4}$<br>$.2021^{-5}$ | .3440<br>$.1191^{-1}$<br>$.5434^{-3}$<br>$.2373^{-4}$<br>$.1789^{-5}$ | .3405<br>$.1161^{-1}$<br>$.5180^{-3}$<br>$.2185^{-4}$<br>$.1611^{-5}$ | .3376<br>$.1135^{-1}$<br>$.4970^{-3}$<br>$.2034^{-4}$<br>$.1470^{-5}$ | .3350<br>$.1113^{-1}$<br>$.4793^{-3}$<br>$.1910^{-4}$<br>$.1356^{-5}$ | .3327<br>$.1095^{-1}$<br>$.4643^{-3}$<br>$.1805^{-4}$<br>$.1262^{-5}$ | .3308<br>$.1078^{-1}$<br>$.4513^{-3}$<br>$.1717^{-4}$<br>$.1184^{-5}$ |
| 11 | .3884<br>$.1410^{-1}$<br>$.6751^{-3}$<br>$.3259^{-4}$<br>$.2510^{-5}$ | .3829<br>$.1360^{-1}$<br>$.6317^{-3}$<br>$.2916^{-4}$<br>$.2171^{-5}$ | .3783<br>$.1319^{-1}$<br>$.5972^{-3}$<br>$.2651^{-4}$<br>$.1918^{-5}$ | .3745<br>$.1286^{-1}$<br>$.5692^{-3}$<br>$.2443^{-4}$<br>$.1724^{-5}$ | .3712<br>$.1257^{-1}$<br>$.5460^{-3}$<br>$.2274^{-4}$<br>$.1571^{-5}$ | .3684<br>$.1233^{-1}$<br>$.5266^{-3}$<br>$.2135^{-4}$<br>$.1447^{-5}$ | .3659<br>$.1212^{-1}$<br>$.5100^{-3}$<br>$.2019^{-4}$<br>$.1345^{-5}$ | .3638<br>$.1194^{-1}$<br>$.4957^{-3}$<br>$.1921^{-4}$<br>$.1260^{-5}$ |
| 12 | .4236<br>$.1547^{-1}$<br>$.7362^{-3}$<br>$.3598^{-4}$<br>$.2693^{-5}$ | .4176<br>$.1492^{-1}$<br>$.6887^{-3}$<br>$.3219^{-4}$<br>$.2326^{-5}$ | .4126<br>$.1447^{-1}$<br>$.6511^{-3}$<br>$.2928^{-4}$<br>$.2052^{-5}$ | .4085<br>$.1410^{-1}$<br>$.6205^{-3}$<br>$.2698^{-4}$<br>$.1842^{-5}$ | .4049<br>$.1379^{-1}$<br>$.5951^{-3}$<br>$.2513^{-4}$<br>$.1676^{-5}$ | .4018<br>$.1353^{-1}$<br>$.5739^{-3}$<br>$.2360^{-4}$<br>$.1542^{-5}$ | .3991<br>$.1330^{-1}$<br>$.5557^{-3}$<br>$.2232^{-4}$<br>$.1433^{-5}$ | .3968<br>$.1310^{-1}$<br>$.5401^{-3}$<br>$.2124^{-4}$<br>$.1341^{-5}$ |
| 13 | .4588<br>$.1684^{-1}$<br>$.7973^{-3}$<br>$.3935^{-4}$<br>$.2881^{-5}$ | .4523<br>$.1624^{-1}$<br>$.7458^{-3}$<br>$.3521^{-4}$<br>$.2485^{-5}$ | .4469<br>$.1576^{-1}$<br>$.7049^{-3}$<br>$.3204^{-4}$<br>$.2190^{-5}$ | .4424<br>$.1535^{-1}$<br>$.6717^{-3}$<br>$.2953^{-4}$<br>$.1964^{-5}$ | .4386<br>$.1501^{-1}$<br>$.6443^{-3}$<br>$.2750^{-4}$<br>$.1785^{-5}$ | .4352<br>$.1472^{-1}$<br>$.6212^{-3}$<br>$.2583^{-4}$<br>$.1641^{-5}$ | .4323<br>$.1447^{-1}$<br>$.6015^{-3}$<br>$.2443^{-4}$<br>$.1523^{-5}$ | .4297<br>$.1426^{-1}$<br>$.5846^{-3}$<br>$.2325^{-4}$<br>$.1425^{-5}$ |

| $\theta_1 \backslash \theta_2$ | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|
| 1 | $.3346^{-1}$<br>$.3487^{-3}$<br>$.7052^{-4}$<br>$-.9824^{-5}$<br>$.3642^{-5}$ | $.3327^{-1}$<br>$.3456^{-3}$<br>$.6910^{-4}$<br>$-.9540^{-5}$<br>$.3510^{-5}$ | $.3310^{-1}$<br>$.3428^{-3}$<br>$.6784^{-4}$<br>$-.9289^{-5}$<br>$.3396^{-5}$ | $.3295^{-1}$<br>$.3404^{-3}$<br>$.6672^{-4}$<br>$-.9066^{-5}$<br>$.3294^{-5}$ | $.3281^{-1}$<br>$.3382^{-3}$<br>$.6571^{-4}$<br>$-.8867^{-5}$<br>$.3204^{-5}$ | $.3268^{-1}$<br>$.3363^{-3}$<br>$.6479^{-4}$<br>$-.8688^{-5}$<br>$.3124^{-5}$ | $.3257^{-1}$<br>$.3345^{-3}$<br>$.6396^{-4}$<br>$-.8526^{-5}$<br>$.3052^{-5}$ | $.3246^{-1}$<br>$.3329^{-3}$<br>$.6321^{-4}$<br>$-.8379^{-5}$<br>$.2986^{-5}$ |
| 2 | $.6630^{-1}$<br>$.1492^{-2}$<br>$.1021^{-3}$<br>$-.2578^{-5}$<br>$.1560^{-5}$ | $.6596^{-1}$<br>$.1475^{-2}$<br>$.9999^{-4}$<br>$-.2532^{-5}$<br>$.1506^{-5}$ | $.6565^{-1}$<br>$.1460^{-2}$<br>$.9816^{-4}$<br>$-.2491^{-5}$<br>$.1460^{-5}$ | $.6537^{-1}$<br>$.1446^{-2}$<br>$.9652^{-4}$<br>$-.2455^{-5}$<br>$.1418^{-5}$ | $.6512^{-1}$<br>$.1434^{-2}$<br>$.9505^{-4}$<br>$-.2421^{-5}$<br>$.1382^{-5}$ | $.6490^{-1}$<br>$.1423^{-2}$<br>$.9373^{-4}$<br>$-.2391^{-5}$<br>$.1349^{-5}$ | $.6469^{-1}$<br>$.1413^{-2}$<br>$.9253^{-4}$<br>$-.2363^{-5}$<br>$.1319^{-5}$ | $.6450^{-1}$<br>$.1404^{-2}$<br>$.9143^{-4}$<br>$-.2337^{-5}$<br>$.1293^{-5}$ |
| 3 | $.9914^{-1}$<br>$.2635^{-2}$<br>$.1415^{-3}$<br>$.1015^{-5}$<br>$.1100^{-5}$ | $.9864^{-1}$<br>$.2604^{-2}$<br>$.1386^{-3}$<br>$.9362^{-6}$<br>$.1061^{-5}$ | $.9820^{-1}$<br>$.2576^{-2}$<br>$.1359^{-3}$<br>$.8682^{-6}$<br>$.1026^{-5}$ | $.9780^{-1}$<br>$.2552^{-2}$<br>$.1336^{-3}$<br>$.8089^{-6}$<br>$.9959^{-6}$ | $.9744^{-1}$<br>$.2530^{-2}$<br>$.1315^{-3}$<br>$.7567^{-6}$<br>$.9689^{-6}$ | $.9711^{-1}$<br>$.2509^{-2}$<br>$.1296^{-3}$<br>$.7105^{-6}$<br>$.9449^{-6}$ | $.9681^{-1}$<br>$.2491^{-2}$<br>$.1279^{-3}$<br>$.6692^{-6}$<br>$.9233^{-6}$ | $.9653^{-1}$<br>$.2474^{-2}$<br>$.1263^{-3}$<br>$.6323^{-6}$<br>$.9038^{-6}$ |
| 4 | .1320<br>$.3778^{-2}$<br>$.1830^{-3}$<br>$.3730^{-5}$<br>$.9484^{-6}$ | .1313<br>$.3733^{-2}$<br>$.1791^{-3}$<br>$.3554^{-5}$<br>$.9118^{-6}$ | .1307<br>$.3693^{-2}$<br>$.1756^{-3}$<br>$.3400^{-5}$<br>$.8800^{-6}$ | .1302<br>$.3658^{-2}$<br>$.1725^{-3}$<br>$.3265^{-5}$<br>$.8519^{-6}$ | .1298<br>$.3625^{-2}$<br>$.1697^{-3}$<br>$.3146^{-5}$<br>$.8271^{-6}$ | .1293<br>$.3596^{-2}$<br>$.1672^{-3}$<br>$.3040^{-5}$<br>$.8050^{-6}$ | .1289<br>$.3569^{-2}$<br>$.1650^{-3}$<br>$.2945^{-5}$<br>$.7851^{-6}$ | .1286<br>$.3545^{-2}$<br>$.1629^{-3}$<br>$.2859^{-5}$<br>$.7672^{-6}$ |
| 5 | .1648<br>$.4921^{-2}$<br>$.2252^{-3}$<br>$.6100^{-5}$<br>$.9047^{-6}$ | .1640<br>$.4862^{-2}$<br>$.2203^{-3}$<br>$.5836^{-5}$<br>$.8672^{-6}$ | .1633<br>$.4810^{-2}$<br>$.2160^{-3}$<br>$.5607^{-5}$<br>$.8345^{-6}$ | .1627<br>$.4763^{-2}$<br>$.2121^{-3}$<br>$.5404^{-5}$<br>$.8059^{-6}$ | .1621<br>$.4721^{-2}$<br>$.2087^{-3}$<br>$.5225^{-5}$<br>$.7806^{-6}$ | .1615<br>$.4683^{-2}$<br>$.2055^{-3}$<br>$.5065^{-5}$<br>$.7580^{-6}$ | .1610<br>$.4648^{-2}$<br>$.2027^{-3}$<br>$.4921^{-5}$<br>$.7379^{-6}$ | .1606<br>$.4616^{-2}$<br>$.2001^{-3}$<br>$.4792^{-5}$<br>$.7197^{-6}$ |
| 6 | .1977<br>$.6064^{-2}$<br>$.2679^{-3}$<br>$.8299^{-5}$<br>$.9107^{-6}$ | .1967<br>$.5992^{-2}$<br>$.2620^{-3}$<br>$.7953^{-5}$<br>$.8705^{-6}$ | .1958<br>$.5927^{-2}$<br>$.2568^{-3}$<br>$.7651^{-5}$<br>$.8356^{-6}$ | .1951<br>$.5869^{-2}$<br>$.2521^{-3}$<br>$.7385^{-5}$<br>$.8051^{-6}$ | .1944<br>$.5817^{-2}$<br>$.2479^{-3}$<br>$.7150^{-5}$<br>$.7781^{-6}$ | .1937<br>$.5769^{-2}$<br>$.2442^{-3}$<br>$.6939^{-5}$<br>$.7542^{-6}$ | .1932<br>$.5726^{-2}$<br>$.2408^{-3}$<br>$.6750^{-5}$<br>$.7328^{-6}$ | .1926<br>$.5686^{-2}$<br>$.2377^{-3}$<br>$.6580^{-5}$<br>$.7135^{-6}$ |
| 7 | .2305<br>$.7207^{-2}$<br>$.3107^{-3}$<br>$.1040^{-4}$<br>$.9434^{-6}$ | .2294<br>$.7121^{-2}$<br>$.3038^{-3}$<br>$.9975^{-5}$<br>$.8997^{-6}$ | .2284<br>$.7044^{-2}$<br>$.2977^{-3}$<br>$.9604^{-5}$<br>$.8618^{-6}$ | .2275<br>$.6975^{-2}$<br>$.2923^{-3}$<br>$.9277^{-5}$<br>$.8287^{-6}$ | .2267<br>$.6912^{-2}$<br>$.2874^{-3}$<br>$.8987^{-5}$<br>$.7995^{-6}$ | .2260<br>$.6856^{-2}$<br>$.2830^{-3}$<br>$.8728^{-5}$<br>$.7736^{-6}$ | .2253<br>$.6804^{-2}$<br>$.2791^{-3}$<br>$.8495^{-5}$<br>$.7505^{-6}$ | .2247<br>$.6757^{-2}$<br>$.2754^{-3}$<br>$.8285^{-5}$<br>$.7297^{-6}$ |
| 8 | .2633<br>$.8350^{-2}$<br>$.3537^{-3}$<br>$.1244^{-4}$<br>$.9921^{-6}$ | .2621<br>$.8250^{-2}$<br>$.3459^{-3}$<br>$.1194^{-4}$<br>$.9444^{-6}$ | .2609<br>$.8161^{-2}$<br>$.3389^{-3}$<br>$.1150^{-4}$<br>$.9031^{-6}$ | .2599<br>$.8080^{-2}$<br>$.3326^{-3}$<br>$.1111^{-4}$<br>$.8670^{-6}$ | .2590<br>$.8008^{-2}$<br>$.3270^{-3}$<br>$.1077^{-4}$<br>$.8352^{-6}$ | .2582<br>$.7942^{-2}$<br>$.3220^{-3}$<br>$.1046^{-4}$<br>$.8071^{-6}$ | .2574<br>$.7883^{-2}$<br>$.3174^{-3}$<br>$.1019^{-4}$<br>$-.6265^{-5}$ | .2567<br>$.7828^{-2}$<br>$.3133^{-3}$<br>$.9939^{-5}$<br>$-.2231^{-4}$ |
| 9 | .2962<br>$.9493^{-2}$<br>$.3969^{-3}$<br>$.1444^{-4}$<br>$.1051^{-5}$ | .2948<br>$.9379^{-2}$<br>$.3880^{-3}$<br>$.1386^{-4}$<br>$.9992^{-6}$ | .2935<br>$.9277^{-2}$<br>$.3801^{-3}$<br>$.1336^{-4}$<br>$.9542^{-6}$ | .2924<br>$.9186^{-2}$<br>$.3730^{-3}$<br>$.1291^{-4}$<br>$.9148^{-6}$ | .2913<br>$.9104^{-2}$<br>$.3667^{-3}$<br>$.1252^{-4}$<br>$.8802^{-6}$ | .2904<br>$.9029^{-2}$<br>$.3611^{-3}$<br>$.1216^{-4}$<br>$-.1711^{-4}$ | .2895<br>$.8961^{-2}$<br>$.3559^{-3}$<br>$.1184^{-4}$<br>$-.3025^{-4}$ | .2887<br>$.8898^{-2}$<br>$.3512^{-3}$<br>$.1156^{-4}$<br>$-.3536^{-4}$ |
| 10 | .3290<br>$.1064^{-1}$<br>$.4400^{-3}$<br>$.1642^{-4}$<br>$.1117^{-5}$ | .3275<br>$.1051^{-1}$<br>$.4301^{-3}$<br>$.1576^{-4}$<br>$.1061^{-5}$ | .3260<br>$.1039^{-1}$<br>$.4213^{-3}$<br>$.1519^{-4}$<br>$.1012^{-5}$ | .3248<br>$.1029^{-1}$<br>$.4135^{-3}$<br>$.1468^{-4}$<br>$.9691^{-6}$ | .3236<br>$.1020^{-1}$<br>$.4065^{-3}$<br>$.1424^{-4}$<br>$-.2396^{-4}$ | .3226<br>$.1012^{-1}$<br>$.4002^{-3}$<br>$.1384^{-4}$<br>$-.3650^{-4}$ | .3216<br>$.1004^{-1}$<br>$.3945^{-3}$<br>$.1348^{-4}$<br>$-.4107^{-4}$ | .3208<br>$.9969^{-2}$<br>$.3892^{-3}$<br>$.1315^{-4}$<br>$-.4259^{-4}$ |
| 11 | .3618<br>$.1178^{-1}$<br>$.4833^{-3}$<br>$.1837^{-4}$<br>$.1189^{-5}$ | .3601<br>$.1164^{-1}$<br>$.4723^{-3}$<br>$.1764^{-4}$<br>$.1127^{-5}$ | .3586<br>$.1151^{-1}$<br>$.4627^{-3}$<br>$.1701^{-4}$<br>$.1074^{-5}$ | .3572<br>$.1140^{-1}$<br>$.4540^{-3}$<br>$.1644^{-4}$<br>$-.2798^{-4}$ | .3560<br>$.1130^{-1}$<br>$.4463^{-3}$<br>$.1594^{-4}$<br>$-.4175^{-4}$ | .3548<br>$.1120^{-1}$<br>$.4393^{-3}$<br>$.1549^{-4}$<br>$-.4647^{-4}$ | .3537<br>$.1112^{-1}$<br>$.4330^{-3}$<br>$.1509^{-4}$<br>$-.4788^{-4}$ | .3528<br>$.1104^{-1}$<br>$.4273^{-3}$<br>$.1473^{-4}$<br>$-.4803^{-4}$ |
| 12 | .3947<br>$.1292^{-1}$<br>$.5265^{-3}$<br>$.2031^{-4}$<br>$.1264^{-5}$ | .3928<br>$.1277^{-1}$<br>$.5146^{-3}$<br>$.1950^{-4}$<br>$.1197^{-5}$ | .3911<br>$.1263^{-1}$<br>$.5040^{-3}$<br>$.1880^{-4}$<br>$-.2884^{-4}$ | .3896<br>$.1250^{-1}$<br>$.4946^{-3}$<br>$.1818^{-4}$<br>$-.4604^{-4}$ | .3883<br>$.1239^{-1}$<br>$.4861^{-3}$<br>$.1763^{-4}$<br>$-.5165^{-4}$ | .3870<br>$.1229^{-1}$<br>$.4785^{-3}$<br>$.1714^{-4}$<br>$-.5320^{-4}$ | .3859<br>$.1220^{-1}$<br>$.4716^{-3}$<br>$.1670^{-4}$<br>$-.5329^{-4}$ | .3848<br>$.1211^{-1}$<br>$.4653^{-3}$<br>$.1630^{-4}$<br>$-.5286^{-4}$ |
| 13 | .4275<br>$.1407^{-1}$<br>$.5698^{-3}$<br>$.2223^{-4}$<br>$.1341^{-5}$ | .4255<br>$.1390^{-1}$<br>$.5568^{-3}$<br>$.2135^{-4}$<br>$-.2469^{-4}$ | .4237<br>$.1374^{-1}$<br>$.5454^{-3}$<br>$.2058^{-4}$<br>$-.4896^{-4}$ | .4221<br>$.1361^{-1}$<br>$.5352^{-3}$<br>$.1991^{-4}$<br>$-.5654^{-4}$ | .4206<br>$.1349^{-1}$<br>$.5260^{-3}$<br>$.1931^{-4}$<br>$-.5857^{-4}$ | .4192<br>$.1338^{-1}$<br>$.5178^{-3}$<br>$.1877^{-4}$<br>$-.5870^{-4}$ | .4180<br>$.1327^{-1}$<br>$.5102^{-3}$<br>$.1829^{-4}$<br>$-.5818^{-4}$ | .4168<br>$.1318^{-1}$<br>$.5035^{-3}$<br>$.1785^{-4}$<br>$-.5763^{-4}$ |

CONTINUED

BIAS OF $\Theta_1^*$ $N^{-1}$ TO $N^{-5}$ TERMS WHEN N = 100 (MOMENTS ESTIMATOR)

| $\theta_1 \backslash \theta_2$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 14 | $.1430^{+1}$<br>$.2369$<br>$.6304^{-1}$<br>$.2281^{-1}$<br>$.1065^{-1}$ | $.8500$<br>$.6961^{-1}$<br>$.8686^{-2}$<br>$.1406^{-2}$<br>$.2986^{-3}$ | $.6900$<br>$.4194^{-1}$<br>$.3723^{-2}$<br>$.4112^{-3}$<br>$.6068^{-4}$ | $.6162$<br>$.3174^{-1}$<br>$.2309^{-2}$<br>$.2017^{-3}$<br>$.2395^{-4}$ | $.5740$<br>$.2659^{-1}$<br>$.1696^{-2}$<br>$.1259^{-3}$<br>$.1291^{-4}$ | $.5467$<br>$.2353^{-1}$<br>$.1365^{-2}$<br>$.8995^{-4}$<br>$.8279^{-5}$ | $.5276$<br>$.2151^{-1}$<br>$.1163^{-2}$<br>$.6978^{-4}$<br>$.5915^{-5}$ | $.5134$<br>$.2009^{-1}$<br>$.1027^{-2}$<br>$.5720^{-4}$<br>$.4541^{-5}$ | $.5026$<br>$.1903^{-1}$<br>$.9304^{-3}$<br>$.4874^{-4}$<br>$.3666^{-5}$ | $.4940$<br>$.1821^{-1}$<br>$.8585^{-3}$<br>$.4270^{-4}$<br>$.3072^{-5}$ | $.4870$<br>$.1756^{-1}$<br>$.8029^{-3}$<br>$.3822^{-4}$<br>$.2647^{-5}$ | $.4812$<br>$.1704^{-1}$<br>$.7589^{-3}$<br>$.3478^{-4}$<br>$.2331^{-5}$ | $.4764$<br>$.1660^{-1}$<br>$.7230^{-3}$<br>$.3206^{-4}$<br>$.2088^{-5}$ |
| 15 | $.1530^{+1}$<br>$.2540$<br>$.6760^{-1}$<br>$.2449^{-1}$<br>$.1143^{-1}$ | $.9100$<br>$.7478^{-1}$<br>$.9322^{-2}$<br>$.1513^{-2}$<br>$.3208^{-3}$ | $.7389$<br>$.4509^{-1}$<br>$.3995^{-2}$<br>$.4427^{-3}$<br>$.6510^{-4}$ | $.6600$<br>$.3412^{-1}$<br>$.2477^{-2}$<br>$.2172^{-3}$<br>$.2565^{-4}$ | $.6147$<br>$.2859^{-1}$<br>$.1818^{-2}$<br>$.1357^{-3}$<br>$.1380^{-4}$ | $.5856$<br>$.2530^{-1}$<br>$.1464^{-2}$<br>$.9693^{-4}$<br>$.8841^{-5}$ | $.5651$<br>$.2313^{-1}$<br>$.1246^{-2}$<br>$.7522^{-4}$<br>$.6308^{-5}$ | $.5500$<br>$.2160^{-1}$<br>$.1101^{-2}$<br>$.6167^{-4}$<br>$.4837^{-5}$ | $.5384$<br>$.2046^{-1}$<br>$.9969^{-3}$<br>$.5254^{-4}$<br>$.3901^{-5}$ | $.5292$<br>$.1958^{-1}$<br>$.9197^{-3}$<br>$.4605^{-4}$<br>$.3266^{-5}$ | $.5217$<br>$.1888^{-1}$<br>$.8601^{-3}$<br>$.4122^{-4}$<br>$.2812^{-5}$ | $.5156$<br>$.1832^{-1}$<br>$.8128^{-3}$<br>$.3751^{-4}$<br>$.2473^{-5}$ | $.5104$<br>$.1785^{-1}$<br>$.7744^{-3}$<br>$.3458^{-4}$<br>$.2214^{-5}$ |
| 16 | $.1630^{+1}$<br>$.2713$<br>$.7215^{-1}$<br>$.2616^{-1}$<br>$.1221^{-1}$ | $.9700$<br>$.7995^{-1}$<br>$.9958^{-2}$<br>$.1620^{-2}$<br>$.3430^{-3}$ | $.7878$<br>$.4823^{-1}$<br>$.4267^{-2}$<br>$.4742^{-3}$<br>$.6953^{-4}$ | $.7037$<br>$.3650^{-1}$<br>$.2644^{-2}$<br>$.2327^{-3}$<br>$.2736^{-4}$ | $.6556$<br>$.3059^{-1}$<br>$.1941^{-2}$<br>$.1454^{-3}$<br>$.1470^{-4}$ | $.6244$<br>$.2707^{-1}$<br>$.1562^{-2}$<br>$.1039^{-3}$<br>$.9407^{-5}$ | $.6027$<br>$.2475^{-1}$<br>$.1330^{-2}$<br>$.8064^{-4}$<br>$.6704^{-5}$ | $.5866$<br>$.2311^{-1}$<br>$.1174^{-2}$<br>$.6612^{-4}$<br>$.5136^{-5}$ | $.5742$<br>$.2189^{-1}$<br>$.1063^{-2}$<br>$.5635^{-4}$<br>$.4139^{-5}$ | $.5644$<br>$.2095^{-1}$<br>$.9809^{-3}$<br>$.4939^{-4}$<br>$.3462^{-5}$ | $.5564$<br>$.2021^{-1}$<br>$.9173^{-3}$<br>$.4421^{-4}$<br>$.2978^{-5}$ | $.5499$<br>$.1960^{-1}$<br>$.8667^{-3}$<br>$.4023^{-4}$<br>$.2619^{-5}$ | $.5443$<br>$.1910^{-1}$<br>$.8257^{-3}$<br>$.3709^{-4}$<br>$.2343^{-5}$ |
| 17 | $.1730^{+1}$<br>$.2885$<br>$.7671^{-1}$<br>$.2784^{-1}$<br>$.1300^{-1}$ | $.1030^{+1}$<br>$.8513^{-1}$<br>$.1059^{-1}$<br>$.1726^{-2}$<br>$.3653^{-3}$ | $.8367$<br>$.5137^{-1}$<br>$.4539^{-2}$<br>$.5056^{-3}$<br>$.7397^{-4}$ | $.7475$<br>$.3888^{-1}$<br>$.2812^{-2}$<br>$.2482^{-3}$<br>$.2907^{-4}$ | $.6964$<br>$.3259^{-1}$<br>$.2063^{-2}$<br>$.1551^{-3}$<br>$.1561^{-4}$ | $.6633$<br>$.2884^{-1}$<br>$.1660^{-2}$<br>$.1109^{-3}$<br>$.9976^{-5}$ | $.6402$<br>$.2636^{-1}$<br>$.1413^{-2}$<br>$.8605^{-4}$<br>$.7103^{-5}$ | $.6231$<br>$.2462^{-1}$<br>$.1248^{-2}$<br>$.7056^{-4}$<br>$.5437^{-5}$ | $.6100$<br>$.2332^{-1}$<br>$.1130^{-2}$<br>$.6014^{-4}$<br>$.4378^{-5}$ | $.5996$<br>$.2232^{-1}$<br>$.1042^{-2}$<br>$.5271^{-4}$<br>$.3660^{-5}$ | $.5912$<br>$.2153^{-1}$<br>$.9745^{-3}$<br>$.4720^{-4}$<br>$.3146^{-5}$ | $.5842$<br>$.2088^{-1}$<br>$.9207^{-3}$<br>$.4295^{-4}$<br>$.2765^{-5}$ | $.5783$<br>$.2035^{-1}$<br>$.8771^{-3}$<br>$.3960^{-4}$<br>$.2472^{-5}$ |
| 18 | $.1830^{+1}$<br>$.3057$<br>$.8127^{-1}$<br>$.2951^{-1}$<br>$.1378^{-1}$ | $.1090^{+1}$<br>$.9030^{-1}$<br>$.1123^{-1}$<br>$.1833^{-2}$<br>$.3875^{-3}$ | $.8856$<br>$.5451^{-1}$<br>$.4811^{-2}$<br>$.5370^{-3}$<br>$.7842^{-4}$ | $.7912$<br>$.4127^{-1}$<br>$.2980^{-2}$<br>$.2637^{-3}$<br>$.3079^{-4}$ | $.7372$<br>$.3458^{-1}$<br>$.2186^{-2}$<br>$.1648^{-3}$<br>$.1652^{-4}$ | $.7022$<br>$.3061^{-1}$<br>$.1759^{-2}$<br>$.1178^{-3}$<br>$.1055^{-4}$ | $.6778$<br>$.2798^{-1}$<br>$.1497^{-2}$<br>$.9145^{-4}$<br>$.7504^{-5}$ | $.6597$<br>$.2613^{-1}$<br>$.1321^{-2}$<br>$.7500^{-4}$<br>$.5740^{-5}$ | $.6458$<br>$.2475^{-1}$<br>$.1196^{-2}$<br>$.6392^{-4}$<br>$.4619^{-5}$ | $.6348$<br>$.2369^{-1}$<br>$.1103^{-2}$<br>$.5604^{-4}$<br>$.3859^{-5}$ | $.6259$<br>$.2285^{-1}$<br>$.1032^{-2}$<br>$.5017^{-4}$<br>$.3316^{-5}$ | $.6185$<br>$.2216^{-1}$<br>$.9747^{-3}$<br>$.4566^{-4}$<br>$.2912^{-5}$ | $.6122$<br>$.2159^{-1}$<br>$.9284^{-3}$<br>$.4210^{-4}$<br>$.2603^{-5}$ |
| 19 | $.1930^{+1}$<br>$.3229$<br>$.8583^{-1}$<br>$.3119^{-1}$<br>$.1456^{-1}$ | $.1150^{+1}$<br>$.9548^{-1}$<br>$.1187^{-1}$<br>$.1939^{-2}$<br>$.4098^{-3}$ | $.9344$<br>$.5765^{-1}$<br>$.5083^{-2}$<br>$.5684^{-3}$<br>$.8288^{-4}$ | $.8350$<br>$.4365^{-1}$<br>$.3148^{-2}$<br>$.2792^{-3}$<br>$.3252^{-4}$ | $.7780$<br>$.3658^{-1}$<br>$.2309^{-2}$<br>$.1745^{-3}$<br>$.1743^{-4}$ | $.7411$<br>$.3238^{-1}$<br>$.1857^{-2}$<br>$.1247^{-3}$<br>$.1112^{-4}$ | $.7153$<br>$.2960^{-1}$<br>$.1580^{-2}$<br>$.9684^{-4}$<br>$.7907^{-5}$ | $.6962$<br>$.2764^{-1}$<br>$.1395^{-2}$<br>$.7943^{-4}$<br>$.6044^{-5}$ | $.6816$<br>$.2618^{-1}$<br>$.1263^{-2}$<br>$.6770^{-4}$<br>$.4861^{-5}$ | $.6700$<br>$.2506^{-1}$<br>$.1165^{-2}$<br>$.5935^{-4}$<br>$.4059^{-5}$ | $.6606$<br>$.2417^{-1}$<br>$.1089^{-2}$<br>$.5315^{-4}$<br>$.3486^{-5}$ | $.6528$<br>$.2344^{-1}$<br>$.1029^{-2}$<br>$.4837^{-4}$<br>$.3061^{-5}$ | $.6462$<br>$.2284^{-1}$<br>$.9798^{-3}$<br>$.4460^{-4}$<br>$.2734^{-5}$ |
| 20 | $.2030^{+1}$<br>$.3400$<br>$.9039^{-1}$<br>$.3286^{-1}$<br>$.1535^{-1}$ | $.1210^{+1}$<br>$.1007$<br>$.1250^{-1}$<br>$.2045^{-2}$<br>$.4321^{-3}$ | $.9833$<br>$.6079^{-1}$<br>$.5355^{-2}$<br>$.5998^{-3}$<br>$.8734^{-4}$ | $.8787$<br>$.4603^{-1}$<br>$.3316^{-2}$<br>$.2947^{-3}$<br>$.3424^{-4}$ | $.8188$<br>$.3858^{-1}$<br>$.2432^{-2}$<br>$.1842^{-3}$<br>$.1834^{-4}$ | $.7800$<br>$.3415^{-1}$<br>$.1956^{-2}$<br>$.1317^{-3}$<br>$.1170^{-4}$ | $.7529$<br>$.3122^{-1}$<br>$.1664^{-2}$<br>$.1022^{-3}$<br>$.8311^{-5}$ | $.7328$<br>$.2915^{-1}$<br>$.1468^{-2}$<br>$.8385^{-4}$<br>$.6350^{-5}$ | $.7174$<br>$.2761^{-1}$<br>$.1330^{-2}$<br>$.7148^{-4}$<br>$.5105^{-5}$ | $.7052$<br>$.2643^{-1}$<br>$.1226^{-2}$<br>$.6267^{-4}$<br>$.4260^{-5}$ | $.6953$<br>$.2549^{-1}$<br>$.1146^{-2}$<br>$.5611^{-4}$<br>$.3658^{-5}$ | $.6871$<br>$.2472^{-1}$<br>$.1083^{-2}$<br>$.5108^{-4}$<br>$.3210^{-5}$ | $.6802$<br>$.2409^{-1}$<br>$.1031^{-2}$<br>$.4710^{-4}$<br>$.2867^{-5}$ |
| 21 | $.2130^{+1}$<br>$.3572$<br>$.9496^{-1}$<br>$.3453^{-1}$<br>$.1613^{-1}$ | $.1270^{+1}$<br>$.1058$<br>$.1314^{-1}$<br>$.2152^{-2}$<br>$.4544^{-3}$ | $.1032^{+1}$<br>$.6393^{-1}$<br>$.5627^{-2}$<br>$.6312^{-3}$<br>$.9181^{-4}$ | $.9225$<br>$.4842^{-1}$<br>$.3484^{-2}$<br>$.3101^{-3}$<br>$.3597^{-4}$ | $.8596$<br>$.4058^{-1}$<br>$.2554^{-2}$<br>$.1939^{-3}$<br>$.1926^{-4}$ | $.8189$<br>$.3591^{-1}$<br>$.2054^{-2}$<br>$.1386^{-3}$<br>$.1227^{-4}$ | $.7904$<br>$.3284^{-1}$<br>$.1747^{-2}$<br>$.1076^{-3}$<br>$.8716^{-5}$ | $.7694$<br>$.3066^{-1}$<br>$.1542^{-2}$<br>$.8827^{-4}$<br>$.6656^{-5}$ | $.7532$<br>$.2905^{-1}$<br>$.1396^{-2}$<br>$.7525^{-4}$<br>$.5349^{-5}$ | $.7404$<br>$.2780^{-1}$<br>$.1287^{-2}$<br>$.6597^{-4}$<br>$.4462^{-5}$ | $.7300$<br>$.2681^{-1}$<br>$.1203^{-2}$<br>$.5908^{-4}$<br>$.3830^{-5}$ | $.7214$<br>$.2601^{-1}$<br>$.1137^{-2}$<br>$.5378^{-4}$<br>$.3360^{-5}$ | $.7141$<br>$.2534^{-1}$<br>$.1083^{-2}$<br>$.4959^{-4}$<br>$.3000^{-5}$ |
| 22 | $.2230^{+1}$<br>$.3744$<br>$.9952^{-1}$<br>$.3621^{-1}$<br>$.1691^{-1}$ | $.1330^{+1}$<br>$.1110$<br>$.1378^{-1}$<br>$.2258^{-2}$<br>$.4767^{-3}$ | $.1081^{+1}$<br>$.6707^{-1}$<br>$.5899^{-2}$<br>$.6625^{-3}$<br>$.9628^{-4}$ | $.9662$<br>$.5080^{-1}$<br>$.3652^{-2}$<br>$.3255^{-3}$<br>$.3770^{-4}$ | $.9004$<br>$.4258^{-1}$<br>$.2677^{-2}$<br>$.2036^{-3}$<br>$.2017^{-4}$ | $.8578$<br>$.3768^{-1}$<br>$.2153^{-2}$<br>$.1455^{-3}$<br>$.1285^{-4}$ | $.8280$<br>$.3445^{-1}$<br>$.1831^{-2}$<br>$.1130^{-3}$<br>$.9123^{-5}$ | $.8059$<br>$.3217^{-1}$<br>$.1616^{-2}$<br>$.9269^{-4}$<br>$.6964^{-5}$ | $.7890$<br>$.3048^{-1}$<br>$.1463^{-2}$<br>$.7902^{-4}$<br>$.5594^{-5}$ | $.7756$<br>$.2917^{-1}$<br>$.1349^{-2}$<br>$.6928^{-4}$<br>$.4665^{-5}$ | $.7647$<br>$.2813^{-1}$<br>$.1261^{-2}$<br>$.6204^{-4}$<br>$.4003^{-5}$ | $.7557$<br>$.2729^{-1}$<br>$.1191^{-2}$<br>$.5648^{-4}$<br>$.3511^{-5}$ | $.7481$<br>$.2658^{-1}$<br>$.1134^{-2}$<br>$.5208^{-4}$<br>$.3134^{-5}$ |
| 23 | $.2330^{+1}$<br>$.3916$<br>$.1041$<br>$.3788^{-1}$<br>$.1770^{-1}$ | $.1390^{+1}$<br>$.1162$<br>$.1442^{-1}$<br>$.2364^{-2}$<br>$.4990^{-3}$ | $.1130^{+1}$<br>$.7021^{-1}$<br>$.6171^{-2}$<br>$.6939^{-3}$<br>$.1007^{-3}$ | $.1010^{+1}$<br>$.5318^{-1}$<br>$.3820^{-2}$<br>$.3410^{-3}$<br>$.3944^{-4}$ | $.9412$<br>$.4458^{-1}$<br>$.2800^{-2}$<br>$.2132^{-3}$<br>$.2109^{-4}$ | $.8967$<br>$.3945^{-1}$<br>$.2251^{-2}$<br>$.1524^{-3}$<br>$.1343^{-4}$ | $.8655$<br>$.3607^{-1}$<br>$.1915^{-2}$<br>$.1184^{-3}$<br>$.9530^{-5}$ | $.8425$<br>$.3368^{-1}$<br>$.1689^{-2}$<br>$.9710^{-4}$<br>$.7272^{-5}$ | $.8248$<br>$.3191^{-1}$<br>$.1529^{-2}$<br>$.8278^{-4}$<br>$.5840^{-5}$ | $.8108$<br>$.3054^{-1}$<br>$.1410^{-2}$<br>$.7258^{-4}$<br>$.4869^{-5}$ | $.7994$<br>$.2945^{-1}$<br>$.1318^{-2}$<br>$.6500^{-4}$<br>$.4176^{-5}$ | $.7900$<br>$.2857^{-1}$<br>$.1245^{-2}$<br>$.5917^{-4}$<br>$.3662^{-5}$ | $.7821$<br>$.2784^{-1}$<br>$.1185^{-2}$<br>$.5457^{-4}$<br>$.3268^{-5}$ |
| 24 | $.2430^{+1}$<br>$.4088$<br>$.1086$<br>$.3956^{-1}$<br>$.1848^{-1}$ | $.1450^{+1}$<br>$.1214$<br>$.1505^{-1}$<br>$.2471^{-2}$<br>$.5213^{-3}$ | $.1179^{+1}$<br>$.7335^{-1}$<br>$.6444^{-2}$<br>$.7252^{-3}$<br>$.1052^{-3}$ | $.1054^{+1}$<br>$.5556^{-1}$<br>$.3988^{-2}$<br>$.3564^{-3}$<br>$.4117^{-4}$ | $.9820$<br>$.4658^{-1}$<br>$.2923^{-2}$<br>$.2229^{-3}$<br>$.2201^{-4}$ | $.9356$<br>$.4122^{-1}$<br>$.2350^{-2}$<br>$.1593^{-3}$<br>$.1401^{-4}$ | $.9031$<br>$.3769^{-1}$<br>$.1998^{-2}$<br>$.1237^{-3}$<br>$.9938^{-5}$ | $.8791$<br>$.3519^{-1}$<br>$.1763^{-2}$<br>$.1015^{-3}$<br>$.7581^{-5}$ | $.8606$<br>$.3334^{-1}$<br>$.1596^{-2}$<br>$.8654^{-4}$<br>$.6086^{-5}$ | $.8460$<br>$.3191^{-1}$<br>$.1471^{-2}$<br>$.7588^{-4}$<br>$.5073^{-5}$ | $.8341$<br>$.3077^{-1}$<br>$.1375^{-2}$<br>$.6796^{-4}$<br>$.4350^{-5}$ | $.8243$<br>$.2985^{-1}$<br>$.1299^{-2}$<br>$.6187^{-4}$<br>$.3814^{-5}$ | $.8160$<br>$.2908^{-1}$<br>$.1237^{-2}$<br>$.5705^{-4}$<br>$.3403^{-5}$ |
| 25 | $.2530^{+1}$<br>$.4260$<br>$.1132$<br>$.4123^{-1}$<br>$.1926^{-1}$ | $.1510^{+1}$<br>$.1265$<br>$.1569^{-1}$<br>$.2577^{-2}$<br>$.5436^{-3}$ | $.1228^{+1}$<br>$.7649^{-1}$<br>$.6716^{-2}$<br>$.7565^{-3}$<br>$.1097^{-3}$ | $.1097^{+1}$<br>$.5795^{-1}$<br>$.4156^{-2}$<br>$.3718^{-3}$<br>$.4291^{-4}$ | $.1023^{+1}$<br>$.4858^{-1}$<br>$.3046^{-2}$<br>$.2325^{-3}$<br>$.2293^{-4}$ | $.9744$<br>$.4299^{-1}$<br>$.2448^{-2}$<br>$.1662^{-3}$<br>$.1459^{-4}$ | $.9406$<br>$.3931^{-1}$<br>$.2082^{-2}$<br>$.1291^{-3}$<br>$.1035^{-4}$ | $.9156$<br>$.3670^{-1}$<br>$.1837^{-2}$<br>$.1059^{-3}$<br>$.7891^{-5}$ | $.8964$<br>$.3477^{-1}$<br>$.1662^{-2}$<br>$.9030^{-4}$<br>$.6333^{-5}$ | $.8812$<br>$.3328^{-1}$<br>$.1533^{-2}$<br>$.7918^{-4}$<br>$.5278^{-5}$ | $.8688$<br>$.3209^{-1}$<br>$.1432^{-2}$<br>$.7091^{-4}$<br>$.4525^{-5}$ | $.8586$<br>$.3113^{-1}$<br>$.1353^{-2}$<br>$.6456^{-4}$<br>$.3966^{-5}$ | $.8500$<br>$.3033^{-1}$<br>$.1288^{-2}$<br>$.5953^{-4}$<br>$.3538^{-5}$ |

| $\theta_1 \backslash \theta_2$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 14 | $.4722$<br>$.1623^{-1}$<br>$.6934^{-3}$<br>$.2986^{-4}$<br>$.1897^{-5}$ | $.4687$<br>$.1592^{-1}$<br>$.6685^{-3}$<br>$.2805^{-4}$<br>$.1742^{-5}$ | $.4655$<br>$.1565^{-1}$<br>$.6473^{-3}$<br>$.2653^{-4}$<br>$.1616^{-5}$ | $.4628$<br>$.1542^{-1}$<br>$.6290^{-3}$<br>$.2525^{-4}$<br>$.1510^{-5}$ | $.4604$<br>$.1521^{-1}$<br>$.6131^{-3}$<br>$.2415^{-4}$<br>$-.1092^{-4}$ | $.4582$<br>$.1503^{-1}$<br>$.5991^{-3}$<br>$.2320^{-4}$<br>$-.4930^{-4}$ | $.4562$<br>$.1486^{-1}$<br>$.5868^{-3}$<br>$.2236^{-4}$<br>$-.6085^{-4}$ | $.4545$<br>$.1471^{-1}$<br>$.5758^{-3}$<br>$.2163^{-4}$<br>$-.6394^{-4}$ | $.4529$<br>$.1458^{-1}$<br>$.5659^{-3}$<br>$.2098^{-4}$<br>$-.6427^{-4}$ | $.4514$<br>$.1446^{-1}$<br>$.5570^{-3}$<br>$.2039^{-4}$<br>$-.6370^{-4}$ | $.4501$<br>$.1435^{-1}$<br>$.5489^{-3}$<br>$.1987^{-4}$<br>$-.6305^{-4}$ | $.4489$<br>$.1425^{-1}$<br>$.5416^{-3}$<br>$.1940^{-4}$<br>$-.6205^{-4}$ |
| 15 | $.5059$<br>$.1745^{-1}$<br>$.7426^{-3}$<br>$.3221^{-4}$<br>$.2010^{-5}$ | $.5021$<br>$.1712^{-1}$<br>$.7159^{-3}$<br>$.3026^{-4}$<br>$.1845^{-5}$ | $.4987$<br>$.1683^{-1}$<br>$.6931^{-3}$<br>$.2863^{-4}$<br>$.1710^{-5}$ | $.4958$<br>$.1657^{-1}$<br>$.6735^{-3}$<br>$.2724^{-4}$<br>$.1598^{-5}$ | $.4932$<br>$.1635^{-1}$<br>$.6564^{-3}$<br>$.2606^{-4}$<br>$-.4427^{-4}$ | $.4909$<br>$.1615^{-1}$<br>$.6414^{-3}$<br>$.2503^{-4}$<br>$-.6385^{-4}$ | $.4888$<br>$.1598^{-1}$<br>$.6282^{-3}$<br>$.2413^{-4}$<br>$-.6914^{-4}$ | $.4869$<br>$.1582^{-1}$<br>$.6164^{-3}$<br>$.2334^{-4}$<br>$-.6998^{-4}$ | $.4852$<br>$.1568^{-1}$<br>$.6058^{-3}$<br>$.2264^{-4}$<br>$-.6943^{-4}$ | $.4836$<br>$.1555^{-1}$<br>$.5962^{-3}$<br>$.2201^{-4}$<br>$-.6872^{-4}$ | $.4822$<br>$.1543^{-1}$<br>$.5876^{-3}$<br>$.2145^{-4}$<br>$-.6754^{-4}$ | $.4809$<br>$.1532^{-1}$<br>$.5797^{-3}$<br>$.2094^{-4}$<br>$-.6647^{-4}$ |
| 16 | $.5396$<br>$.1868^{-1}$<br>$.7918^{-3}$<br>$.3455^{-4}$<br>$.2125^{-5}$ | $.5355$<br>$.1832^{-1}$<br>$.7632^{-3}$<br>$.3246^{-4}$<br>$.1950^{-5}$ | $.5320$<br>$.1800^{-1}$<br>$.7389^{-3}$<br>$.3071^{-4}$<br>$.1806^{-5}$ | $.5288$<br>$.1773^{-1}$<br>$.7180^{-3}$<br>$.2923^{-4}$<br>$-.2719^{-4}$ | $.5260$<br>$.1749^{-1}$<br>$.6998^{-3}$<br>$.2796^{-4}$<br>$-.6383^{-4}$ | $.5236$<br>$.1728^{-1}$<br>$.6838^{-3}$<br>$.2686^{-4}$<br>$-.7372^{-4}$ | $.5213$<br>$.1710^{-1}$<br>$.6696^{-3}$<br>$.2590^{-4}$<br>$-.7575^{-4}$ | $.5193$<br>$.1693^{-1}$<br>$.6570^{-3}$<br>$.2505^{-4}$<br>$-.7539^{-4}$ | $.5175$<br>$.1677^{-1}$<br>$.6457^{-3}$<br>$.2430^{-4}$<br>$-.7468^{-4}$ | $.5158$<br>$.1663^{-1}$<br>$.6355^{-3}$<br>$.2363^{-4}$<br>$-.7328^{-4}$ | $.5143$<br>$.1651^{-1}$<br>$.6263^{-3}$<br>$.2302^{-4}$<br>$-.7202^{-4}$ | $.5129$<br>$.1639^{-1}$<br>$.6179^{-3}$<br>$.2248^{-4}$<br>$-.7088^{-4}$ |
| 17 | $.5733$<br>$.1990^{-1}$<br>$.8410^{-3}$<br>$.3689^{-4}$<br>$.2242^{-5}$ | $.5689$<br>$.1951^{-1}$<br>$.8106^{-3}$<br>$.3466^{-4}$<br>$.2056^{-5}$ | $.5652$<br>$.1918^{-1}$<br>$.7848^{-3}$<br>$.3280^{-4}$<br>$.1904^{-5}$ | $.5618$<br>$.1889^{-1}$<br>$.7625^{-3}$<br>$.3122^{-4}$<br>$-.5666^{-4}$ | $.5589$<br>$.1864^{-1}$<br>$.7431^{-3}$<br>$.2986^{-4}$<br>$-.7666^{-4}$ | $.5563$<br>$.1841^{-1}$<br>$.7261^{-3}$<br>$.2869^{-4}$<br>$-.8131^{-4}$ | $.5539$<br>$.1821^{-1}$<br>$.7110^{-3}$<br>$.2766^{-4}$<br>$-.8153^{-4}$ | $.5518$<br>$.1803^{-1}$<br>$.6976^{-3}$<br>$.2675^{-4}$<br>$-.8049^{-4}$ | $.5498$<br>$.1787^{-1}$<br>$.6856^{-3}$<br>$.2595^{-4}$<br>$-.7933^{-4}$ | $.5481$<br>$.1772^{-1}$<br>$.6748^{-3}$<br>$.2523^{-4}$<br>$-.7785^{-4}$ | $.5465$<br>$.1759^{-1}$<br>$.6650^{-3}$<br>$.2459^{-4}$<br>$-.7651^{-4}$ | $.5450$<br>$.1746^{-1}$<br>$.6560^{-3}$<br>$.2401^{-4}$<br>$-.7530^{-4}$ |
| 18 | $.6069$<br>$.2112^{-1}$<br>$.8902^{-3}$<br>$.3922^{-4}$<br>$.2359^{-5}$ | $.6024$<br>$.2071^{-1}$<br>$.8580^{-3}$<br>$.3686^{-4}$<br>$.2163^{-5}$ | $.5984$<br>$.2036^{-1}$<br>$.8306^{-3}$<br>$.3487^{-4}$<br>$-.3157^{-4}$ | $.5948$<br>$.2005^{-1}$<br>$.8070^{-3}$<br>$.3320^{-4}$<br>$-.7534^{-4}$ | $.5917$<br>$.1978^{-1}$<br>$.7865^{-3}$<br>$.3175^{-4}$<br>$-.8604^{-4}$ | $.5889$<br>$.1954^{-1}$<br>$.7684^{-3}$<br>$.3051^{-4}$<br>$-.8773^{-4}$ | $.5864$<br>$.1933^{-1}$<br>$.7525^{-3}$<br>$.2942^{-4}$<br>$-.8690^{-4}$ | $.5841$<br>$.1914^{-1}$<br>$.7383^{-3}$<br>$.2845^{-4}$<br>$-.8572^{-4}$ | $.5821$<br>$.1896^{-1}$<br>$.7255^{-3}$<br>$.2760^{-4}$<br>$-.8397^{-4}$ | $.5803$<br>$.1881^{-1}$<br>$.7140^{-3}$<br>$.2684^{-4}$<br>$-.8241^{-4}$ | $.5786$<br>$.1867^{-1}$<br>$.7036^{-3}$<br>$.2615^{-4}$<br>$-.8100^{-4}$ | $.5770$<br>$.1853^{-1}$<br>$.6942^{-3}$<br>$.2554^{-4}$<br>$-.7972^{-4}$ |
| 19 | $.6406$<br>$.2234^{-1}$<br>$.9394^{-3}$<br>$.4155^{-4}$<br>$.2477^{-5}$ | $.6358$<br>$.2191^{-1}$<br>$.9054^{-3}$<br>$.3904^{-4}$<br>$.2271^{-5}$ | $.6316$<br>$.2153^{-1}$<br>$.8765^{-3}$<br>$.3695^{-4}$<br>$-.6285^{-4}$ | $.6279$<br>$.2121^{-1}$<br>$.8515^{-3}$<br>$.3517^{-4}$<br>$-.8829^{-4}$ | $.6246$<br>$.2092^{-1}$<br>$.8298^{-3}$<br>$.3365^{-4}$<br>$-.9360^{-4}$ | $.6216$<br>$.2067^{-1}$<br>$.8108^{-3}$<br>$.3232^{-4}$<br>$-.9353^{-4}$ | $.6190$<br>$.2045^{-1}$<br>$.7939^{-3}$<br>$.3117^{-4}$<br>$-.9205^{-4}$ | $.6166$<br>$.2024^{-1}$<br>$.7789^{-3}$<br>$.3015^{-4}$<br>$-.9046^{-4}$ | $.6145$<br>$.2006^{-1}$<br>$.7655^{-3}$<br>$.2925^{-4}$<br>$-.8862^{-4}$ | $.6125$<br>$.1989^{-1}$<br>$.7533^{-3}$<br>$.2844^{-4}$<br>$-.8697^{-4}$ | $.6107$<br>$.1974^{-1}$<br>$.7423^{-3}$<br>$.2772^{-4}$<br>$-.8548^{-4}$ | $.6090$<br>$.1961^{-1}$<br>$.7323^{-3}$<br>$.2706^{-4}$<br>$-.8413^{-4}$ |
| 20 | $.6743$<br>$.2356^{-1}$<br>$.9886^{-3}$<br>$.4388^{-4}$<br>$.2597^{-5}$ | $.6692$<br>$.2310^{-1}$<br>$.9528^{-3}$<br>$.4123^{-4}$<br>$-.1941^{-4}$ | $.6648$<br>$.2271^{-1}$<br>$.9223^{-3}$<br>$.3902^{-4}$<br>$-.8354^{-4}$ | $.6609$<br>$.2237^{-1}$<br>$.8961^{-3}$<br>$.3714^{-4}$<br>$-.9808^{-4}$ | $.6574$<br>$.2207^{-1}$<br>$.8732^{-3}$<br>$.3553^{-4}$<br>$-.1001^{-3}$ | $.6543$<br>$.2180^{-1}$<br>$.8532^{-3}$<br>$.3414^{-4}$<br>$-.9899^{-4}$ | $.6515$<br>$.2156^{-1}$<br>$.8354^{-3}$<br>$.3292^{-4}$<br>$-.9738^{-4}$ | $.6491$<br>$.2135^{-1}$<br>$.8196^{-3}$<br>$.3185^{-4}$<br>$-.9521^{-4}$ | $.6468$<br>$.2116^{-1}$<br>$.8054^{-3}$<br>$.3089^{-4}$<br>$-.9327^{-4}$ | $.6447$<br>$.2098^{-1}$<br>$.7926^{-3}$<br>$.3004^{-4}$<br>$-.9153^{-4}$ | $.6428$<br>$.2082^{-1}$<br>$.7811^{-3}$<br>$.2927^{-4}$<br>$-.8997^{-4}$ | $.6411$<br>$.2068^{-1}$<br>$.7705^{-3}$<br>$.2858^{-4}$<br>$-.8855^{-4}$ |
| 21 | $.7080$<br>$.2478^{-1}$<br>$.1038^{-2}$<br>$.4620^{-4}$<br>$.2716^{-5}$ | $.7026$<br>$.2430^{-1}$<br>$.1000^{-2}$<br>$.4342^{-4}$<br>$-.5863^{-4}$ | $.6980$<br>$.2389^{-1}$<br>$.9682^{-3}$<br>$.4109^{-4}$<br>$-.9821^{-4}$ | $.6939$<br>$.2353^{-1}$<br>$.9406^{-3}$<br>$.3911^{-4}$<br>$-.1061^{-3}$ | $.6902$<br>$.2321^{-1}$<br>$.9166^{-3}$<br>$.3742^{-4}$<br>$-.1061^{-3}$ | $.6870$<br>$.2293^{-1}$<br>$.8955^{-3}$<br>$.3595^{-4}$<br>$-.1043^{-3}$ | $.6841$<br>$.2268^{-1}$<br>$.8769^{-3}$<br>$.3467^{-4}$<br>$-.1022^{-3}$ | $.6815$<br>$.2245^{-1}$<br>$.8602^{-3}$<br>$.3354^{-4}$<br>$-.9995^{-4}$ | $.6791$<br>$.2225^{-1}$<br>$.8454^{-3}$<br>$.3253^{-4}$<br>$-.9792^{-4}$ | $.6769$<br>$.2207^{-1}$<br>$.8319^{-3}$<br>$.3164^{-4}$<br>$-.9609^{-4}$ | $.6749$<br>$.2190^{-1}$<br>$.8198^{-3}$<br>$.3083^{-4}$<br>$-.9445^{-4}$ | $.6731$<br>$.2175^{-1}$<br>$.8087^{-3}$<br>$.3010^{-4}$<br>$-.9296^{-4}$ |
| 22 | $.7416$<br>$.2600^{-1}$<br>$.1087^{-2}$<br>$.4853^{-4}$<br>$.2837^{-5}$ | $.7360$<br>$.2550^{-1}$<br>$.1048^{-2}$<br>$.4560^{-4}$<br>$-.8511^{-4}$ | $.7312$<br>$.2506^{-1}$<br>$.1014^{-2}$<br>$.4315^{-4}$<br>$-.1094^{-3}$ | $.7269$<br>$.2469^{-1}$<br>$.9852^{-3}$<br>$.4108^{-4}$<br>$-.1131^{-3}$ | $.7231$<br>$.2435^{-1}$<br>$.9600^{-3}$<br>$.3930^{-4}$<br>$-.1118^{-3}$ | $.7197$<br>$.2406^{-1}$<br>$.9379^{-3}$<br>$.3776^{-4}$<br>$-.1098^{-3}$ | $.7166$<br>$.2380^{-1}$<br>$.9183^{-3}$<br>$.3641^{-4}$<br>$-.1071^{-3}$ | $.7139$<br>$.2356^{-1}$<br>$.9009^{-3}$<br>$.3523^{-4}$<br>$-.1047^{-3}$ | $.7114$<br>$.2335^{-1}$<br>$.8853^{-3}$<br>$.3417^{-4}$<br>$-.1026^{-3}$ | $.7091$<br>$.2315^{-1}$<br>$.8712^{-3}$<br>$.3323^{-4}$<br>$-.1007^{-3}$ | $.7070$<br>$.2298^{-1}$<br>$.8585^{-3}$<br>$.3239^{-4}$<br>$-.9893^{-4}$ | $.7051$<br>$.2282^{-1}$<br>$.8469^{-3}$<br>$.3162^{-4}$<br>$-.9737^{-4}$ |
| 23 | $.7753$<br>$.2722^{-1}$<br>$.1136^{-2}$<br>$.5084^{-4}$<br>$-.3002^{-4}$ | $.7695$<br>$.2669^{-1}$<br>$.1095^{-2}$<br>$.4778^{-4}$<br>$-.1040^{-3}$ | $.7644$<br>$.2624^{-1}$<br>$.1060^{-2}$<br>$.4522^{-4}$<br>$-.1185^{-3}$ | $.7599$<br>$.2584^{-1}$<br>$.1030^{-2}$<br>$.4305^{-4}$<br>$-.1194^{-3}$ | $.7559$<br>$.2550^{-1}$<br>$.1003^{-2}$<br>$.4118^{-4}$<br>$-.1172^{-3}$ | $.7524$<br>$.2519^{-1}$<br>$.9803^{-3}$<br>$.3957^{-4}$<br>$-.1147^{-3}$ | $.7492$<br>$.2491^{-1}$<br>$.9598^{-3}$<br>$.3816^{-4}$<br>$-.1119^{-3}$ | $.7463$<br>$.2467^{-1}$<br>$.9416^{-3}$<br>$.3692^{-4}$<br>$-.1094^{-3}$ | $.7437$<br>$.2444^{-1}$<br>$.9253^{-3}$<br>$.3581^{-4}$<br>$-.1072^{-3}$ | $.7413$<br>$.2424^{-1}$<br>$.9105^{-3}$<br>$.3483^{-4}$<br>$-.1052^{-3}$ | $.7392$<br>$.2406^{-1}$<br>$.8972^{-3}$<br>$.3394^{-4}$<br>$-.1034^{-3}$ | $.7372$<br>$.2389^{-1}$<br>$.8851^{-3}$<br>$.3314^{-4}$<br>$-.1844^{-3}$ |
| 24 | $.8090$<br>$.2844^{-1}$<br>$.1186^{-2}$<br>$.5316^{-4}$<br>$-.6988^{-4}$ | $.8029$<br>$.2789^{-1}$<br>$.1143^{-2}$<br>$.4996^{-4}$<br>$-.1182^{-3}$ | $.7976$<br>$.2742^{-1}$<br>$.1106^{-2}$<br>$.4728^{-4}$<br>$-.1263^{-3}$ | $.7929$<br>$.2700^{-1}$<br>$.1074^{-2}$<br>$.4501^{-4}$<br>$-.1254^{-3}$ | $.7888$<br>$.2664^{-1}$<br>$.1047^{-2}$<br>$.4306^{-4}$<br>$-.1230^{-3}$ | $.7851$<br>$.2632^{-1}$<br>$.1023^{-2}$<br>$.4138^{-4}$<br>$-.1197^{-3}$ | $.7817$<br>$.2603^{-1}$<br>$.1001^{-2}$<br>$.3990^{-4}$<br>$-.1168^{-3}$ | $.7788$<br>$.2577^{-1}$<br>$.9823^{-3}$<br>$.3860^{-4}$<br>$-.1142^{-3}$ | $.7760$<br>$.2554^{-1}$<br>$.9652^{-3}$<br>$.3745^{-4}$<br>$-.1119^{-3}$ | $.7736$<br>$.2533^{-1}$<br>$.9498^{-3}$<br>$.3642^{-4}$<br>$-.1098^{-3}$ | $.7713$<br>$.2514^{-1}$<br>$.9359^{-3}$<br>$.3549^{-4}$<br>$-.1150^{-3}$ | $.7692$<br>$.2496^{-1}$<br>$.9233^{-3}$<br>$.3466^{-4}$<br>$-.2559^{-3}$ |
| 25 | $.8427$<br>$.2966^{-1}$<br>$.1235^{-2}$<br>$.5548^{-4}$<br>$-.9815^{-4}$ | $.8363$<br>$.2908^{-1}$<br>$.1190^{-2}$<br>$.5214^{-4}$<br>$-.1294^{-3}$ | $.8308$<br>$.2859^{-1}$<br>$.1152^{-2}$<br>$.4934^{-4}$<br>$-.1332^{-3}$ | $.8259$<br>$.2816^{-1}$<br>$.1119^{-2}$<br>$.4697^{-4}$<br>$-.1311^{-3}$ | $.8216$<br>$.2778^{-1}$<br>$.1090^{-2}$<br>$.4494^{-4}$<br>$-.1281^{-3}$ | $.8178$<br>$.2745^{-1}$<br>$.1065^{-2}$<br>$.4318^{-4}$<br>$-.1247^{-3}$ | $.8143$<br>$.2715^{-1}$<br>$.1043^{-2}$<br>$.4164^{-4}$<br>$-.1216^{-3}$ | $.8112$<br>$.2688^{-1}$<br>$.1023^{-2}$<br>$.4029^{-4}$<br>$-.1189^{-3}$ | $.8083$<br>$.2663^{-1}$<br>$.1005^{-2}$<br>$.3908^{-4}$<br>$-.1165^{-3}$ | $.8058$<br>$.2642^{-1}$<br>$.9892^{-3}$<br>$.3801^{-4}$<br>$-.1143^{-3}$ | $.8034$<br>$.2621^{-1}$<br>$.9747^{-3}$<br>$.3704^{-4}$<br>$-.2074^{-3}$ | $.8012$<br>$.2603^{-1}$<br>$.9615^{-3}$<br>$.3617^{-4}$<br>$-.3117^{-3}$ |

TABLE 2

BIASES OF $\hat{\theta}_1$ AND $\theta_1^*$ (m.l. and Moment Resp.)

SAMPLE SIZE $n = 100$ ENTRIES ARE (a) $E(\hat{\theta}_1 - \theta_1)/\theta_1$, (b) $E(\theta_1^* - \theta_1)/\theta_1$

| $\theta_1$ | $\theta_2$ | 0.5 | 1.0 | 5.0 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|---|---|
| 0.5 | a | .250 + .163 | .064 + .012 | .0024 + .0001 | .00036 - .00001 | .00027 + .00002 | .00025 + .00003 | .00019 + .00003 |
| | b | .410 + .194 | .160 + .013 | .0464 - .0008 | .03760 - .00056 | .03494 - .00047 | .03364 - .00043 | .03290 - .00040 |
| 1.0 | a | .191 + .096 | .057 + .009 | .0052 + .0003 | .00194 + .00012 | .00109 + .00013 | .00064 + .00009 | .00039 + .00005 |
| | b | .310 + .128 | .130 + .014 | .0436 + .0006 | .03640 + .00040 | .03418 + .00036 | .03310 + .00034 | .03246 + .00033 |
| 5.0 | a | .191 + .080 | .087 + .014 | .0398 + .0015 | .03408 + .00110 | .02488 + .01835 | .01271 + .01533 | .00589 + .00611 |
| | b | .230 + .096 | .106 + .016 | .0413 + .0017 | .03544 + .00118 | .03358 + .00103 | .03266 + .00096 | .03212 + .00092 |
| 10.0 | a | .200 + .087 | .094 + .016 | .0396 + .0019 | .03467 + .00058 | .03470 - .00939 | .03818 - .03878 | .03972 + .02658 |
| | b | .220 + .093 | .103 + .017 | .0411 + .0019 | .03532 + .00127 | .03350 + .00111 | .03760 + .00104 | .03708 + .00100 |
| 15 | a | .204 + .088 | .096 + .016 | .0399 + .0019 | .03454 + .00133 | .03291 + .00737 | .03244 - .00477 | .03335 - .03881 |
| | b | .217 + .092 | .102 + .017 | .0410 + .0019 | .03528 + .00131 | .03347 + .00114 | .03259 + .00107 | .03206 + .00102 |
| 20 | a | .205 + .089 | .097 + .017 | .0410 + .0020 | .03470 + .00135 | .03298 + .00117 | .03215 + .00086 | .03173 - .00179 |
| | b | .215 + .092 | .102 + .017 | .0409 + .0019 | .03526 + .00132 | .03346 + .00116 | .03258 + .00108 | .03205 + .00103 |
| 25 | a | .206 + .089 | .098 + .017 | .0402 + .0020 | .03480 + .00135 | .03307 + .00118 | .03222 + .00110 | .03172 + .00092 |
| | b | .214 + .092 | .101 + .017 | .0409 + .0019 | .03525 + .00133 | .03345 + .00116 | .03257 + .00109 | .03205 + .00104 |

TABLE 3

BIASES OF $\hat{\theta}_2$ AND $\theta_2^*$ (m.l. and Moment Resp.)

SAMPLE SIZE n = 20 ENTRIES ARE (a) $E(\hat{\theta}_2 - \theta_2)/\theta_2$, (b) $E(\theta_2^* - \theta_2)/\theta_2$

| $\theta_1$ | $\theta_2$ | 0.5 | 1.0 | 5.0 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|---|---|
| 0.5 | a | -.0735 - .0316 | -.0306 - .0051 | .0112 + .0024 | .0106 + .0043 | .0108 + .0030 | .00247 + .00146 | .00117 + .00067 |
| | b | -.3500 - .0100 | -.2000 - .0050 | -.0800 - .0010 | -.0650 - .0005 | -.0600 - .0003 | -.05750 - .00025 | -.05600 - .00020 |
| 1.0 | a | -.0741 - .0334 | -.0386 - .0068 | .0028 + .0009 | .0055 + .0028 | .0023 + .0010 | .00079 + .00011 | .00030 - .00002 |
| | b | -.2500 - .0000 | -.1500 - .0000 | -.0700 + .0000 | -.0600 + .0000 | -.0567 + .0000 | -.05500 + .00000 | -.05400 - .00000 |
| 5.0 | a | -.1411 + .0053 | -.1282 + .0118 | -.0876 + .0335 | -.0872 + .1311 | -.0756 - .2085 | -.04042 - .26013 | -.01842 - .10967 |
| | b | -.1700 + .0008 | -.1100 + .0004 | -.0620 + .0001 | -.0560 + .0000 | -.0540 + .0000 | -.05300 + .00002 | -.05240 + .00002 |
| 10 | a | -.1506 - .0037 | -.1064 - .0018 | -.0637 + .0002 | -.0591 + .0193 | -.0677 + .3215 | -.09707 + 1.4525 | -.12675 + .57186 |
| | b | -.1600 + .0000 | -.1050 + .0002 | -.0610 + .0000 | -.0555 + .0000 | -.0537 + .0000 | -.05277 + .00001 | -.05220 + .00001 |
| 15 | a | -.1513 - .0058 | -.1039 - .0011 | -.0615 + .0002 | -.0558 + .0003 | -.0543 + .0117 | -.05547 + .16737 | -.06330 + 1.18492 |
| | b | -.1567 + .0003 | -.1034 + .0002 | -.0607 + .0000 | -.0553 + .0000 | -.0536 + .0000 | -.05267 + .00001 | -.05213 + .00000 |
| 20 | a | -.1512 - .0051 | -.1028 - .0006 | -.0610 + .0001 | -.0555 + .0001 | -.0537 + .0003 | -.05287 - .00666 | -.05275 + .07959 |
| | b | -.1550 + .0002 | -.1025 + .0001 | -.0605 + .0000 | -.0552 + .0000 | -.0535 + .0000 | -.05262 - .00000 | -.05210 + .00000 |
| 25 | a | -.1511 - .0043 | -.1021 - .0003 | -.0607 + .0001 | -.0554 + .0000 | -.0536 + .0000 | -.05270 + .00022 | -.05220 + .00382 |
| | b | -.1540 + .0002 | -.1020 + .0001 | -.0604 + .0000 | -.0552 + .0000 | -.0535 + .0000 | -.05260 + .00000 | -.05208 + .00000 |

TABLE 4

VARIANCES: (a) VAR $\hat{\theta}_1/\theta_1^2$ AND (b) VAR $\theta_1^*/\theta_1^2$

SAMPLE SIZE n = 100

| $\theta_1$ | $\theta_2$ | 0.5 | 1.0 | 5.0 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|---|---|
| 0.5 | a | .2550 + .3947 | .0788 + .0270 | .0247 + .0041 | .02248 + .00025 | .02135 + .00019 | .02074 + .00012 | .02043 + .00008 |
| | b | .3600 + .5736 | .1400 + .0528 | .0504 + .0025 | .04460 + .00203 | .04292 + .00193 | .04216 + .00189 | .04170 + .00187 |
| 1.0 | a | .1859 + .2338 | .0592 + .0207 | .0158 + .0007 | .01304 + .00038 | .01154 + .00023 | .01080 + .00012 | .01045 + .00007 |
| | b | .2700 + .3704 | .1100 + .0466 | .0396 + .0037 | .03440 + .00271 | .03284 + .00244 | .03210 + .00232 | .03166 + .00225 |
| 5.0 | a | .1643 + .2097 | .0673 + .0355 | .0263 + .0056 | .01864 + .00826 | .01178 + .01108 | .00663 + .00530 | .00421 + .00185 |
| | b | .1980 + .2572 | .0860 + .0442 | .0310 + .0047 | .02624 + .00324 | .02477 + .00284 | .02406 + .00266 | .02364 + .00255 |
| 10 | a | .1712 + .2210 | .0741 + .0396 | .0279 + .0047 | .02384 + .00335 | .02215 + .00538 | .01977 + .02140 | .01537 + .05128 |
| | b | .1890 + .2462 | .0830 + .0441 | .0299 + .0048 | .02522 + .00331 | .02376 + .00289 | .02305 + .00270 | .02264 + .00259 |
| 15 | a | .1740 + .2257 | .0762 + .0411 | .0283 + .0048 | .02405 + .00331 | .02271 + .00295 | .02201 + .00359 | .02136 + .01031 |
| | b | .1860 + .2426 | .0820 + .0441 | .0295 + .0049 | .02488 + .00334 | .02342 + .00291 | .02272 + .00271 | .02230 + .00260 |
| 20 | a | .1755 + .2281 | .0772 + .0418 | .0284 + .0048 | .02409 + .00334 | .02274 + .00292 | .02208 + .00275 | .02168 + .00296 |
| | b | .1845 + .2409 | .0815 + .0441 | .0293 + .0049 | .02471 + .00334 | .02325 + .00292 | .02255 + .00272 | .02213 + .00261 |
| 25 | a | .1765 + .2296 | .0778 + .0423 | .0285 + .0049 | .02411 + .00334 | .02275 + .00292 | .02208 + .00273 | .02168 + .00263 |
| | b | .1835 + .2398 | .0812 + .0440 | .0292 + .0049 | .02461 + .00335 | .02315 + .00292 | .02245 + .00273 | .02203 + .00261 |

TABLE 5

COVARIANCES: (a) $COV(\hat{\Theta}_1,\hat{\Theta}_2)/(\Theta_1\Theta_2)$ AND (b) $COV(\Theta_1^*,\Theta_2^*)/(\Theta_1\Theta_2)$

SAMPLE SIZE n = 100

| $\Theta_1$ | $\Theta_2$ | 0.5 | 1.0 | 5.0 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|---|---|
| 0.5 | a | -.2350 - .1580 | -.0588 - .0116 | -.00468 - .00019 | -.00248 - .00016 | -.00135 - .00014 | -.00074 - .000092 | -.00043 - .000057 |
| | b | -.3400 - .1650 | -.1200 - .0066 | -.03040 + .00156 | -.02460 + .00107 | -.02293 + .00091 | -.02215 + .000828 | -.02170 + .000781 |
| 1.0 | a | -.1759 - .0921 | -.0492 - .0086 | -.00580 - .00031 | -.00304 - .00024 | -.00154 - .00017 | -.00080 - .000090 | -.00045 - .000048 |
| | b | -.2600 - .1126 | -.1000 - .0103 | -.02960 + .00001 | -.02440 + .00003 | -.02285 + .00026 | -.02210 + .000021 | -.02166 + .000018 |
| 5.0 | a | -.1623 - .0795 | -.0653 - .0131 | -.02232 - .00218 | -.01664 - .00574 | -.00977 - .00963 | -.00463 - .004823 | -.00221 - .001698 |
| | b | -.1960 - .0882 | -.0840 - .0141 | -.02896 - .00121 | -.02424 - .00078 | -.02277 - .00067 | -.02206 - .000617 | -.02164 - .000587 |
| 10 | a | -.1702 - .0810 | -.0731 - .0138 | -.02688 - .00136 | -.02284 - .00094 | -.02115 - .00300 | -.01877 - .018515 | -.01437 - .048444 |
| | b | -.1880 - .0863 | -.0820 - .0147 | -.02888 - .00136 | -.02422 - .00088 | -.02277 - .00075 | -.02205 - .000696 | -.02164 - .000662 |
| 15 | a | -.1733 - .0819 | -.0755 - .0143 | -.02759 - .00145 | -.02338 - .00095 | -.02205 - .00085 | -.02135 - .001569 | -.02069 - .008165 |
| | b | -.1853 - .0857 | -.0813 - .0148 | -.02885 - .00141 | -.02421 - .00092 | -.02276 - .00078 | -.02205 - .000722 | -.02163 - .000688 |
| 20 | a | -.1750 - .0825 | -.0767 - .0145 | -.02790 - .00147 | -.02360 - .00096 | -.02224 - .00083 | -.02158 - .000784 | -.02118 - .001061 |
| | b | -.1840 - .0854 | -.0810 - .0149 | -.02884 - .00144 | -.02421 - .00093 | -.02276 - .00080 | -.02205 - .000735 | -.02163 - .000700 |
| 25 | a | -.1760 - .0829 | -.0774 - .0147 | -.02809 - .00148 | -.02372 - .00097 | -.02235 - .00083 | -.02168 - .000766 | -.02128 - .000741 |
| | b | -.1832 - .0853 | -.0808 - .0150 | -.02883 - .00145 | -.02421 - .00094 | -.02276 - .00081 | -.02206 - .000744 | -.02163 - .000708 |

TABLE 6

VARIANCES (a) VAR $\hat{\theta}_2/\theta_2^2$ AND (b) VAR $\theta_2^*/\theta_2^2$

SAMPLE SIZE n = 100

| $\theta_1$ | $\theta_2$ | 0.5 | 1.0 | 5.0 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|---|---|
| 0.5 | a | .2750 + .01247 | .07876 + .00271 | .00868 + .00031 | .00448 + .00021 | .00268 + .00016 | .00174 + .00010 | .00123 + .000068 |
| | b | .3800 - .03180 | .14000 - .00880 | .03440 - .00140 | .02660 - .00096 | .02427 - .00083 | .02315 - .00077 | .02250 - .000734 |
| 1.0 | a | .1959 + .00667 | .05922 + .00157 | .00780 + .00027 | .00404 + .00022 | .00220 + .00016 | .00130 + .00008 | .00085 + .000048 |
| | b | .2800 - .01540 | .11000 - .00470 | .03160 - .00086 | .02540 - .00061 | .02351 - .00054 | .02260 - .00050 | .02206 - .000479 |
| 5.0 | a | .1663 + .00154 | .06727 - .00049 | .02272 + .00004 | .01684 + .00401 | .00991 + .00856 | .00473 + .00448 | .00229 + .001602 |
| | b | .2000 - .00430 | .08600 - .00156 | .02936 - .00041 | .02444 - .00032 | .02291 - .00029 | .02216 - .00028 | .02172 - .000270 |
| 10 | a | .1722 - .00124 | .07411 - .00115 | .02708 - .00041 | .02294 - .00033 | .02122 + .00161 | .01882 + .01658 | .01441 + .046167 |
| | b | .1900 - .00303 | .08300 - .00118 | .02908 - .00327 | .02432 - .00028 | .02283 - .00026 | .02210 - .00025 | .02168 - .000243 |
| 15 | a | .1747 - .00170 | .07621 - .00102 | .02772 - .00031 | .02345 - .00026 | .02209 - .00022 | .02138 + .00052 | .02072 + .006943 |
| | b | .1867 - .00252 | .08200 - .00105 | .02899 - .00033 | .02428 - .00027 | .02280 - .00025 | .02209 - .00024 | .02166 - .000234 |
| 20 | a | .1760 - .00181 | .07721 - .00095 | .02800 - .00030 | .02364 - .00025 | .02227 - .00023 | .02160 - .00021 | .02120 + .000103 |
| | b | .1850 - .00241 | .08150 - .00099 | .02894 - .00032 | .02426 - .00026 | .02279 - .00024 | .02208 - .00023 | .02165 - .000230 |
| 25 | a | .1768 - .00184 | .07779 - .00091 | .02817 - .00030 | .02376 - .00025 | .02237 - .00023 | .02169 - .00022 | .02129 - .000207 |
| | b | .1840 - .00229 | .08120 - .00095 | .02891 - .00031 | .02425 - .00026 | .02279 - .00024 | .02207 - .00023 | .02165 - .000227 |

TABLE 7

DOMINANT TERMS IN ASYMPTOTIC EFFICIENCY IN JOINT ESTIMATION OF $\Theta_1$, $\Theta_2$, BY MOMENTS

SAMPLE SIZE n = 100

| $\Theta_1$ \ $\Theta_2$ | 0.5 | 1.0 | 5.0 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|---|
| 0.5 | 2.52<br>0.703<br>4.45 | 0.35<br>0.529<br>0.88 | 0.05<br>0.238<br>0.14 | 0.05<br>0.163<br>0.11 | 0.06<br>0.108<br>0.10 | 0.06<br>0.073<br>0.10 | 0.06<br>0.053<br>0.10 |
| 1.0 | 2.67<br>0.685<br>5.12 | 0.44<br>0.516<br>1.21 | 0.06<br>0.239<br>0.22 | 0.07<br>0.156<br>0.17 | 0.08<br>0.092<br>0.16 | 0.07<br>0.056<br>0.16 | 0.06<br>0.038<br>0.15 |
| 5.0 | 9.50<br>0.829<br>13.51 | 2.43<br>0.780<br>3.81 | 0.55<br>0.774<br>0.80 | 0.61<br>0.689<br>0.61 | 1.06<br>0.432<br>0.56 | 1.02<br>0.213<br>0.54 | 0.73<br>0.105<br>0.52 |
| 10 | 20.05<br>0.906<br>24.29 | 5.65<br>0.892<br>7.02 | 1.30<br>0.931<br>1.49 | 1.03<br>0.943<br>1.15 | 1.02<br>0.929<br>1.05 | 1.68<br>0.851<br>1.00 | 3.75<br>0.665<br>0.97 |
| 15 | 30.80<br>0.936<br>35.08 | 8.87<br>0.929<br>10.22 | 2.00<br>0.956<br>2.18 | 1.58<br>0.966<br>1.68 | 1.45<br>0.969<br>1.54 | 1.42<br>0.968<br>1.47 | 1.67<br>0.956<br>1.43 |
| 20 | 41.58<br>0.951<br>45.88 | 12.08<br>0.947<br>13.43 | 2.70<br>0.968<br>2.87 | 2.11<br>0.975<br>2.22 | 1.93<br>0.977<br>2.02 | 1.85<br>0.978<br>1.92 | 1.82<br>0.979<br>1.88 |
| 25 | 52.37<br>0.961<br>56.68 | 15.28<br>0.958<br>16.63 | 3.39<br>0.974<br>3.57 | 2.64<br>0.980<br>2.75 | 2.42<br>0.982<br>2.51 | 2.31<br>0.983<br>2.39 | 2.25<br>0.984<br>2.33 |

(Entries in this table are: Efficiency = Ratio of Covariance determinants to order $n^{-2}$, expressed as $\hat{C}/C$, where $\hat{C}$ and C refer to maximum likelihood and moments respectively. Moreover writing

$$\hat{C}/C \sim E_{ff}\left(\frac{1 + C_1/n}{1 + C_2/n}\right), \quad \text{an entry is } E_{ff}, \ \frac{C_1}{C_2}.)$$

## 5. COMMENTS ON THE ESTIMATORS

5.1 Biases of $\theta_1$. These are given in Table 2, the entries being $E(\hat{\theta}_1 - \theta_1)/\theta_1$ and $E(\theta_1^* - \theta_1)/\theta_1$ for n = 100. It will be seen that the second order term becomes important (or even dominant) when (a) $\theta_2$ decreases for fixed $\theta_1$ for $\hat{\theta}_1$ and $\theta_1^*$, (b) $\theta_1$, $\theta_2$ are in a wedge-shaped region bounded by a parabolic arc with axis sloping at approximately 45°, and vertex at about $\theta_1 = 5$, $\theta_2 = 15$, for $\hat{\theta}_1$ only. With regard to (a) this condition corresponds to the tangency of the NTAd to the Poisson distribution as $\theta_2 \to 0$. As for (b), the unexpected resurgence of the second order asymptotic for $\hat{\theta}_1$ seems to correspond to the region of high modality of the distribution. Intuitively one would anticipate estimation problems with distributions which tend to oscillate, but it is strange that the property is not shared by the moment estimator; of course the m.l. estimator has a more complicated structure, and this makes it difficult to appreciate its characteristics. Further information on the biases is given graphically in Figures 4a and 4b. As a rough rule, for the parameter space tabulated, a sample size of a hundred seems adequate to dilute the second-order terms except in the Poisson region ($\theta_2$ small) and in the region of high modality. However in any given case the tables and charts should be consulted for more precise information.

Some remarks on modality are relegated to the Appendix.

5.2 Biases of $\theta_2$. These are given in Table 3 and Figures 5a and 5b, the entries being $E(\hat{\theta}_2 - \theta_2)/\theta_2$, and $E(\theta_2^* - \theta_2)/\theta_2$ for samples of size n = 20. As before the second order asymptotic for $\hat{\theta}_2$ is held in check

apart from (a) a region in which $\theta_2$ is small and particularly so for $\theta_1$ also small; (b) a wedge-shaped region similar to the one for $E\hat{\theta}_1$, and bounded by a parabolic arc with vertex at about $\theta_1 = 5$, $\theta_2 = 10$. The second-order asymptotic for $E\theta_2^*$ is for the most part held at all the tabulated points with $n = 20$. As a rough rule, a sample of $n = 20$ or thereabouts is sufficient for stabilization, with the exception of the m.l. estimator in the parameter space favoring Poissonicity or high modality.

5.3 Variances of $\theta_1$. These are given in Table 4 and Figures 1a and 1b, the entries being Var $(\hat{\theta}_1/\theta_1)$ and Var $(\theta_1^*/\theta_1)$. The results are similar to those for $E\hat{\theta}_1$ and $E\theta_1^*$, except that second order terms play a more dominant role. Both the moment and m.l. estimators show marked instability in the Poisson region, and a larger sample size is needed for stabilization as compared to the biases. Another trouble spot occurs for the m.l. estimator only, in the wedge-shaped region previously mentioned. Again the relative magnitude of the terms for the two estimators shows remarkable invariance especially in the Poisson region; moreover the tendency to limiting values as $\theta_1$ increases ($\theta_2$ fixed) is noteworthy.

5.4 Covariances. These are given in Table 5 and Figures 2a and 2b, the entries being $\mathrm{Cov}(\hat{\theta}_1/\theta_1, \hat{\theta}_2/\theta_2)$ and $\mathrm{Cov}(\theta_1^*/\theta_1, \theta_2^*/\theta_2)$. Both estimators need samples of more than a hundred for stabilization in the Poisson region, especially if both parameters are small. In addition the m.l. estimator suffers some instability in the region of high modality. The high correlation between the terms in the m.l. and moment expansions in a region avoiding $\theta_1$ small is noteworthy.

5.5 Variances of $\theta_2$. These are given in Table 6 and Figures 3a and 3b, the entries being $\mathrm{Var}(\hat{\theta}_2/\theta_2)$ and $\mathrm{Var}(\theta_2^*/\theta_2)$. In this case the second order asymptotics are much less important, especially in the Poisson region; however the m.l. estimator again has a trouble spot in the high modality parameter space.

5.6 Asymptotic Efficiency. This is given in Table 7 and Figure 6, and is based on the ratio

$$\text{Eff.} = \frac{|\text{Cov. matrix for m.l. estimators}|}{|\text{Cov. matrix for moment estimators}|},$$

$$= \frac{\hat{C}}{C}.$$

where

$$C = \begin{vmatrix} \mathrm{Var}\ \theta_1^* & \mathrm{Cov.}(\theta_1^*, \theta_2^*) \\ \mathrm{Cov.}(\theta_1^*, \theta_2^*) & \mathrm{Var}\ \theta_2^* \end{vmatrix}$$

and similarly for $\hat{C}$. Each of the terms in $C$ and $\hat{C}$ is approximated to order $n^{-2}$, and finally we have to order $n^{-1}$

$$\text{Eff.} \sim E_{ff}\left(\frac{1 + C_1/n}{1 + C_2/n}\right) .$$

The $E_{ff}$ factor is the usual asymptotic for the efficiency of the method of moments relative to the method of maximum likelihood; the $C_1$ and $C_2$ terms may be regarded as correction terms. The tabulated values of $E_{ff}$ have been truncated to 3 d.p.'s, but the original values to 4 d.p.'s and they agreed to 3 d.p.'s with those given by Katti and Gurland (1962). If we relied entirely on the usual asymptotic for the efficiency we

should be ignorant of the fact that high efficiency usually entails highly unstable moments; this is evident in the tabulation. Again at the moderate values $\theta_1 = \theta_2 = 1$, we have

$$\text{Eff} \sim 0.5\left(\frac{1 + 0.44 + \ldots}{1 + 1.21 + \ldots}\right) \qquad n = 100 .$$

The situation with the same sample size deteriorates rapidly as the efficiency $E_{ff}$ increases towards unity. This perverse behaviour has also been noticed with respect to a similar problem of estimation for the negative binomial distribution. It seems to the authors that it is unsound practice to advise the use of one set of estimators against another set merely because of a high relative efficiency, unless some reference is made to sample size over and above a mere qualitative appeal. Thus in the example just quoted, we might advise a sample of about 500 or more. In the Poisson region (for the parameter values of the tabulation) first order efficiencies are in general high; in no case could one justify a sample size of less than about 750. Even in the parameter space remote from the Poisson region, high 'efficiencies' are possible but they are of doubtful validity and perhaps meaningless with samples of less than 500-1000.

5.7 General Remarks. Both the moment and m.l. estimators have unstable asymptotics in the Poisson region for samples of n = 100. A suggested hierarchical trouble ordering would be Var $\hat{\theta}_1$, Cov$(\hat{\theta}_1, \hat{\theta}_2)$ and $E\hat{\theta}_1$; the intensity of trouble is about the same for both estimators. In addition the m.l. estimator moments get into trouble in a region of high modality for the distribution; roughly this is the inside region of a parabolic

arc sweeping away from the origin. The intensity of trouble is about the same for all the moments considered <u>with the exception of</u> $E\Theta_2$; here, for a direct comparison with the entries in the other tables, we have for $n = 100$

$$\Theta_1 = 5 \qquad \Theta_2 = 20 \qquad E(\hat{\Theta}_2 - \Theta_2) \sim -.0081 - .0106$$

$$\Theta_1 = 10 \qquad \Theta_2 = 20 \qquad E(\hat{\Theta}_2 - \Theta_2) \sim -.0194 + .0581$$

$$\Theta_1 = 15 \qquad \Theta_2 = 20 \qquad E(\hat{\Theta}_2 - \Theta_2) \sim -.0111 + .0067.$$

Evidently the second order terms here are much more important than in the other cases considered. This feature may be related to the fact that the modes of the distribution tend to be spaced at intervals of $\Theta_2$ in the region of high modality; further reference to modality is made in the appendix (see also in Figure 7).

Our remarks on the estimators for $\Theta_1$, and $\Theta_2$, concern the parameter space $0.5 \leq \Theta_1 \leq 25$, $0.5 \leq \Theta_2 \leq 25$. Considerable caution is needed in extrapolating into the region $0 < \Theta_1 < 0.5$, $0 < \Theta_2 < 0.5$. With regard to extreme values, parameter values exceeding 25 are not likely to be met in practical problems, and in any case an alternative distribution might be sought in this event.

Returning to cases in which a parameter is small we quote a few examples to illustrate the instability of the moments.

$$\Theta_1 = 0.8 \qquad \Theta_2 = 0.2 \qquad n = 1$$

$E\hat{\Theta}_1 \sim 0.8 + 94 + 28110$ $\qquad$ $E\hat{\Theta}_2 \sim 0.2 - .70 - 31.04$

$E\Theta_1^* \sim 0.8 + 117 + 32030$ $\qquad$ $E\Theta_2^* \sim 0.2 - 2.45 - .31$

$Var\hat{\Theta}_1 \sim 74 + 59340$ $\quad$ $Cov(\hat{\Theta}_1, \hat{\Theta}_2) \sim - 18 - 5530$ $\quad$ $Var\ \hat{\Theta}_2 \sim 4.8 + 31.8$

$Var\Theta_2^* \sim 87 + 71750$ $\quad$ $Cov(\Theta_1^*, \Theta_2^*) \sim - 21 - 6119$ $\quad$ $Var\ \Theta_2^* \sim 5.6 - 57.3$

$E_{ff} = .8527$ $\qquad$ $C_1 = 3891$ $\qquad$ $C_2 = 5211$

.....

$\Theta_1 = 10$ $\qquad$ $\Theta_2 = 0.1$ $\qquad$ $n = 1$

$E\hat{\Theta}_1 \sim 10 + 2632 + 1896627$ $\qquad$ $E\hat{\Theta}_2 \sim 0.1 - 1.0 - 24.6$

$E\Theta_1^* \sim 10 + 2740 + 1949254$ $\qquad$ $E\Theta_2^* \sim 0.1 - 1.2 + 0.1$

$Var\hat{\Theta}_1 \sim 25342 + 49979846$ $\quad$ $Cov(\hat{\Theta}_1, \hat{\Theta}_2) \sim - 253 - 186801$ $\quad$ $Var\hat{\Theta}_2$ $\quad$ $2.5+16.0$

$Var\Theta_1^* \sim 26210 + 52057780$ $\quad$ $Cov(\Theta_1^*, \Theta_2^*) \sim - 262 - 191638$ $\quad$ $Var\Theta_2^* \sim 2.6-6.6$

.....

$\Theta_1 = 0.1$ $\qquad$ $\Theta_2 = 20$ $\qquad$ $n = 1$

$E\hat{\Theta}_1 \sim 0.1 - 0.007 - 0.088$ $\qquad$ $E\hat{\Theta}_2 \sim 20 + 3.32 + 90.00$

$E\Theta_1^* \sim 0.1 + 0.381 - 6.621$ $\qquad$ $E\Theta_2^* \sim 20 - 31.00 - 90.00$

$Var\hat{\Theta}_1 \sim .10 + .03$ $\quad$ $Cov(\hat{\Theta}_1, \hat{\Theta}_2) \sim - .19 - 7.22$ $\quad$ $Var\hat{\Theta}_2 \sim 237 + 3205$

$Var\Theta_1^* \sim .12 - .15$ $\quad$ $Cov(\Theta_1^*, \Theta_2^*) \sim - 4.51 + 153.21$ $\quad$ $Var\Theta_2^* \sim 1102 - 10273$

.....

$\Theta_1 = 0.1$ $\qquad$ $\Theta_2 = 0.1$ $\qquad$ $n = 1000$

$E\hat{\Theta}_1 \sim 0.100 + 0.217 + 1.033$

$E\Theta_1^* \sim 0.100 + 0.235 + 1.049 + 7.851 + 78.034 + 1004.090$

$E\hat{\Theta}_2 \sim 0.10000 - 0.00308 - 0.00037$

$E\Theta_2^* \sim 0.10000 - 0.01110 - 0.00009 - 0.00000$

$\mathrm{Var}\ \theta_1^* \sim 0.0225 + 0.3010$ $\qquad$ $\mathrm{Var}\ \hat{\theta}_1 \sim 0.0215 + 0.2896$

$\mathrm{Cov}\ (\theta_1^*, \theta_2^*) \sim -0.0224 - 0.1023 - 0.7735$ $\qquad$ $\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2) \sim -0.0214 - 0.026$

$\mathrm{Var}\theta_2^* \sim 0.0234 - 0.0024$ $\qquad$ $\mathrm{Var}\ \hat{\theta}_2 \sim 0.0224 + 0.0011$

.....

$\theta_1 = 1.0$ $\qquad$ $\theta_2 = 0.1$ $\qquad$ $n = 1000$

$E\hat{\theta}_1 \sim 1.00 + 0.41 + 0.43$

$E\theta_1^* \sim 1.00 + 0.46 + 0.47 + 0.80 + 1.91 + 5.87$

$E\hat{\theta}_2 = 0.100 - 0.001 + 0.000$

$E\theta_2^* \sim 0.100 - 0.002 + 0.000$

$\mathrm{Var}\ \theta_1^* \sim 0.443 + 1.299$ $\qquad$ $\mathrm{Var}\ \hat{\theta}_1 \sim 0.408 + 1.176$

$\mathrm{Cov}(\theta_1^*, \theta_2^*) \sim -0.044 - 0.046 - 0.079$ $\qquad$ $\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2) \sim -0.041 - 0.043$

$\mathrm{Var}\ \theta_2^* \sim 0.0045 - 0.0001$ $\qquad$ $\mathrm{Var}\ \hat{\theta}_2 \sim 0.0042 + 0.0000$ .

Evidently the border for which $\theta_2$ is small is associated with high instability and a Poisson distribution should be considered as an alternative. The other border does not present the same difficulties.

The remarks we have made on estimation problems associated with the NTAd are intended as a step towards a better understanding of the underlying distributional properties. They are not final and there is certainly scope for further investigation. If the study focusses more attention on this problem of statistical asymptotics and leads to a more careful appraisal of large sample theory in practical situations, then perhaps a slight advance will have been made.

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# APPENDIX

## A1. Formulae Involving Differences of Zero.

A1.1 Cumulants, Factorial Moments, Probability Function. The cumulants of the NTAd satisfy

$$K_{r+1} = \theta_2 \{K_r + \binom{r}{1} K_{r-1} + \binom{r}{2} K_{r-2} + \cdots + \binom{r}{r-1} K_1 + \binom{r}{r}\theta_1\} \tag{10a}$$

and

$$K_{r+1} = \theta_2 \{K_r + \frac{\partial K_r}{\partial \theta_2}\} \tag{10b}$$

(Shenton, 1949)

In fact,

$$\begin{aligned}
K_1 &= \lambda' \\
K_2 &= \lambda' ( 1+\theta_2 ) \\
K_3 &= \lambda' ( 1+3\theta_2 + \theta_2^2 ) \\
K_4 &= \lambda' ( 1+7\theta_2 + 6\theta_2^2 + \theta_2^3 ) \\
K_5 &= \lambda' ( 1+15\theta_2 + 25\theta_2^2 + 10\theta_2^3 + \theta_2^4 ) \\
K_6 &= \lambda' ( 1+31\theta_2 + 90\theta_2^2 + 65\theta_2^3 + 15\theta_2^4 + \theta_2^5 ) \\
K_7 &= \lambda' ( 1+63\theta_2 + 301\theta_2^2 + 350\theta_2^3 + 140\theta_2^4 + 21\theta_2^5 + \theta_2^6 ) \\
K_8 &= \lambda' ( 1+127\theta_2 + 966\theta_2^2 + 1701\theta_2^3 + 1050\theta_2^4 + 266\theta_2^5 + 28\theta_2^6 \\
&\quad + \theta_2^7 ) \\
K_9 &= \lambda' ( 1+255\theta_2 + 3025\theta_2^2 + 7770\theta_2^3 + 6951\theta_2^4 + 2646\theta_2^5 + \\
&\quad 462\theta_2^6 + 36\theta_2^7 + \theta_2^8 ) \\
K_{10} &= \lambda' ( 1+511\theta_2 + 9330\theta_2^2 + 34105\theta_2^3 + 42525\theta_2^4 + 22827\theta_2^5 + \\
&\quad 5880\theta_2^6 + 750\theta_2^7 + 45\theta_2^8 + \theta_2^9 ). \qquad (\lambda' = \theta_1\theta_2)
\end{aligned} \tag{11}$$

Thus in terms of advancing differences

$$K_r = \lambda' e^{\theta_2 \Delta} 1^{r-1} \qquad (r=1,2,\ldots). \tag{12}$$

There are exactly similar formulae for the factorial moments and the probability function itself. Indeed

Factorial Moments

$$\mu_{(r)} = \theta_1 \theta_2^{\,r} e^{\theta_1 \Delta} 1^{r-1} \qquad (r=1,2,\ldots). \tag{13}$$

Probability Function

$$p_r = \frac{e^{-\theta_2}\theta_2^{\,r}}{r!} \quad \theta_1 p_o e^{\lambda\Delta} 1^{r-1} \quad \left( \begin{array}{c} r=1,2,\ldots; \\ \lambda=\theta_1 e^{-\theta_2} \end{array} \right) . \tag{14}$$

We thought these representations were not too well known but Barton (1957) gives a result equivalent to (14). The coefficients in the polynomials in (12)-(14) are related to Stirling's numbers of the second kind, and in this connection reference may be made to David and Barton (1962); a short tabulation in the statistical literature may be found in Thomson (1952).

## A2 COMPARISON OF TAIL LENGTHS

Anscombe (1950) has arranged several discrete compound distributions in order of skewness and tail length. Confining attention to two parameter cases parametrized so that the mean is kp and the variance kp(p+1), we extract the following from his paper:

| Distribution | $k_{(3)}/(kp^3)$ | $k_{(4)}/(kp^4)$ |
|---|---|---|
| Thomas | $3/4+1/8p + O(p)$ | $1/2+o(p)$ |
| Neyman Type A | 1 | 1 |
| Negative Binomial | 2 | 6 |

( $k_{(s)}$ is a factorial cumulant: the Thomas distribution is given in Thomas (1949) )

Another tail-length comparison is displayed in Table A1 where the entry refers to the largest value of x such that $P_x$ is less than about $10^{-20}$. The three entries relate to the Poisson, NTA, and Negative Binomial distributions, and this turns out to be the precise ordering of the distributions with respect to tail-length over the region of tabulation.

The fact that the negative binomial distribution has a long tail is useful to remember in connection with estimators (involving moments) for its parameters.

TABLE A1

A COMPARISON OF THE NUMBER OF TERMS TO ATTAIN GIVEN ACCURACY FOR THREE DISTRIBUTIONS

($P_x \leq 10^{-20}$)

| | | $\Theta_1$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Theta_2$ = 1 | (a) | 21 | 27 | 31 | 35 | 38 | 41 | 44 | 47 | 50 | 52 | 64 | 75 | 85 |
| | (b) | 46 | 54 | 60 | 64 | 69 | 73 | 77 | 80 | 84 | 87 | 102 | 115 | 127 |
| | (c) | 66 | 71 | 75 | 79 | 83 | 86 | 90 | 93 | 96 | 99 | 112 | 124 | 136 |
| 2 | (a) | 27 | 35 | 41 | 47 | 52 | 57 | 62 | 66 | 70 | 75 | 94 | 112 | 129 |
| | (b) | 70 | 83 | 93 | 102 | 110 | 117 | 124 | 130 | 136 | 142 | 169 | 194 | 216 |
| | (c) | 111 | 120 | 128 | 135 | 142 | 148 | 154 | 159 | 165 | 170 | 194 | 217 | 238 |
| 3 | (a) | 31 | 41 | 50 | 57 | 64 | 70 | 77 | 83 | 88 | 94 | 121 | 145 | 169 |
| | (b) | 92 | 111 | 125 | 138 | 149 | 159 | 169 | 178 | 187 | 196 | 235 | 270 | 302 |
| | (c) | 156 | 169 | 180 | 190 | 199 | 208 | 216 | 225 | 232 | 240 | 275 | 308 | 338 |
| 4 | (a) | 35 | 47 | 57 | 66 | 75 | 83 | 90 | 98 | 105 | 112 | 145 | 176 | 206 |
| | (b) | 114 | 138 | 157 | 173 | 187 | 201 | 214 | 226 | 237 | 248 | 299 | 345 | 387 |
| | (c) | 200 | 217 | 231 | 244 | 256 | 268 | 279 | 289 | 299 | 309 | 356 | 398 | 438 |
| 5 | (a) | 38 | 52 | 64 | 75 | 85 | 94 | 103 | 112 | 121 | 129 | 169 | 206 | 242 |
| | (b) | 135 | 164 | 187 | 207 | 225 | 242 | 258 | 272 | 287 | 350 | 363 | 419 | 471 |
| | (c) | 243 | 264 | 282 | 298 | 313 | 327 | 340 | 353 | 366 | 378 | 435 | 488 | 537 |
| 6 | (a) | 41 | 57 | 70 | 83 | 94 | 105 | 115 | 126 | 135 | 145 | 191 | 235 | 277 |
| | (b) | 156 | 191 | 218 | 242 | 263 | 283 | 301 | 319 | 336 | 352 | 426 | 493 | 556 |
| | (c) | 287 | 311 | 332 | 351 | 369 | 386 | 402 | 417 | 432 | 447 | 515 | 577 | 636 |
| 7 | (a) | 44 | 62 | 77 | 90 | 103 | 115 | 127 | 139 | 150 | 161 | 213 | 263 | 311 |
| | (b) | 177 | 217 | 249 | 276 | 301 | 324 | 345 | 365 | 385 | 404 | 490 | 507 | 640 |
| | (c) | 330 | 358 | 383 | 405 | 425 | 445 | 463 | 481 | 498 | 515 | 594 | 666 | 735 |
| 8 | (a) | 47 | 66 | 83 | 98 | 112 | 126 | 139 | 152 | 164 | 176 | 235 | 290 | 344 |
| | (b) | 198 | 243 | 279 | 310 | 338 | 364 | 388 | 412 | 434 | 455 | 553 | 641 | 723 |
| | (c) | 373 | 405 | 433 | 458 | 481 | 503 | 524 | 545 | 564 | 584 | 673 | 755 | 833 |
| 9 | (a) | 50 | 70 | 88 | 105 | 121 | 135 | 150 | 164 | 178 | 191 | 256 | 317 | 377 |
| | (b) | 219 | 270 | 310 | 344 | 376 | 405 | 432 | 458 | 482 | 506 | 616 | 715 | 807 |
| | (c) | 416 | 452 | 483 | 511 | 537 | 562 | 585 | 608 | 630 | 652 | 752 | 844 | 932 |
| 10 | (a) | 52 | 75 | 94 | 112 | 129 | 145 | 161 | 176 | 191 | 206 | 277 | 344 | 409 |
| | (b) | 239 | 296 | 340 | 378 | 413 | 445 | 475 | 504 | 531 | 557 | 679 | 788 | 890 |
| | (c) | 459 | 499 | 533 | 564 | 593 | 620 | 646 | 671 | 696 | 720 | 831 | 935 | 1030 |
| 15 | (a) | 64 | 94 | 121 | 145 | 169 | 191 | 213 | 235 | 256 | 277 | 377 | 473 | 566 |
| | (b) | 322 | 425 | 491 | 547 | 598 | 646 | 690 | 733 | 773 | 812 | 992 | 1154 | 1306 |
| | (c) | 671 | 730 | 781 | 827 | 870 | 910 | 949 | 986 | 1023 | 1058 | 1223 | 1375 | 1518 |
| 20 | (a) | 75 | 112 | 145 | 176 | 206 | 235 | 263 | 290 | 317 | 344 | 473 | 597 | 719 |
| | (b) | 444 | 554 | 640 | 715 | 783 | 845 | 904 | 960 | 1014 | 1066 | 1304 | 1446 | 1720 |
| | (c) | 882 | 960 | 1027 | 1088 | 1145 | 1199 | 1250 | 1299 | 1347 | 1394 | 1613 | 1814 | 2004 |
| 25 | (a) | 85 | 129 | 169 | 206 | 242 | 277 | 311 | 344 | 377 | 409 | 566 | 719 | 867 |
| | (b) | 546 | 682 | 789 | 882 | 966 | 1044 | 1118 | 1187 | 1254 | 1319 | 1615 | 1883 | 2133 |
| | (c) | 1092 | 1189 | 1272 | 1348 | 1419 | 1485 | 1549 | 1611 | 1671 | 1729 | 2001 | 2252 | 2489 |

(a) = Poisson (b) = Neyman Type A (c) = Negative binomial

## A3 MODALITY

It was suggested by Anscombe (1950) that the NTAd could have an indefinitely large number of modes, and that in the region of the parameter space of high modality the modes would occur at variate values spaced at multiples of $\theta_2$. The multimodality suggestion of Anscombe has been studied and demonstrated by Barton (1957), who gives a diagram of the double points of the bounding arcs with specified modality. We give here (Table A2) an analysis of the location of the modes for integer values of the parameters in the region $0<\theta_1,\theta_2<25$. The spacing of the modes in multiples of $\theta_2$ is amply illustrated. It is also curious to notice that this regular spacing breaks down if $\theta_2$ exceeds $\theta_1$ by a certain critical value. The reader will find it instructive to follow the table through for a given $\theta_2$, and $\theta_1$ increasing from unity. In general the number of modes goes through a maximum value and decreases to unity as $\theta_1$ increases. The regular spacing is disrupted by the loss of modes near the largest one (by smallest and largest mode we of course mean the respective abscissa of the modes). Some illustrations of the modality are given in Figure 7.

It appears to be an extremely difficult problem to analyse the frequency and location of modes, for some of the larger ones are scarcely distinguishable from stationary points. Thus although there are interesting asymptotic formulae available, for example the one due to Douglas (1965), namely,

$$p_x \sim \frac{e^{-\Theta_1}}{\sqrt{(2\pi)}} \; \frac{\Theta_2^x \exp\,(x/z_x)}{z_x^x \sqrt{\{x(1+z_x)\}}} \tag{15}$$

where $z_x \exp z_x = x/(\Theta_1 \exp(-\Theta_2))$, they are not sufficiently sensitive to use in difference analysis. In this connection another formula we have tried is one due to Dr. John Philpot who kindly permits us to quote

$$p_x \sim \frac{p_o \Theta_2^x}{x!} \; \mu_x^* ,$$

where

$$\mu_x^* = (\exp(f(x_o) - m))/\sqrt{\lambda(x_o)},$$

$$m = \Theta_1 \exp(-\Theta_2),$$

$$f(x_o) = x_o + (x - 1/2) \quad \{\log m + \frac{x - 1/2}{x_o} - 1\} ,$$

$$\lambda(x_o) = \frac{x - 1/2}{x_o^2} + \frac{1}{x_o} ,$$

$x_o$ is a root of

$$\frac{x - 1/2}{x_o} = \log \frac{x_o}{m} .$$

TABLE A2

MODALITY OF NEYMAN TYPE A DISTRIBUTION. NUMBER OF MODES AND THEIR LOCATION

| $\theta_2$ \ $\theta_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 2 | 0,2 | 0,2 | 0,4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 |
| 3 | 0,3 | 0,4 | 0,7 | 0,10 | 0,13 | 0,16 | 19 | 22 | 25 | 28 | 31 | 34 | 37 | 40 | 43 | 46 | 49 | 52 | 55 | 58 | 61 | 64 | 67 | 70 | 73 |
| 4 | 0,4 | 0,5 | 0,9 | 0,13 | 0,17 | 0,21 | 0,25 | 0,29 | 0,33 | 0,37 | 0,41 | 0,45 | 0,49 | 53 | 57 | 61 | 65 | 69 | 73 | 77 | 81 | 85 | 89 | 93 | 97 |
| 5 | 0,5 | 0,5 | 0,12 | 0,17 | 0,21 | 0,27 | 0,32 | 0,37 | 0,42 | 0,47 | 0,52 | 0,57 | 0,62 | 0,67 | 0,72 | 0,77 | 0,82 | 0,87 | 0,92 | 0,97 | 0,102 | 0,107 | 0,112 | 0,117 | 0,122 |
| 6 | 0,6 | 0,6 | 0,14 | 0,20 | 0,26 | 0,32 | 0,38 | 0,44 | 0,50 | 0,56 | 0,62 | 0,68 | 0,74 | 0,80 | 0,86 | 0,92 | 0,98 | 0,104 | 0,110 | 0,116 | 0,122 | 0,128 | 0,134 | 0,140 | 0,146 |
| 7 | 0,7 | 0,7<br>13 | 0,8<br>16 | 0,23 | 0,30 | 0,37 | 0,44 | 0,51 | 0,59 | 0,66 | 0,73 | 0,80 | 0,87 | 0,94 | 0,101 | 0,108 | 0,115 | 0,122 | 0,129 | 0,136 | 0,143 | 0,150 | 0,157 | 0,164 | 0,171 |
| 8 | 0,8 | 0,8,<br>16 | 0,8,<br>17 | 0,27 | 0,35 | 0,43 | 0,51 | 0,59 | 0,67 | 0,75 | 0,83 | 0,91 | 0,99 | 0,107 | 0,115 | 0,123 | 0,131 | 0,139 | 0,147 | 0,155 | 0,163 | 0,171 | 0,179 | 0,187 | 0,195 |
| 9 | 0,9 | 0,9,<br>18 | 0,9,<br>19 | 0,10,<br>30 | 0,39 | 0,48 | 0,57 | 0,66 | 0,75 | 0,84 | 0,93 | 0,102 | 0,112 | 0,121 | 0,130 | 0,139 | 0,148 | 0,157 | 0,166 | 0,175 | 0,184 | 0,193 | 0,202 | 0,211 | 0,220 |
| 10 | 0,10 | 0,10,<br>20 | 0,10,<br>21 | 0,10,<br>33 | 0,11,<br>44 | 0,54 | 0,64 | 0,74 | 0,84 | 0,94 | 0,104 | 0,114 | 0,124 | 0,134 | 0,144 | 0,154 | 0,164 | 0,174 | 0,184 | 0,194 | 0,204 | 0,214 | 0,224 | 0,234 | 0,244 |
| 11 | 0,11,<br>21 | 0,11,<br>22 | 0,11,<br>23,32 | 0,11,<br>35 | 0,11,<br>48 | 0,12,<br>59 | 0,70 | 0,81 | 0,92 | 0,103 | 0,114 | 0,125 | 0,136 | 0,147 | 0,158 | 0,169 | 0,180 | 0,192 | 0,203 | 0,214 | 0,225 | 0,236 | 0,247 | 0,258 | 0,269 |
| 12 | 0,12,<br>23 | 0,12,<br>24 | 0,12,<br>25,35 | 0,12,<br>26,38 | 0,12,<br>52 | 0,12,<br>65 | 0,12,<br>77 | 0,89 | 0,101 | 0,113 | 0,125 | 0,137 | 0,149 | 0,161 | 0,173 | 0,185 | 0,197 | 0,209 | 0,221 | 0,233 | 0,245 | 0,257 | 0,269 | 0,281 | 0,293 |
| 13 | 0,13,<br>25 | 0,13,<br>26 | 0,13,<br>26,39 | 0,13,<br>27,41 | 0,13,<br>56 | 0,13,<br>70 | 0,13,<br>83 | 0,13,<br>96 | 0,14,<br>109 | 0,122 | 0,135 | 0,148 | 0,161 | 0,174 | 0,187 | 0,200 | 0,213 | 0,226 | 0,239 | 0,252 | 0,265 | 0,278 | 0,292 | 0,305 | 0,318 |
| 14 | 0,14,<br>27 | 0,14,<br>28 | 0,14,<br>28,42 | 0,14,<br>29,44,<br>54 | 0,14,<br>30,59 | 0,14,<br>75 | 0,14,<br>90 | 0,14,<br>104 | 0,14,<br>118 | 0,14,<br>118 | 0,14,<br>146 | 0,15,<br>160 | 0,174 | 0,188 | 0,202 | 0,216 | 0,230 | 0,244 | 0,258 | 0,272 | 0,286 | 0,300 | 0,314 | 0,328 | 0,342 |
| 15 | 0,15,<br>30 | 0,15,<br>30 | 0,15,<br>30,45 | 0,15,<br>31,46,<br>59 | 0,15,<br>31,63 | 0,15,<br>80 | 0,15,<br>96 | 0,15,<br>111 | 0,15,<br>126 | 0,15,<br>141 | 0,15,<br>156 | 0,15,<br>171 | 0,15,<br>186 | 0,15,<br>201 | 0,16,<br>216 | 0,231 | 0,246 | 0,261 | 0,276 | 0,291 | 0,306 | 0,321 | 0,336 | 0,351 | 0,366 |
| 16 | 0,16,<br>32 | 0,16,<br>32,46 | 0,16,<br>32,48 | 0,16,<br>32,49,<br>63 | 0,16,<br>33,51,<br>67 | 0,16,<br>34,84 | 0,16,<br>102 | 0,16,<br>119 | 0,16,<br>135 | 0,16,<br>151 | 0,16,<br>167 | 0,16,<br>183 | 0,16,<br>199 | 0,16,<br>215 | 0,16,<br>231 | 0,16,<br>247 | 0,16,<br>263 | 0,16,<br>279 | 0,17,<br>295 | 0,311 | 0,327 | 0,343 | 0,359 | 0,375 | 0,391 |
| 17 | 0,17,<br>34 | 0,17,<br>34,50 | 0,17,<br>34,51 | 0,17,<br>34,52,<br>68 | 0,17,<br>35,53,<br>70,82 | 0,17,<br>35,89 | 0,17,<br>36,108 | 0,17,<br>126 | 0,17,<br>143 | 0,17,<br>160 | 0,17,<br>177 | 0,17,<br>194 | 0,17,<br>211 | 0,17,<br>228 | 0,17,<br>245 | 0,17,<br>262 | 0,17,<br>279 | 0,17,<br>296 | 0,17,<br>313 | 0,17,<br>330 | 0,17,<br>347 | 0,17,<br>364 | 0,17,<br>381 | 0,18,<br>398 | 0,415 |
| 18 | 0,18,<br>36 | 0,18,<br>36,53 | 0,18,<br>36,54 | 0,18,<br>36,55,<br>72 | 0,18,<br>36,56,<br>74,88 | 0,18,<br>37,94 | 0,18,<br>37,113 | 0,18,<br>133 | 0,18,<br>151 | 0,18,<br>170 | 0,18,<br>188 | 0,18,<br>206 | 0,18,<br>224 | 0,18,<br>242 | 0,18,<br>260 | 0,18,<br>278 | 0,18,<br>296 | 0,18,<br>314 | 0,18,<br>332 | 0,18,<br>350 | 0,18,<br>368 | 0,18,<br>386 | 0,18,<br>404 | 0,18,<br>422 | 0,18,<br>440 |
| 19 | 0,19,<br>38 | 0,19,<br>38,56 | 0,19,<br>38,57 | 0,19,<br>38,58,<br>76 | 0,19,<br>38,58,<br>78,94 | 0,19,<br>38,60,<br>98 | 0,19,<br>39,119 | 0,19,<br>39,140 | 0,19,<br>40,160 | 0,19,<br>179 | 0,19,<br>198 | 0,19,<br>217 | 0,19,<br>236 | 0,19,<br>255 | 0,19,<br>274 | 0,19,<br>293 | 0,19,<br>312 | 0,19,<br>331 | 0,19,<br>350 | 0,19,<br>369 | 0,19,<br>388 | 0,19,<br>407 | 0,19,<br>426 | 0,19,<br>445 | 0,19,<br>464 |
| 20 | 0,20,<br>40 | 0,20,<br>40,59 | 0,20,<br>40,60,<br>77 | 0,20,<br>40,61,<br>80 | 0,20,<br>40,61,<br>82,99 | 0,20,<br>40,62,<br>84,103,<br>116 | 0,20,<br>41,125 | 0,20,<br>41,147 | 0,20,<br>41,168 | 0,20,<br>42,188 | 0,20,<br>209 | 0,20,<br>229 | 0,20,<br>249 | 0,20,<br>296 | 0,20,<br>289 | 0,20,<br>309 | 0,20,<br>329 | 0,20,<br>349 | 0,20,<br>369 | 0,20,<br>390 | 0,20,<br>409 | 0,20,<br>429 | 0,20,<br>449 | 0,20,<br>469 | 0,20,<br>489 |
| 21 | 0,21,<br>42,61 | 0,21,<br>42,62 | 0,21,<br>42,63,<br>82 | 0,21,<br>42,63,<br>84 | 0,21,<br>42,54,<br>85,104 | 0,21,<br>42,65,<br>87,108,<br>124 | 0,21,<br>42,66,<br>130 | 0,21,<br>43,153 | 0,21,<br>43,176 | 0,21,<br>43,198 | 0,21,<br>43,219 | 0,21,<br>240 | 0,21,<br>261 | 0,21,<br>282 | 0,21,<br>303 | 0,21,<br>324 | 0,21,<br>345 | 0,21,<br>366 | 0,21,<br>387 | 0,21,<br>408 | 0,21,<br>429 | 0,21,<br>450 | 0,21,<br>471 | 0,21,<br>492 | 0,21,<br>513 |
| 22 | 0,22,<br>44,64 | 0,22,<br>44,65 | 0,22,<br>44,66,<br>87 | 0,22,<br>44,66,<br>88 | 0,22,<br>44,67,<br>89,110 | 0,22,<br>44,67,<br>91,112,<br>130 | 0,22,<br>44,68<br>136 | 0,22,<br>44,160 | 0,22,<br>44,184 | 0,22,<br>45,207 | 0,22,<br>45,229 | 0,22,<br>45,252 | 0,22,<br>46,274 | 0,22,<br>296 | 0,22,<br>318 | 0,22,<br>340 | 0,22,<br>362 | 0,22,<br>384 | 0,22,<br>406 | 0,22,<br>428 | 0,22,<br>450 | 0,22,<br>472 | 0,22,<br>494 | 0,22,<br>516 | 0,22,<br>538 |
| 23 | 0,23,<br>46,68 | 0,23,<br>46,69 | 0,23,<br>46,69,<br>91 | 0,23,<br>46,69,<br>92 | 0,23,<br>46,70,<br>93,115 | 0,23,<br>46,70,<br>94,117,<br>137 | 0,23,<br>46,71,<br>141,157 | 0,23,<br>46,72<br>166 | 0,23,<br>46,191 | 0,23,<br>46,216 | 0,23,<br>47,240 | 0,23,<br>47,263 | 0,23,<br>47,286 | 0,23,<br>47,309 | 0,23,<br>48,332 | 0,23,<br>355 | 0,23,<br>378 | 0,23,<br>401 | 0,23,<br>424 | 0,23,<br>447 | 0,23,<br>470 | 0,23,<br>493 | 0,23,<br>516 | 0,23,<br>539 | 0,23,<br>562 |
| 24 | 0,24,<br>48,71 | 0,24,<br>48,72 | 0,24,<br>48,72,<br>95 | 0,24,<br>48,72,<br>96,117 | 0,24,<br>48,72,<br>97,120 | 0,24,<br>48,73,<br>98,122,<br>143 | 0,24,<br>48,73,<br>99,125,<br>147,165 | 0,24,<br>48,74,<br>173 | 0,24,<br>48,75,<br>199 | 0,24,<br>48,225 | 0,24,<br>48,250 | 0,24,<br>48,274 | 0,24,<br>49,299 | 0,24,<br>49,323 | 0,24,<br>49,347 | 0,24,<br>49,371 | 0,24,<br>50,395 | 0,24,<br>419 | 0,24,<br>443 | 0,24,<br>467 | 0,24,<br>491 | 0,24,<br>515 | 0,24,<br>539 | 0,24,<br>563 | 0,24,<br>587 |
| 25 | 0,25,<br>50,74 | 0,25,<br>50,75,<br>97 | 0,25,<br>50,75,<br>99 | 0,25,<br>50,75,<br>100,123 | 0,25,<br>50,75,<br>101,125 | 0,25,<br>50,76,<br>102,127,<br>149 | 0,25,<br>50,76,<br>103,129,<br>153,173 | 0,25,<br>50,76,<br>179 | 0,25,<br>50,77,<br>206 | 0,25,<br>50,233 | 0,25,<br>50,260 | 0,25,<br>50,286 | 0,25,<br>50,311 | 0,25,<br>50,336 | 0,25,<br>51,361 | 0,25,<br>51,386 | 0,25,<br>51,411 | 0,25,<br>51,436 | 0,25,<br>51,461 | 0,25,<br>52,486 | 0,25,<br>511 | 0,25,<br>536 | 0,25,<br>561 | 0,25,<br>586 | 0,25,<br>611 |

## A4 APPROXIMATIONS TO THE NTAd

Various attempts have been made to approximate to the NTA probability function; for example Martin and Katti (1962) have considered a Poisson 'with zeros' distribution for one region of the parameter space, and the normal distribution in another region of the parameter space. It may not be so well known that it can be approximated by a finite number of Poisson terms. We attempt a linear approximation in terms of Poisson kernels

$$p_x \sim \sum_{y=1}^{s} A_y e^{-m_y} m_y^{\,x}/x! \qquad (x=0,1,\ldots) \tag{16}$$

where $m_y$, $A_y>0$. Fitting by factorial moments we must solve the equations

$$\sum_{y=1}^{s} A_y m_y^{\,r} = \theta_1 \theta_2^{\,r} e^{\theta_1 \Delta_1} r^{-1} \qquad (r=0,1,\ldots.2s-1) \tag{17}$$

This is a point mass moment problem with a close analogy to quadrature formulae of the Gaussian type. It is not difficult to show that the m's are such that $m_1/\theta_2$, $m_2/\theta_2,\ldots,m_r/\theta_2$ are roots of

$$x^{(r)} - \binom{r}{1}\theta_1 x^{(r-1)} + \binom{r}{2}\theta_1^{\,2} x^{(r-2)} + \ldots = 0$$

or $$e^{-\theta_1 \Delta} x^{(r)} = 0 \qquad (x^{(r)} = x(x-1)\ldots(x-r+1))$$

and that

$$\Lambda_r = \frac{(s-1)!\,\Theta_1^{\,s-1}}{T_{s-1}(\lambda_r)T_s'(\lambda_r)} \qquad (\lambda_r = m_r/\Theta_2)$$

where

$$T_s(x) = e^{-\Theta_1 \Delta} x^{(s)},$$

$$T_s'(x) = \frac{dTs}{dn}.$$

In fact the T's are orthogonal polynomials with respect to a Poisson kernel, so the zeros are real and distinct and non-negative. Moreover the Λ's correspond to Christofell numbers of quadrature formulae, so that they are necessarily non-negative. For example,

s=2

$$\begin{cases} \dfrac{m_1}{\Theta_2} = \dfrac{1+2\Theta_1-\sqrt{(1+4\Theta_1)}}{2} \\[2ex] \dfrac{m_2}{\Theta_2} = \dfrac{1+2\Theta_1+\sqrt{(1+4\Theta_1)}}{2} \end{cases};$$

$$\begin{cases} \Lambda_1 = 1/2 - \dfrac{1}{2\sqrt{(1+4\Theta_1)}}, \\[2ex] \Lambda_2 = 1/2 + \dfrac{1}{2\sqrt{(1+4\Theta_1)}}. \end{cases}$$

One can see from these linear approximations to the NTA distribution that if $\theta_2$ is large in comparison to $\theta_1$ then the roots $m_1, m_2, \ldots,$ are approximately $\theta_2, 2\theta_2, \ldots, (r-1)\theta_2$.

## A5 ORTHOGONAL PARAMETERS

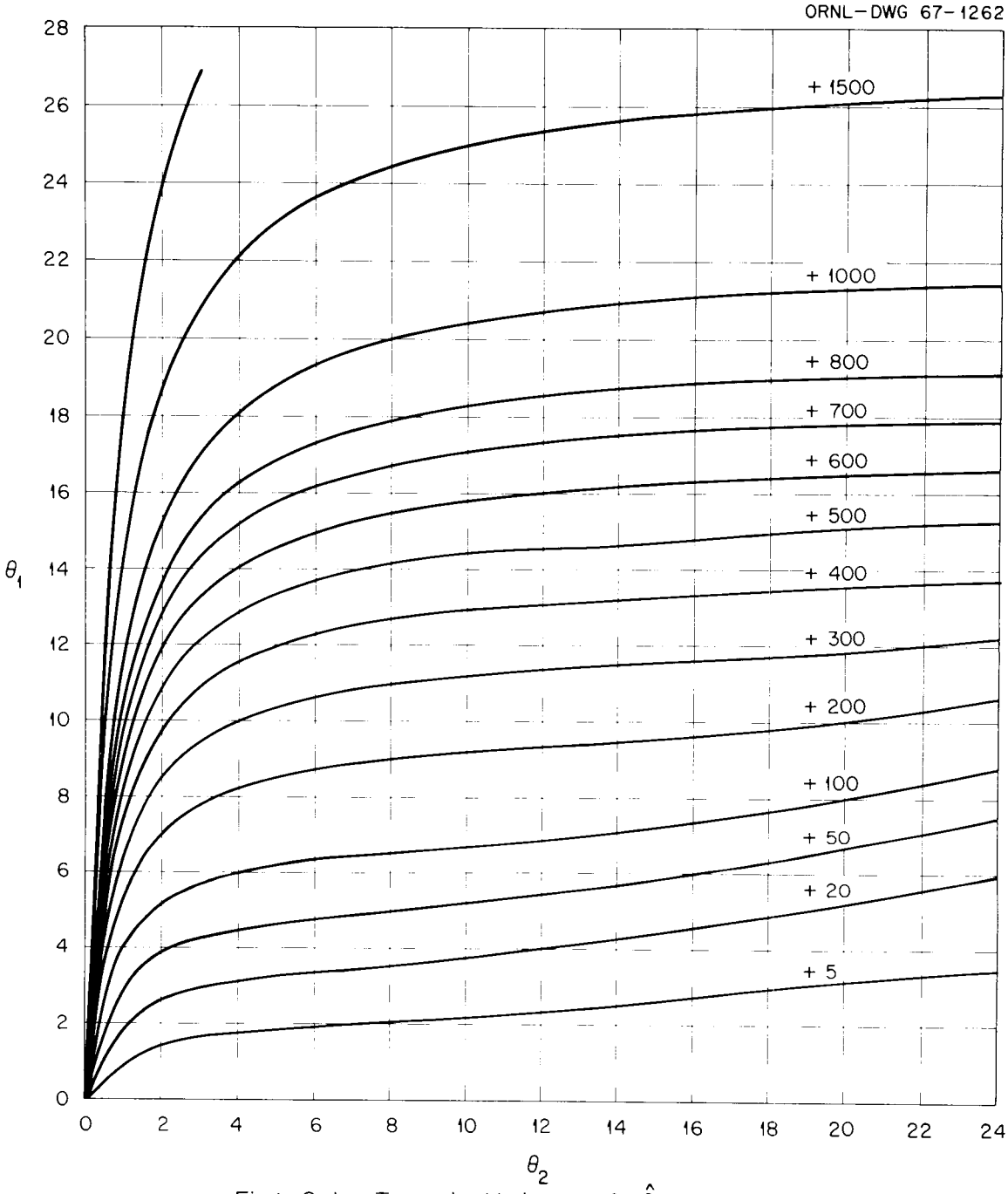
These were suggested by Jeffreys as a procedure for simplifying the maximum likelihood equations. For example, instead of estimating $(\theta_1, \theta_2)$ in a two-parameter distribution we estimate $f_1\ (\theta_1, \theta_2,)$ and $f_2\ (\theta_1, \theta_2)$, where

$$E\left\{ \frac{\partial^2 p_x(\theta_1, \theta_2)}{\partial f_1 \partial f_2} \right\} = 0 \quad .$$

With this parametric form the estimators are asymptotically uncorrelated. For example, the mean and index are orthogonal parameters for the negative binomial distribution. For the NTAd, Philpot (1964) has shown that the parameters

$$\beta_1 = \theta_1\theta_2 \quad , \qquad \beta_2 = \theta_1 e^{-\theta_2}$$

are orthogonal; however the slight gain in asymptotic independence (as measured by correlation) is largely offset by the loss in simplicity.




First Order Term in Variance of $\hat{\theta}_1$ Max Likelihood.

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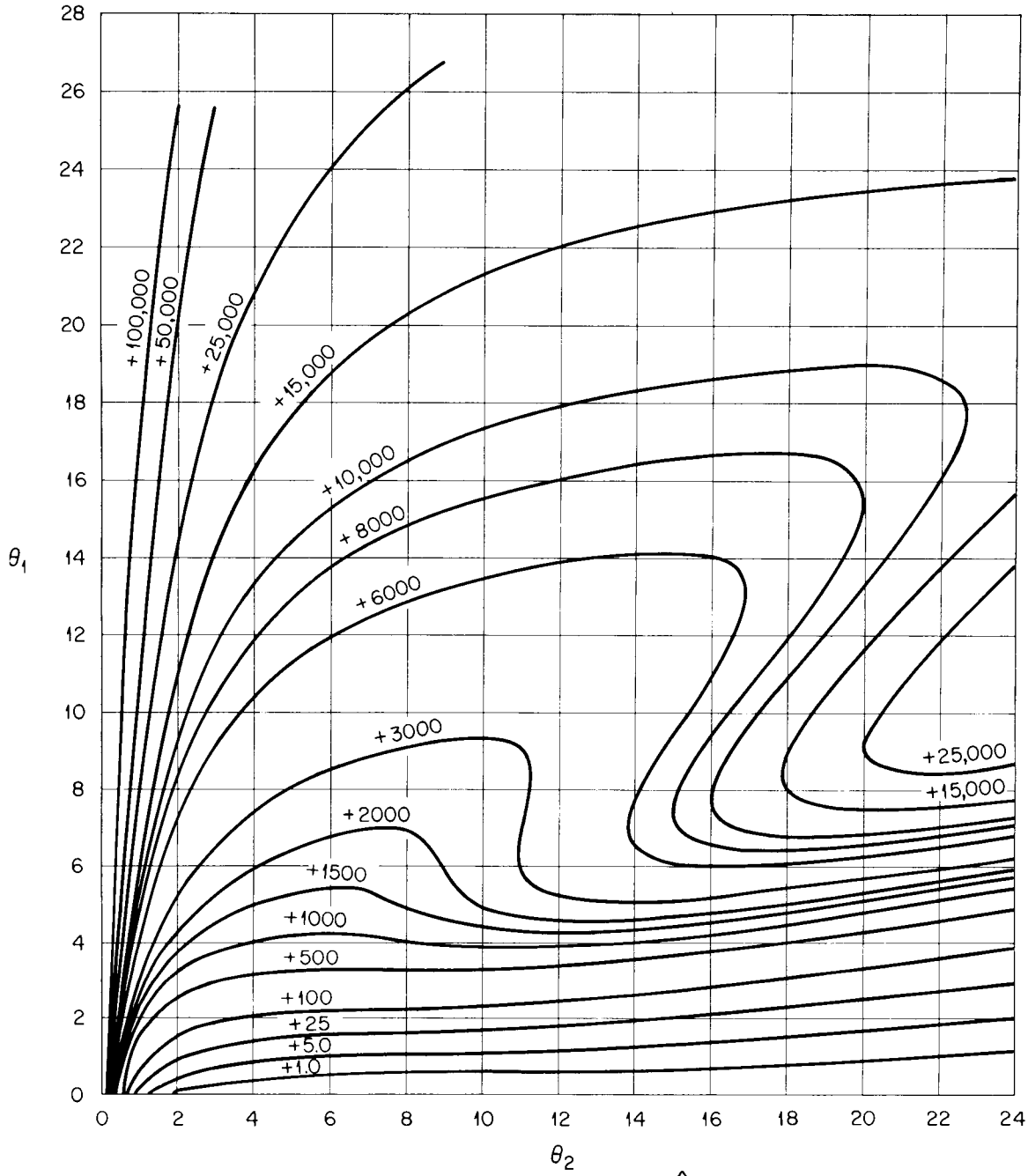
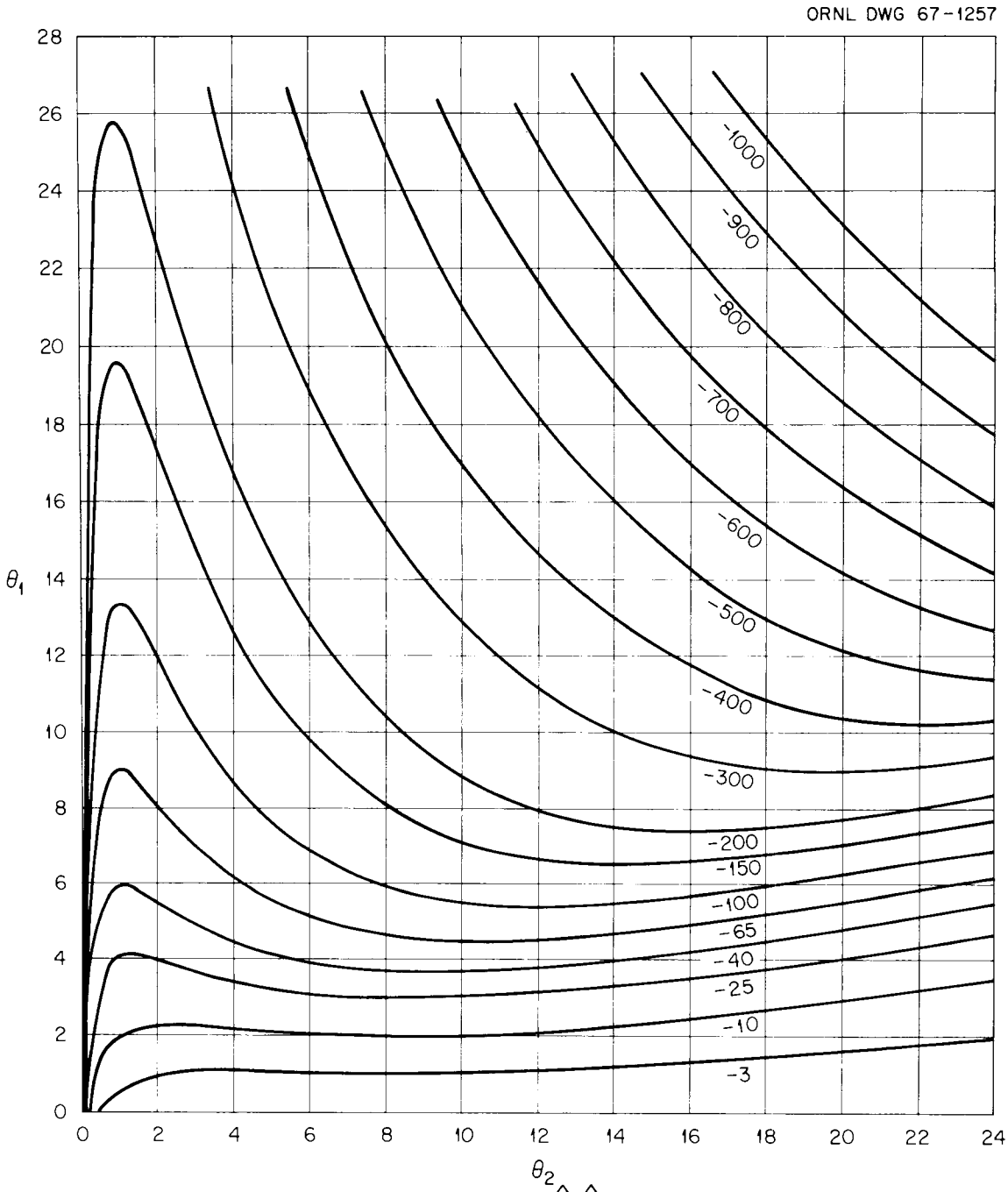
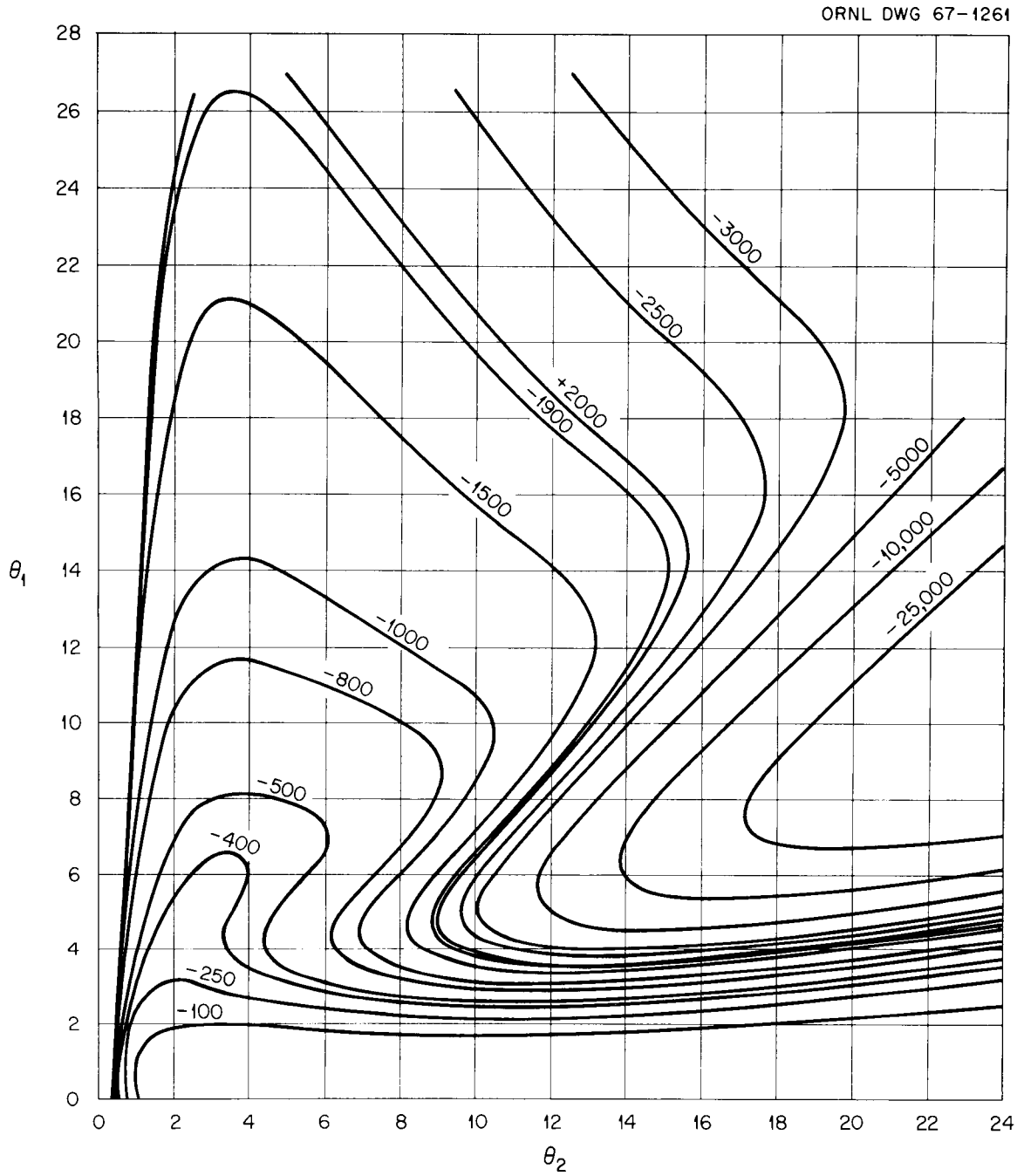



Second-Order Term in Variance of $\hat{\theta}_1$, Max. Likelihood




First Order Term in Cov$(\hat{\theta}_1, \hat{\theta}_2)$ Max. Likelihood




Second Order Term in Cov $(\hat{\theta}_1, \hat{\theta}_2)$ Max. Likelihood

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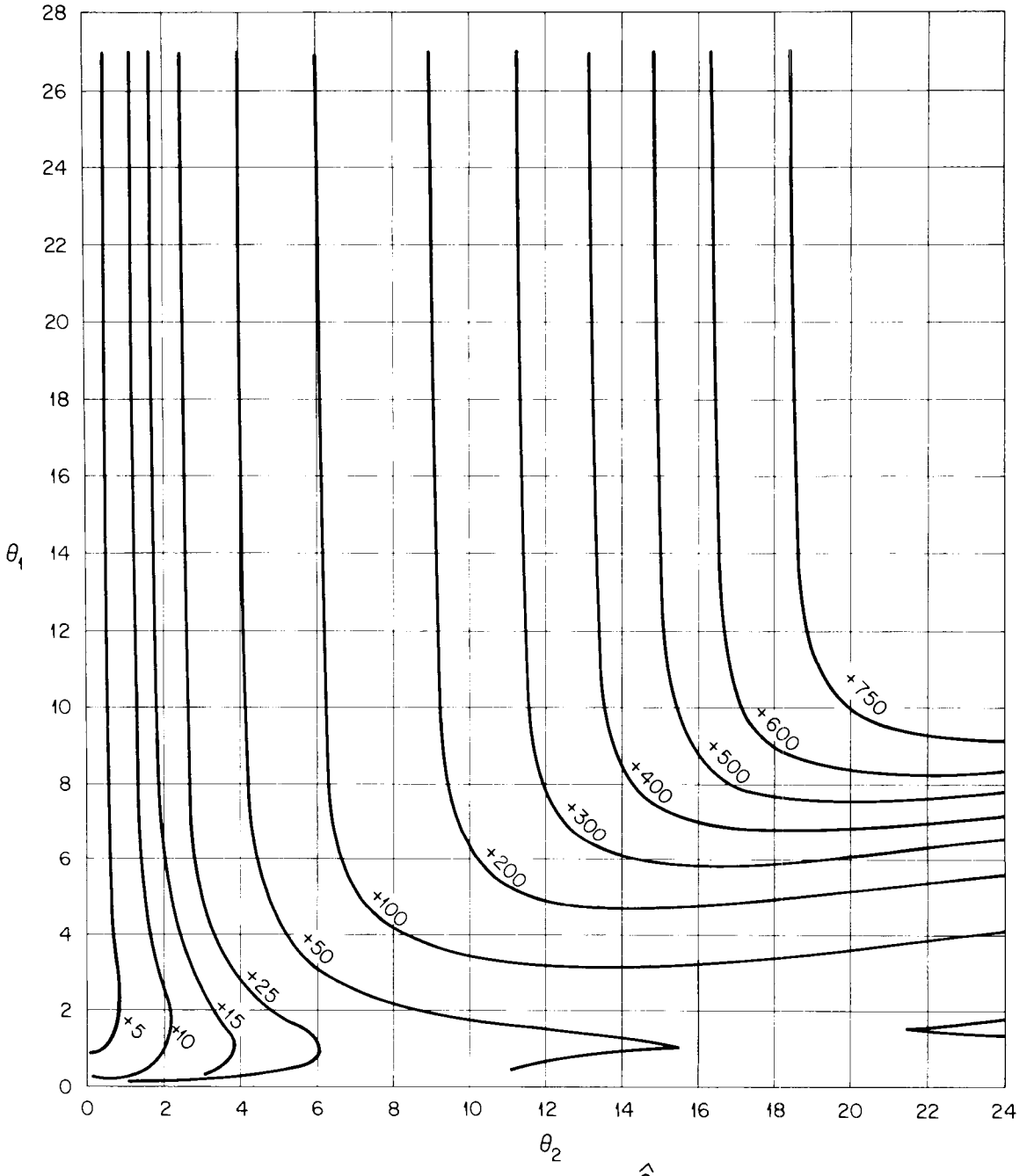


First Order Term in Variance of $\hat{\theta}_2$ Max. Likelihood

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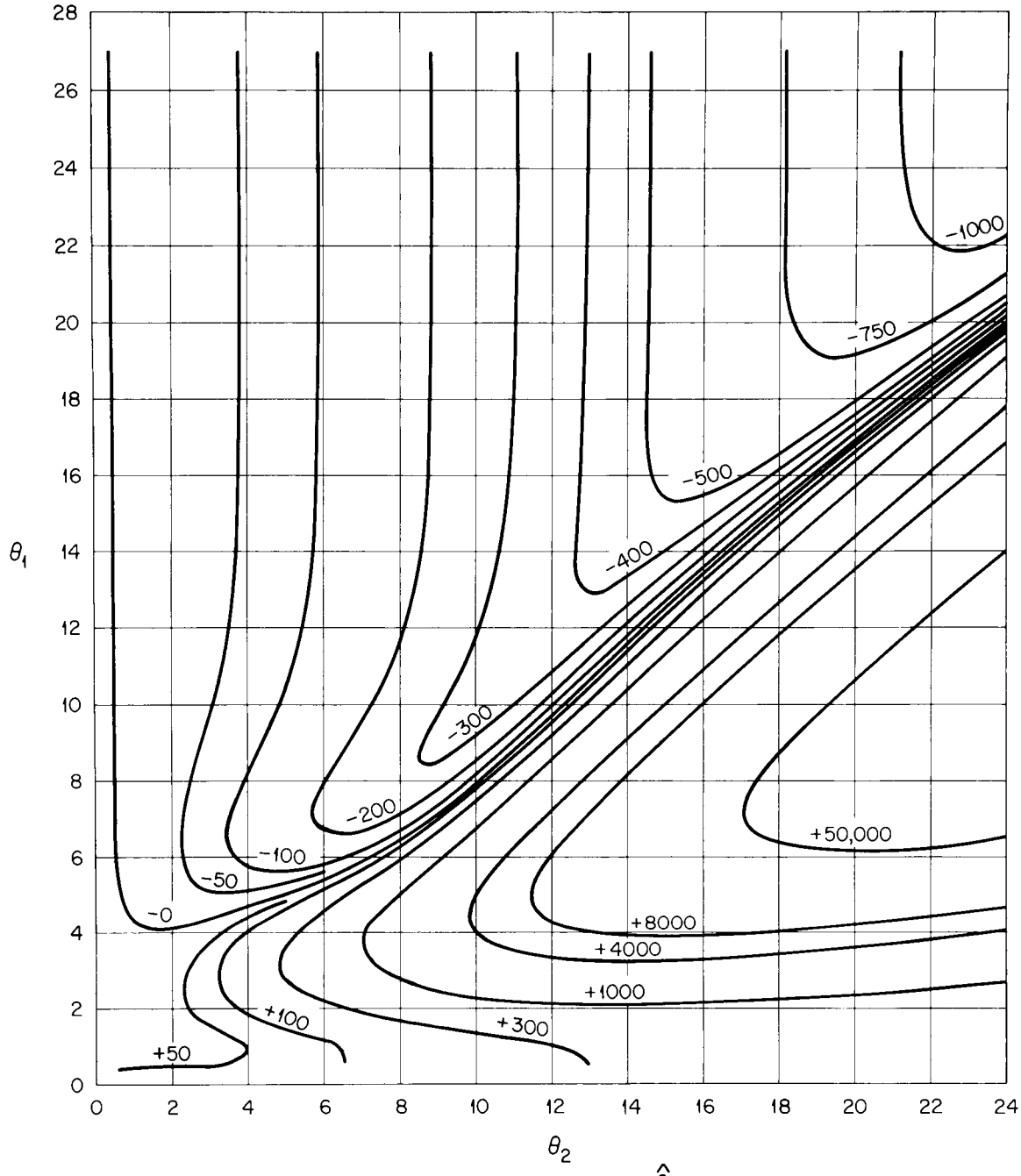
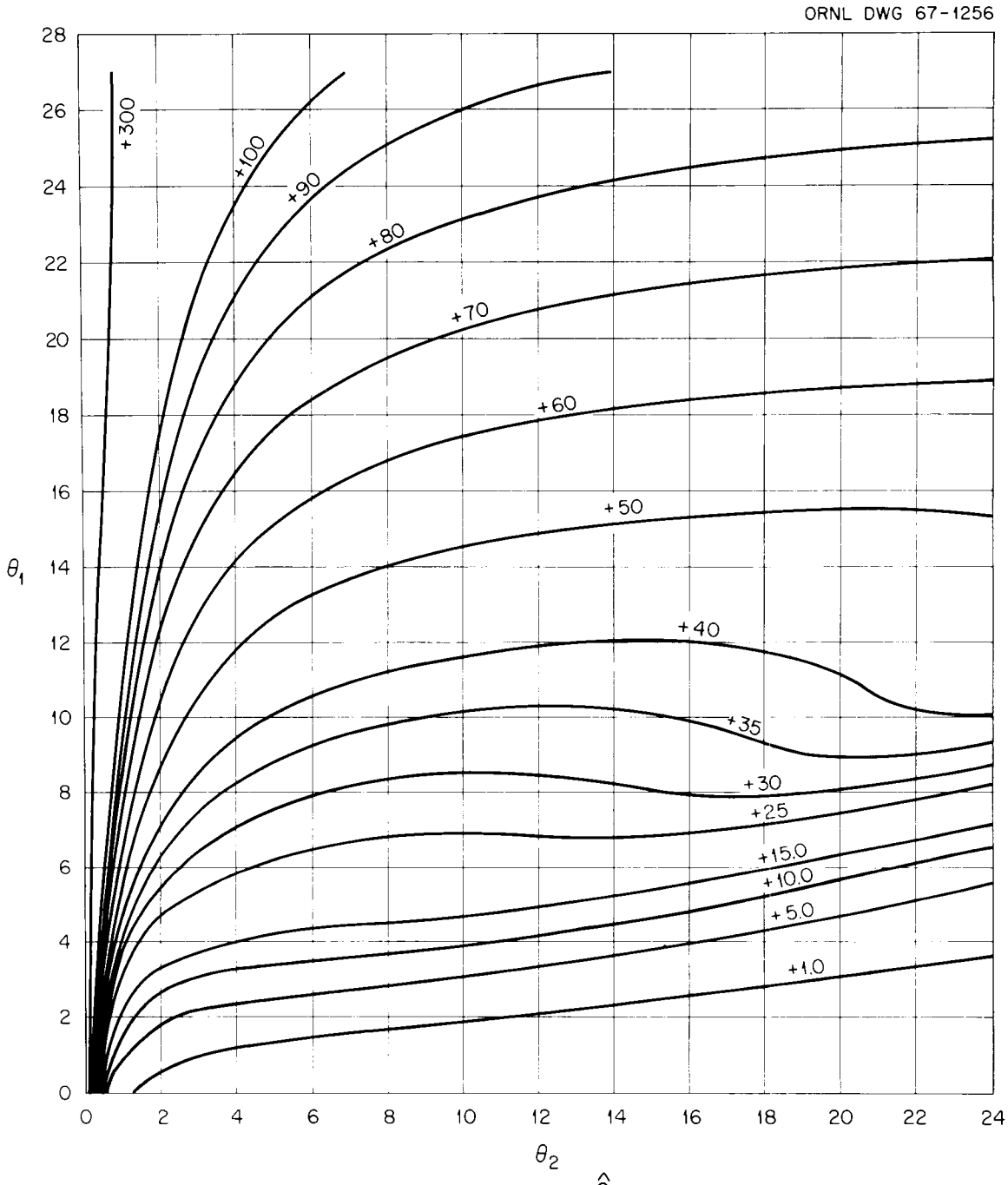
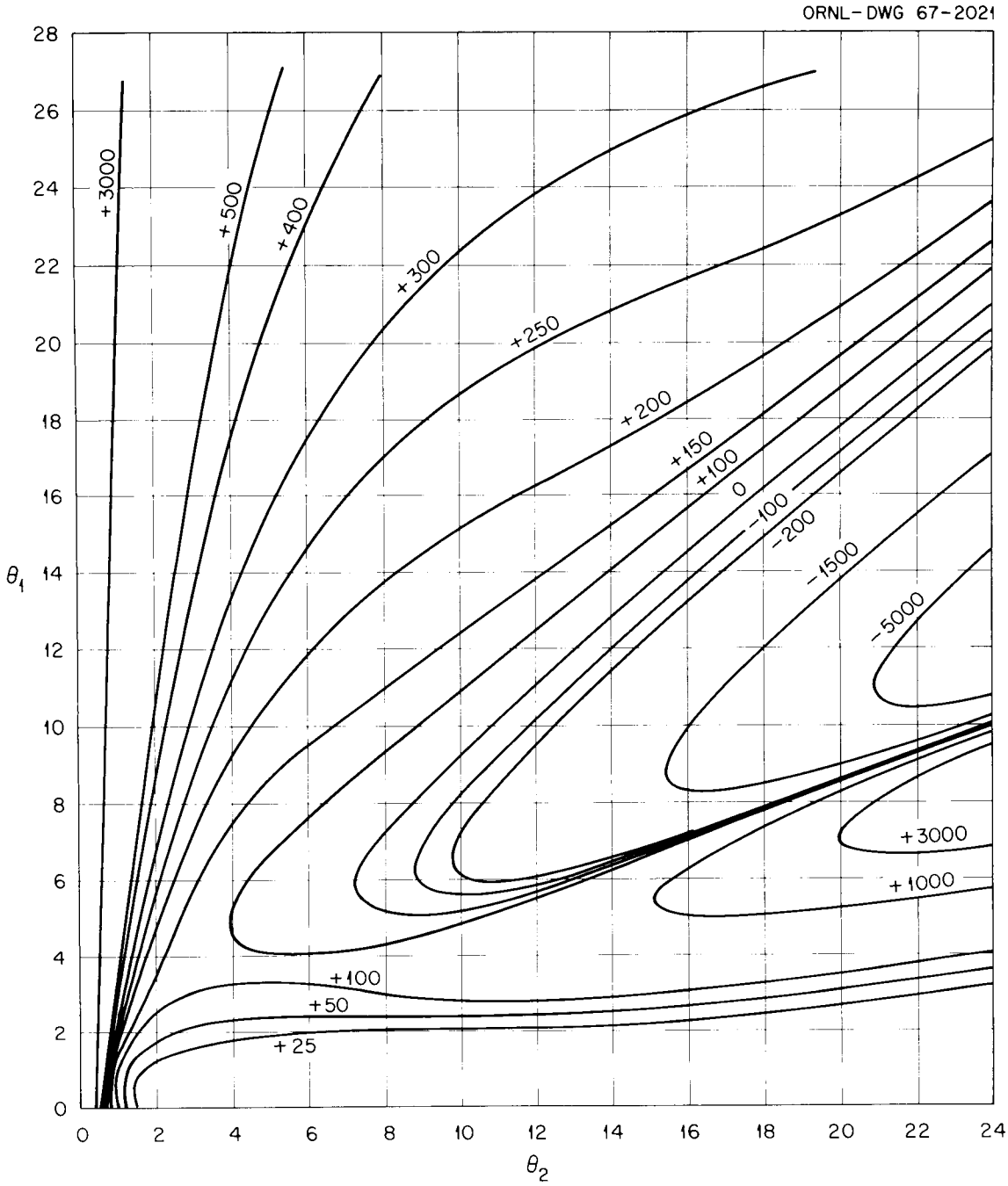


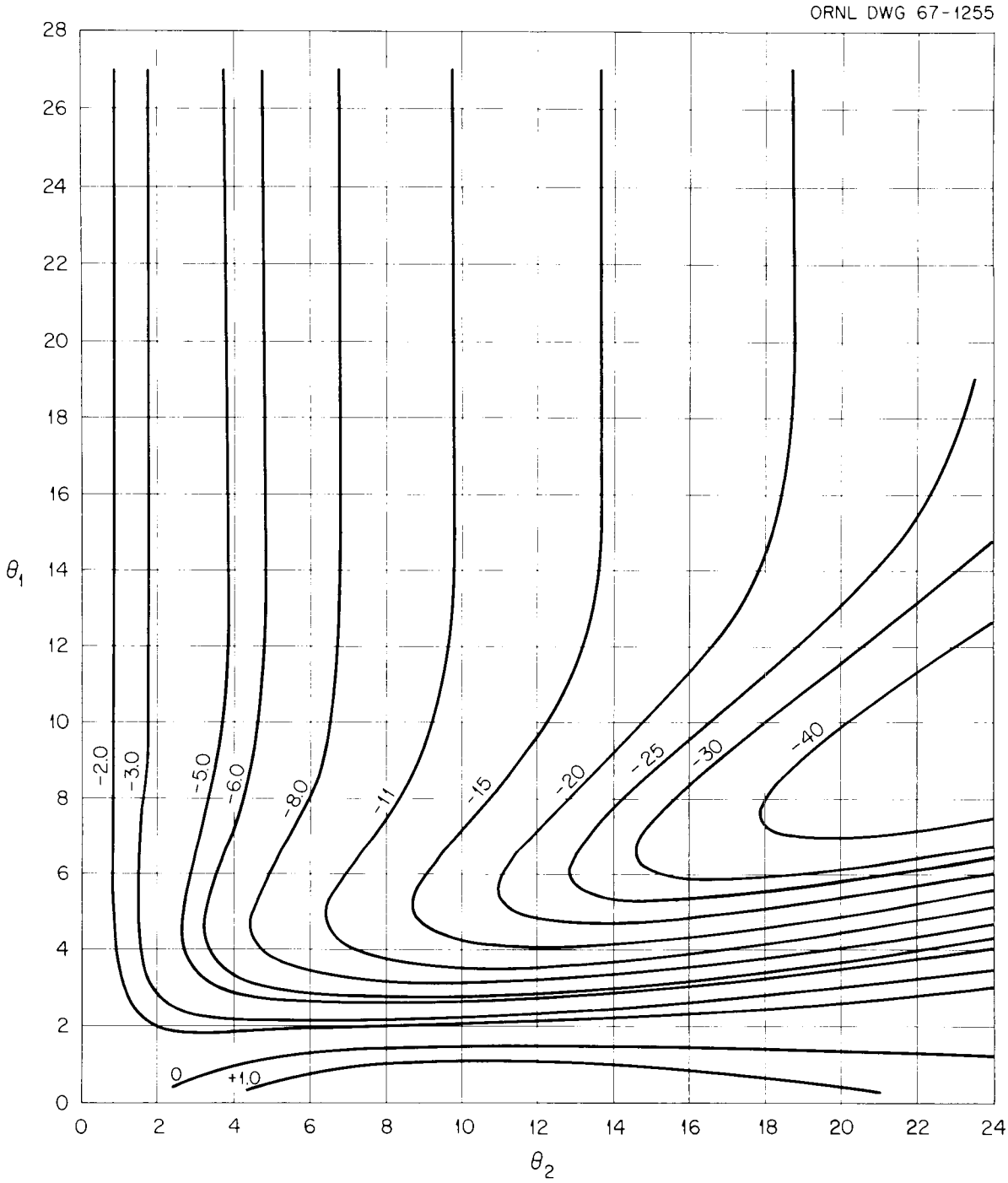

Second Order Term in Variance of $\hat{\theta}_2$ Max. Likelihood



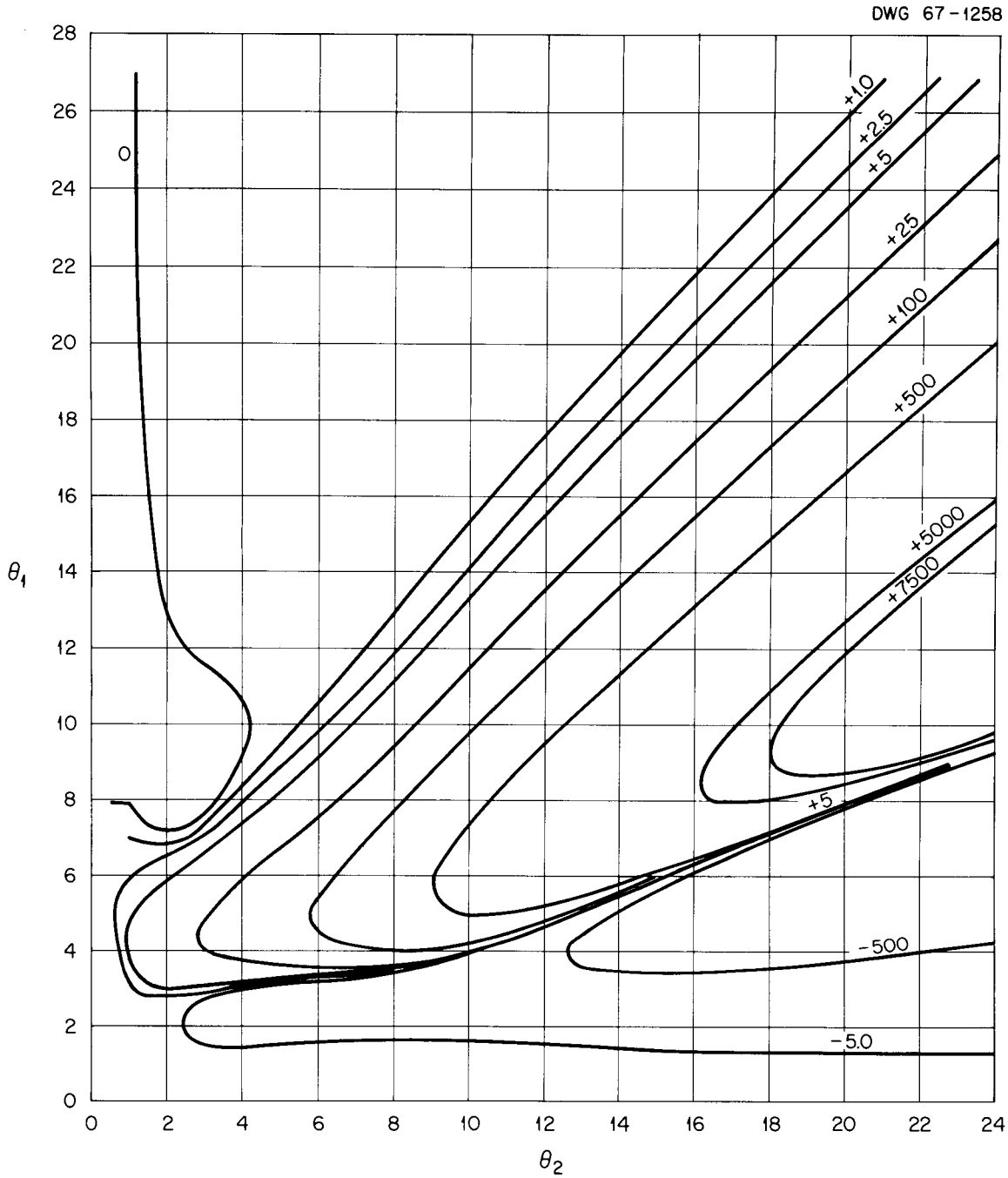

First Order Term in Bias of $\hat{\theta}_1$ Max. Likelihood




Second Order Term in Bias of $\hat{\theta}_1$ Max. Likelihood




First Order Term in Bias $\hat{\theta}_2$, Max. Likelihood




Second Order Term in Bias of $\hat{\theta}_2$ Max. Likelihood

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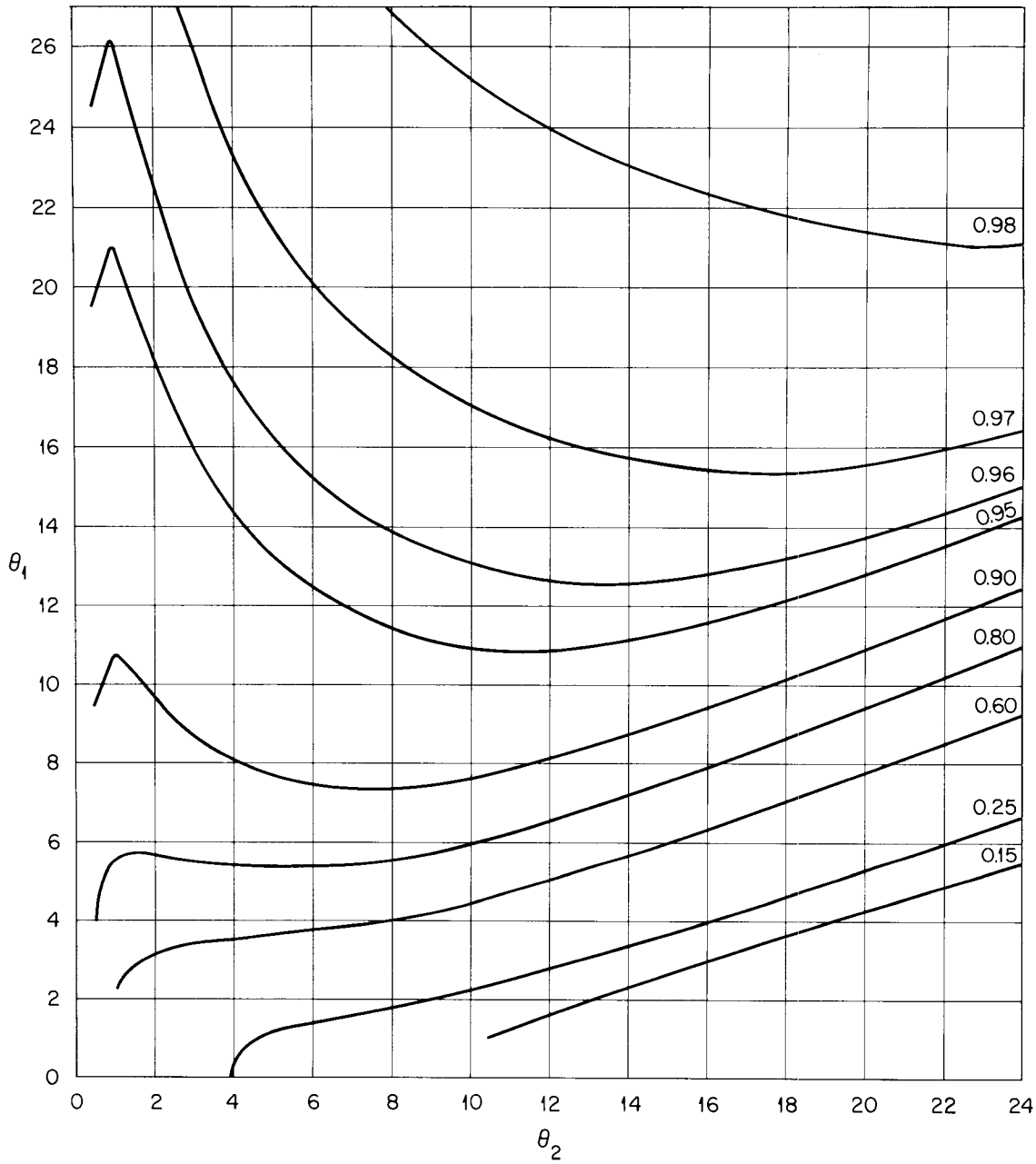


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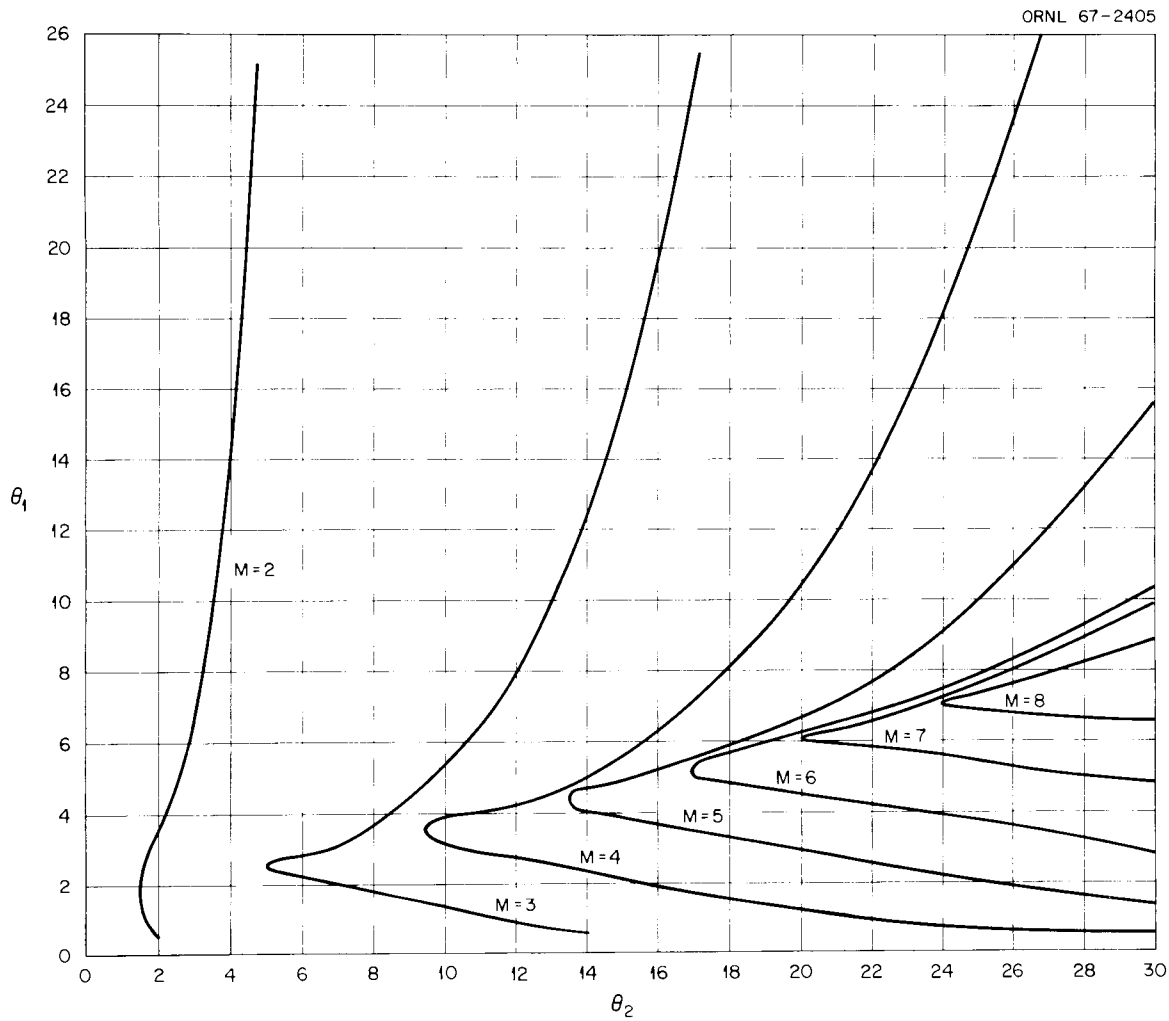


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