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ON THE NONLINEAR EVOLUTION OF AN UNSTABLE LOSS CONE FLUTE MODE

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ABSTRACT

A nonlinear theory is developed which describes the early behavior of a particularly simple unstable mode in a magnetized plasma. The mode is a single electrostatic traveling wave propagating directly across the magnetic field, with the energy for the growth of the wave furnished by an inverted population in velocity space, while the wave propagation is primarily supported by a cold isotropic plasma component. The nonlinear theory is constructed from particle orbits; it predicts a fast stabilization of the wave by the heating of the cold component. The theory agrees qualitatively and quantitatively with the results of a recent computer simulation, predicting the temperature of the cold component and the energy in the electric field. The general conclusion is that these modes are self-stabilizing, with final fluctuating field levels that are not particularly dangerous to plasma confinement.

I. INTRODUCTION

A computer simulation of plasma dynamics is most useful when it can be compared to an analytic theory. Since in the simulation many parameters can be varied independently the regions of validity of analytic approximations can be explored. The masses of data produced in the simulation can be used to check the minute details of the theory, and when successful the theory can be generalized and used to predict the behavior of real plasma systems. The early time evolution of plasma instabilities can be easily followed in a simulation, while in a laboratory experiment only the final state of the system can usually be observed. The theory presented here has been developed by close comparison with the details of the simulation of such a plasma instability. These details are being published separately[1]; a short summary of the theoretical results has also appeared.[2]

The particular instability studied here is a version of the loss-cone flute mode. It is an electrostatic mode, propagating directly across the magnetic field. The energy needed for the growth of the instability is furnished by an inverted population in velocity space[3], produced by the loss cone in open ended mirror traps and also occurring in many other containment devices. In addition to the hot plasma contained by the static magnetic field the mode requires a cold plasma component which appears naturally in many ways. The mode is basically the lower hybrid mode, supported by the cold plasma component. It becomes unstable when it can couple to the negative energy Bernstein modes of the hot plasma, near one of the cyclotron harmonics. The linear theory was first discussed by Hall, Heckrotte, and Kammash[4], and has been explored more completely by Guest, Farr and Dory.[5]

The nonlinear evolution of the mode has been previously studied by Aamodt and Bodner[6], with emphasis on the regime with a small proportion of hot plasma. They found the system to be stabilized by a nonlinear smearing of the cyclotron harmonics. In the plasma simulation of Crume, Meier, and Eldridge[1], the amounts of hot and cold plasma were roughly comparable, and the system was stabilized by heating of the cold component. The saturation level of the electric field energy was found to be substantially lower then that predicted by Aamodt and Bodner or by the empirical formula of Byers and Grewal.[7]

This instability is important as a prototype of a class of instabilities which appear as a necessary result of containment in an open ended mirror, although it can appear in a variety of containment devices. Since a component $k_{\parallel}$ of the wave vector along field lines allows Landau damping to occur and decreases growth rates, the flute modes with $k_{\parallel} = 0$ are particularly dangerous. Maximum growth rates of a tenth of the real frequency are predicted. The finite length of open ended devices does not change the characteristics appreciably. The large growth rates which are possible, and more to the point, the large amount of free energy which can appear in the fluctuating electric field, make the study of the nonlinear development of these modes particularly important. The amount of energy in the electric field spectrum has a direct bearing on the feasibility of open ended mirrors as containment devices. The simulation studies are limited to a number of special cases, but with an analytic theory, the effects of parameter variation can be predicted.

In the computer simulation a single unstable traveling wave is dirven to an amplitude above the noise level and then allowed to evolve in a self consistent way. The magnetic field is constant and the plasma initially uniform in space. Particle trajectories are followed in two velocity dimensions and one spatial dimension perpendicular to the magnetic field. These approximations are also used for the linear analysis. The simulation plasma has a finite spatial extent so that wavelengths are limited to submultiples of the basic length. As a consequence only one unstable wave appears in the simulation; it is chosen to be the wave with maximum growth rate for the plasma parameters. Actually in a large plasma a fairly broad spectrum of waves are unstable, but the fastest growing mode dominates the spectrum after a short while.

The result of the computer experiment is an efficient and rapid stabilization, in association with a rapid heating of the cold plasma component. The typical time dependence of the electric field amplitude is shown in Fig. 1 for a case with frequency nearly twice the cyclotron frequency. The amplitude increases exponentially, as predicted by linear theory; then it saturates and develops a slow nonlinear oscillation. The simulation amplitude is always positive, but a jump in phase by π at the first minimum indicates that an equally valid interpretation is that the second peak is negative. Later in time a backward wave with the same wavelength appears as an additive component at twice the wave frequency. Spatial harmonics also are seen. As shown in reference 1 the temperature of the cold plasma component increases coherently with the field amplitude, reaching a maximum as the amplitude goes to zero. The cyclotron heating at the second gyroharmonic changes the particle distribution and stabilizes the system.

The nonlinear theory is based on a calculation of orbits, with a general representation of the electric field, and the construction of the distribution function from the orbits. The dominant interaction is taken to be resonant; the frequency is assumed to be nearly a multiple of the cyclotron frequency. This is shown to be quite a good approximation for this class of modes. These orbit solutions show a strong coherent heating of cold particles. From the Poisson equation a nonlinear differential equation is derived for the wave amplitude. Solutions of this equation exhibit a behavior quite similar to the early time behavior of the simulation. The quantitative predictions of cold component temperature and maximum field amplitude agree with the simulation over a range of plasma parameters.

The detailed calculations of this paper apply only to the special case of an instability near the second gyroharmonic. The analysis is carried as far as possible for this case, and critical comparisons with the results of the computer simulation are made. The success of the analysis shows that the approximations of the theory are valid, but it does not prove that instabilities at other harmonics are stabilized so quickly and easily. The instabilities at higher frequencies will be analyzed in a future paper, there are reasons to believe that they too are stabilizied by heating of the cold plasma.

In section II the linear stability criteria and details of the model are presented. Orbits are calculated in section III, and the charge density and temperature are calculated in section IV. In section V the differential equation for the field amplitude is derived and solved, and the generation of spatial harmonics predicted.

II. LINEAR BEHAVIOR

The model system is a uniform plasma in a uniform magnetic field. For flute modes there is no variation in the direction of the magnetic field so that the system is essentially two dimensional; velocity distributions are considered to be integrated over the axial velocity. Only one species of particle is considered with a neutralizing background of opposite charge. The particles are taken to be ions, although it turns out that the equations apply equally well to electrons. The equilibrium distribution function is taken to be

$$F(v) = n_c/[2\pi v_\theta{}^2]\exp[-v^2/2v_\theta{}^2] + n_h/[\pi\alpha^4]v^2\exp[-v^2/\alpha^2], \qquad (1)$$

where the cold component with density n_c is Maxwellian, and the hot component distribution with density n_h is designed to approximate a loss cone distribution near the midplane of a mirror trap with mirror ratio of two-to-one. The average energy per particle is given in terms of the thermal velocities v_θ and α by $W_c = mv_\theta{}^2$ for the cold component and $W_h = m\alpha^2$ for the hot component.

The dispersion relation[5] for a small amplitude electrostatic wave with frequency ω and wave number k is

$$\varepsilon(k,\omega) = 1 - \frac{\omega_{pc}{}^2}{\Omega k^2 a_c{}^2} \sum_{n=-\infty}^{\infty} \frac{nI_n(k^2a_c{}^2)}{\omega - n\Omega} \exp(-k^2a_c{}^2)$$

$$+ \frac{\omega_{ph}{}^2}{\Omega} \sum_{n=-\infty}^{\infty} \frac{n[I_n(k^2a_h^2) - I'_n(k^2a_h^2)]}{\omega - n\Omega} \exp(-k^2a_h^2) = 0$$

where I_n is the Bessel function of imaginary argument, Ω is the cyclotron frequency and the cyclotron radii are $a_c = v_\theta/\Omega$ and $a_h = \alpha/\sqrt{2}\,\Omega$.

There are no simple expressions for the frequencies of these modes that are generally valid; roots are usually found numerically. There are an infinite number of solutions with real frequency close to the cyclotron harmonics. Whenever one of these frequencies coincides with the lower hybrid frequency an instability is possible. For the case of an instability near the N'th cyclotron harmonic, with a growth rate γ large enough so that $\Omega >> \gamma >> \omega - N\Omega$, an approximate expression for the growth rate is

$$\gamma^2 = \frac{N[\omega_{ph}{}^2(I_N - I_N{}')\exp(-k^2 a_h{}^2) - \omega_{pc}{}^2(k^2 a_c{}^2/2)^{N-1}/N!]}{[\Omega\partial\varepsilon/\partial\omega]} \tag{3}$$

Here the argument of the Bessel functions is $k^2 a_h{}^2$ and the Bessel functions with argument $k^2 a_c{}^2$ have been approximated by the small argument expansion. The dielectric function of Eq. 2 has the terms with n=N missing in this approximation.

The denominator of Eq. 3 is positive since the basic mode is a positive energy wave. Formally the stability of the mode depends upon a balance between the contributions of the hot and cold plasma components.[8] When the cold component cyclotron radius a_c is large enough the mode is stable, and an initially unstable mode is stabilized by heating of the cold component. The stabilization by heating is hard to explain in a more physical way. The lower hybrid mode and the negative energy Bernstein mode are not decoupled in the sense that two separate oscillations develop, but energy is no longer fed into the growing wave. There is no such mystery about the heating mechanism; it is simply cyclotron harmonic heating.

Most of the following analysis will be developed primarily for the special case of an instability at the second gyroharmonic. Representative parameters are $\omega_{ph}{}^2/\Omega^2 \cong 20$, $\omega_{ph}{}^2/\omega_{pc}{}^2 \cong 4$, $k^2 a_h{}^2 \cong 10$, and $k^2 a_c{}^2 \cong 0.1$. One finds in this moderate density regime and in the high density regime that the real frequency is always close to a gyroharmonic for unstable modes.

III. ION ORBITS

The orbit equations cannot be solved exactly, so the real problem is that of finding a valid approximation for each region of interest. For the cold population the ion cyclotron radius is smaller than the wavelength and a small radius expansion is valid. For the hot population the ion cyclotron radius is neither large nor small, but changes in orbit parameters are slow. The discussion here depends somewhat on the analysis of Aamodt and Bodner,[6] but the development owes more to the continued testing of approximations in the course of the computer experiment.

A. Equations of Motion

The equations of motion of an ion in an constant magnetic field $\vec{B} = B\hat{z}$ and an electric field $\vec{E} = E(x,t)\hat{x}$ are

$$\frac{dv_x}{dt} = \Omega\, v_y + \frac{e}{m} E(x,t),$$

and

$$\frac{dv_y}{dt} = -\Omega v_x, \tag{4}$$

where $\Omega = eB/mc$ is the cyclotron frequency. The form of the electric field is taken to be

$$E(x,t) = E_1(t)\cos(kx-\omega t), \tag{5}$$

representing a single plane wave of real frequency ω and wave vector $\vec{k} = k\hat{x}$, with an arbitrary time dependent amplitude $E_1(t)$. When later it is found that spatial harmonics are generated, these amplitudes will be added as perturbations.

The guiding center of an ion orbit moves in the y direction, so that this motion is ignorable. The x component of the guiding center position $\xi = x + v_y/\Omega$ is a constant. The velocity v(t) and phase $\psi(t)$ are introduced with the transformation $v_x = v\cos(\psi-\Omega t)$ and $v_y = v\sin(\psi-\Omega t)$. In terms of the dimensionless velocity $u = kv/\Omega$ and the dimensionless amplitude $A(t) = ekE_1/2m\Omega^2$ the orbit equations are

$$\frac{du}{dt} = 2\Omega A\cos(\psi-\Omega t)\cos[k\xi-\omega t-u\sin(\psi-\Omega t)] = -\frac{1}{u}\frac{\partial}{\partial\psi}\{2\Omega A\sin[k\xi-\omega t-u\sin(\psi-\Omega t)]\} \quad (6)$$

and

$$\frac{d}{dt} = -\frac{2\Omega A}{u}\sin(\psi-\alpha t)\cos[k\xi-\omega t-u\sin(\psi-\Omega t)]$$

$$= \frac{1}{u}\frac{\partial}{\partial u}\{2\Omega A\sin[k\xi-\omega t - u\sin(\psi-\Omega t)] . \quad (7)$$

By the use of the generating function for Bessel functions $J_n(u)$ the time dependence of the field at the position of an ion is made explicit, taking the form of a Fourier series with discrete frequencies $\omega - n\Omega$:

$$\frac{du}{dt} = \frac{2\Omega A}{u}\sum_{n=-\infty}^{\infty} nJ_n(u)\cos\phi_n, \quad (8)$$

and

$$\frac{d\psi}{dt} = \frac{2\Omega A}{u}\sum_{n=-\infty}^{\infty} J_n'(u)\sin\phi_n, \quad (9)$$

with the wave phase $\phi_n = k\xi - \omega t+n\Omega t-n\psi$ for each term.

B. The Resonant Approximation

During the early stages of the growth of an unstable wave with small amplitude increasing like $A(t) = A_o \exp\gamma t$ a perturbation solution is valid with u and ψ almost constant. By iteration one finds

$$u \cong u_o + \frac{2\Omega A}{u} \sum_{n=-\infty}^{\infty} \frac{nJ_n(u)}{\gamma^2 + (\omega-n\Omega)^2} [\gamma\cos\phi_n - (\omega-n\Omega)\sin\phi_n] \tag{10}$$

where u_o is the initial value. For a wave with large growth rate that is almost resonant with the N'th cyclotron harmonic, the quantity $\Delta=\omega-N\Omega$ is small and $\gamma>\Delta$. The time dependence of the velocity is dominated by the near resonant N'th term but the actual value of the frequency is not very important since the resonance is very broad. A fourth frequency, the rate of change of the ion phase $d\psi/dt \cong \Omega A$, initially is very small. Early in time the ordering of the relevant frequencies is taken to be $\Omega>\gamma>\Delta>\Omega A$.

As the wave grow the amplitude can become so large that the ordering is changed to $\Omega>\Omega A>\gamma>\Delta$ where $\gamma = 1/A(dA/dt)$. Still the value of Δ is relatively unimportant. The fundamental approximation of this theory is to take only resonant contributions to the ion orbits and to set $\Delta = 0$. A new time variable

$$\tau = \int_o^t dt' A(t') \tag{11}$$

is introduced and the orbit equations become

$$\frac{du}{d\tau} = \frac{2\Omega}{u} NJ_N(u) \cos(k\xi - N\psi), \tag{12}$$

$$\frac{d\psi}{d\tau} = \frac{2}{u} J_N'(u)\sin(k\xi - N\psi). \tag{13}$$

As was pointed out by Aamodt and Bodner[6] these equations are in Hamiltonian form with momentum $u^2/2$, coordinate ψ and constant Hamiltonian

$$H = 2\Omega J_N(u)\ \sin(k\xi - N\psi), \tag{14}$$

so that a single ordinary differential equation determines the orbits.

C. Solutions for Cold Ions

The heating of cold ions stabilizes the system. For this plasma component the cyclotron radii are small and $u = kv/\Omega << 1$ even late in time. With this approximation $J_N(u) \cong (u/2)^N/N!$ and

$$\frac{d}{d\tau}(u^2) = [u^{2N} - u_o^{\ 2N}\sin^2(k\xi - N\psi_o)]^{1/2} 4N\Omega/[2^N N!] \tag{15}$$

where u_o and ψ_o are initial values. Solutions in terms of elementary functions are possible for N = 1 and 2 and in terms of elliptic functions for N = 3, 4, 5, 6, and 8.

For N=1, the fundamental resonance, the solution is

$$u^2 = u_o^2 + 2\Omega\tau u_o \cos(k\xi - \psi_o) + \Omega^2\tau^2$$

and

$$\cos(k\xi - \psi) = \frac{[u_o \cos(k\xi - \psi_o) + \Omega\tau]}{[u_o^2 + 2\Omega\tau u_o \cos(k\xi - \psi_o) + \Omega^2\tau^2]^{1/2}}.$$

Initially the energy can either increase or decrease but eventually it increases as τ^2. The ion phase becomes synchronized with the wave phase so that $\cos(k\xi - \psi) \to 1$ as $\tau \to \infty$. Each harmonic must be treated separately, but it may be shown that the synchronization of phases occurs for each N so that in the long time limit $\cos(k\xi - N\psi) \to 1$. The velocity also increases with time, linearly for $N = 1$, exponentially for $N = 2$, and even faster for larger N. Stabilization of the mode seems to occur because each cold ion has its phase synchronized so that its energy is increasing at the fastest possible rate.

The case with $N = 1$ is not useful here since the instability cannot occur near the first gyroharmonic for the densities of interest. To compare with the simulation results the case with $N = 2$ is needed. The solution is

$$u^2 = u_o^2 \, [\cosh \Omega\tau + \cos(k\xi - 2\psi_o) \sinh \Omega\tau]$$

and

$$\cos(k\xi - 2\psi) = \frac{\cos(k\xi - 2\psi_o)\cosh \Omega\tau + \sinh \Omega\tau}{\cosh \Omega\tau + \cos(k\xi - 2\psi_o) \sinh \Omega\tau} \quad (17)$$

Again the remarkable synchronization of phases is seen and in this case an exponential increase in energy.

D. Solutions for Hot Ions.

The region of particular interest for the hot plasma component is near $ka_h \cong 3$, where neither an asymptotic expansion nor a series expansion of the Bessel functions of Eqs. 12 and 13 is valid. The orbits in the asymptotic regime have been found approximately by Aamodt and Bodner.[6] The solutions show that the ions oscillate in the pseudo potential well given by the Hamiltonian of Eq. 14. The relative changes in ion velocity are small and the periods of oscillation are long compared to the time scale for changes in the electric field amplitude. Since the time scale is long the orbits may be calculated by taking the velocity and phase to be constant to first approximation.

To estimate the period of oscillation consider large velocity, u>>1, where the asymptotic expansion of the Bessel function is valid,

$$J_N(u) \cong (2/\pi u)^{1/2} \cos(u - N\pi/2 - \pi/4) .$$

The orbit equation has the form

$$\frac{du}{d\tau} \cong \pm \frac{2N\Omega}{u} \left[\frac{2}{\pi u} \cos^2 (u - N\pi/2 - \pi/4) - \frac{2}{\pi u_o} \cos^2(u_o - N\pi/2 - \pi/4)\sin^2(k\xi - N\psi_o)\right]^{1/2}.$$

As is implicit in the derivation of the asympotic expansion, the functional dependence on the velocity u is primarily an oscillation, so that the velocity may be replaced by its initial value u_o except within the oscillating term. With this substitution the solution is

$$\sin(u - N\pi/2 - \pi/4) = \kappa\, \mathrm{sn}[F(\chi,\kappa) + \alpha\tau], \tag{19}$$

where $F(\chi,\kappa)$ is the elliptic integral of the first kind[9] of argument χ and modulus κ, sn is the Jacobi elliptic function, $\sin\chi = \sin(u_o - N\pi/2 - \pi/4)/\kappa$, $\kappa^2 = 1 - \sin^2(k\xi - N\psi_o)\cos^2(u_o - N\pi/2 - \pi/4)$, and $\alpha = 2N\Omega(2/\pi u_o{}^3)^{1/2}$. The period of oscillation of the velocity is $4K(\kappa)/\alpha$ where $K(\kappa)$ is the complete elliptic integral of the first kind. The shortest period occurs for the particular combination of initial conditions for which $\kappa=0$; it is $4K(0)/\alpha = (\pi/N\alpha)(\pi u_o^3/2)^{1/2}$ which for $u_o = 3$ and $N = 2$ is $10.5/\Omega$. All other initial conditions lead to oscillations with longer periods.

Most of the hot ions will have completed an oscillation in the pseudo-potential well when $\Omega\tau = 10.5$. However, by referring to Eq. 17 one sees that the velocity of the cold ions has changed enormously by this time. The relative changes in hot ion velocity are small during the initial stages of evolution of the wave. Of course, the energy acquired by the cold distribution does come from the hot distribution. Conceivably one could use conservation of energy to calculate the saturation values of field energy, but it is much easier to calculate the charge density and use Poisson's equation.

The appropriate resonant solution for the hot components is, from these arguments,

$$u \cong u_o + \frac{2N\Omega\tau}{u} J_N(u) \cos(k\xi - N\psi), \tag{20}$$

obtained by holding u and ψ constant during integration.

E. Nonresonant Solutions

The nonresonant contribution to the ion orbits is calculated with perturbation theory. The amplitude of the electric field is expanded in a Taylor series

$$A(t) \cong A_o + A't. \tag{21}$$

For these terms the time dependence of the wave phase is fast enough so that the ion velocity and phase may be held constant. From Eq. 8 a velocity increment is found which is to be added to the resonant solutions of Eqs. 17 and 20:

$$\delta u = -\frac{2\Omega A}{u} \sum_{n\neq N} \frac{nJ_n(u)\sin\phi_n}{\omega-n\Omega} + \frac{2\Omega A'}{u} \sum_{n\neq N} \frac{nJ_n(u)\cos\phi_n}{(\omega-n\Omega)^2} \tag{22}$$

The Taylor expansion is valid for only a short while, but this is sufficient since the orbits are to be used to find a differential equation in the variables $A(t)$, dA/dt, and $\tau = \int A dt$.

IV. Distribution and Charge Density

A solution of the Vlasov equation is generated from the orbits without further approximation. The orbit variables have been given as functions of time with initial values v_o and ψ_o. The orbits are inverted to express the initial values as functions $v_o(v,\psi,\xi,t)$ and $\psi_o(v,\psi,\xi,t)$. A solution of the Vlasov equation is[10]

$$f(v,\psi,\xi,t) = F[v_o(v,\psi,\xi,t);\ \psi_o(v,\psi,\xi,t);\xi],$$

where F is normalizable and non negative, but otherwise arbitrary.

A. The Cold Component

The initial cold distribution of Eq. 1 depends only on velocity. From Eqs. 17 and 22 one finds for N=2,

$$f_c(v,\psi,\xi,t) = n_c/(2\pi v_\theta^2)\exp\{-[v-\delta v]^2[\cosh \Omega\tau - \cos(k\xi-2\psi)\sinh \Omega\tau]/2V_\theta^2\}. \tag{23}$$

with $\delta v = \Omega\delta u/k$.

By a Taylor expansion one finds

$$f_c = f_{co} - \delta v\ \partial f_{co}/\partial v \tag{24}$$

where

$$f_{co} = n_c/(2\pi v_\theta^2)\exp[-v^2 \cosh \Omega\tau/2v_\theta^2]\cdot\{I_o(v^2\sinh \Omega\tau/2v_\theta^2)$$
$$+ 2\sum_{p=1}^{\infty} I_p(v^2\sinh\Omega\tau/2v_\theta^2)\cos[pkx+pu\sin(\psi-\Omega t) - 2p\psi] \ .$$

By using the generating function for Bessel functions J_n one finds the explicit form

$$f_{co}(v,\psi,x,t) = n_c/(2\pi v_\theta^2)\exp[-v^2 \cosh \Omega\tau/2v_\theta^2]$$

$$\cdot\{I_o(v^2\sinh\,\Omega\tau/2v_\theta^2)+2\sum_{p=1}^{\infty}\sum_{n=-\infty}^{\infty} I_p(v^2\sinh\,\Omega\tau/2v_\theta^2)\;J_n(pu)$$

$$\cdot\;\cos[pkx + n(\psi-\Omega t) - 2p\psi]\quad . \tag{26}$$

To be consistent with the cold orbits the series expansion of J_n must be used again. The integrals needed to find macroscopic quantities have the form[11]

$$\int_o^\infty dt\; t^q\,\exp[-t\cosh\alpha]\; I_p\;(t\sinh\alpha) = (q-p)!\,\mathcal{P}_q^p(\cosh\alpha) \tag{27}$$

which is valid for $q\geq p\geq 0$. The functions $\mathcal{P}_q^p(\cosh\,\alpha)$ are the associated Legendre functions, defined for arguments greater than unity.

By direct integration the temperature of the cold component is

$$\Theta(t) = \frac{1}{N_c}\int_o^\infty vdv\int_o^{2\pi} d\psi\int_o^L dx\; mv^2 f_{co} = \Theta_o\;\cosh\,\Omega\tau, \tag{28}$$

where L is the length of the plasma, N_c is the number of cold ions, and $\Theta_o = mv_\theta^2$ is the initial temperature.

The charge density calculated from the resonant part of the distribution is

$$\rho_o(x,t) = en_c\{1+2\sum_{p=1}^{\infty}\frac{1}{(2p)!}\left(\frac{pkv_\theta}{2^{1/2}\Omega}\right)^{2p}\mathcal{P}_p^p(\cosh\Omega\tau)\cos(pkx-2p\Omega t)\}$$

$$= en_c\{1 + 2\sum_{p=1}^{\infty}\frac{1}{p!}\left(\frac{pkv_\theta}{2\Omega}\right)^{2p}\sinh^p\Omega\tau\;\cos(pkx-2p\Omega t)\}.$$

The term with p=1 acts to stabilize the wave while the terms with p>1 generate harmonics which do not propagate.

The nonresonant contribution to the charge density is

$$\delta\rho_c = -e\int_0^\infty v\,dv\int_0^{2\pi} d\psi\;\delta v\,\frac{\partial f_{co}}{\partial v}\,,$$

with δv given in Eq. 22. The calculation is rather lengthy, and only partial results are given here. The part of the charge density that has the wave phase $kx-\omega t$ is

$$\delta\rho_c \cong en_c \sum_{\substack{n=1\\ n\neq 2}}^{\infty} \left(\frac{k^2v_\theta^2}{2\Omega^2}\right)^{n-1} \frac{1}{n!}\,\mathcal{P}^{o}_{n-1}(\cosh\Omega\tau) \tag{30}$$

$$\cdot\left[\frac{-2An^2\Omega^2}{\omega^2-n^2\Omega^2}\sin(kx-\omega t)+4\,\frac{dA}{dt}\,\frac{n^2\Omega^2\omega}{\omega^2-n^2\Omega^2}\;2\;\cos(kx-\omega t)\right].$$

Only the term with $n=1$ is large enough to contribute to stabilization or dispersion, even though the terms with $n\geq 3$ are increasing rapidly with increasing τ. The part neglected here generates harmonics.

B. The Hot Component.

From Eqs. 20 and 22 the initial velocity of the hot component is

$$v_o(v,\psi,\xi,t) = v - \frac{4\Omega^3\tau}{k^2v}\;J_2(kv/\Omega)\;\cos(k\xi-2\psi) - \delta v$$

where changes in velocity are small. The distribution is approximately

$$f_h = F_h - \left[\frac{4\Omega^3\tau}{k^2v}\,J_2(kv/\Omega)\cos(k\xi-2\psi) - \delta v\right]\frac{\partial F_h}{\partial v}\,, \tag{31}$$

where F_h is the equilibrium distribution of Eq. 1. The charge density is found by direct integration to be

$$\rho_h = en_h\{1-4\Omega\tau[I_2 - I_2']\exp(-k^2a_h{}^2)\cdot\cos(kx-2\Omega t)$$

$$-2\frac{dA}{dt} \sum_{n\neq 2} \frac{n}{(\omega-n\Omega)^2} [I_n - I_n']\exp(-k^2a_h^2) \cdot \cos(kx-\omega t) \tag{32}$$

$$+2A \sum_{n\neq 2} \frac{n\Omega}{\omega-n\Omega} [I_n - I_n']\exp(-k^2a_h^2) \cdot \sin(kx-\omega t).\} ,$$

where the argument of the Bessel functions is $k^2a_h^2$.

V. STABILIZATION

The fields must now be made self consistent by using Poisson's equation, $\partial E/\partial x = 4\pi\rho$, which with the field representation of Eq. 5 is

$$-\frac{2m\Omega^2}{e} A(t) \sin(kx-\omega t) = 4\pi\rho \tag{31}$$

The charge density is the sum of partial densities from Eqs. 29, 30 and 32. Consistency is obtained by matching the coefficients of sin(kx-ωt) and cos(kx-ωt) in this equation, with $\omega \cong 2\Omega$. One result is the dispersion relation of Eq. 2, in the small cyclotron radius limit for the cold component, with the n=2 terms missing, and with real frequency. The second result is an equation linking the time dependent quantities dA/dt, and $\tau = \int A dt$. This may be written as a second order differential equation in the dimensionless variable $Z(t) = \Omega\tau$:

$$\frac{d^2Z}{dt^2} = \gamma^2\left[\frac{Z-K \sinh Z}{1-K}\right] , \tag{32}$$

where γ is the growth rate of Eq. 3, and $K = [n_c k^2 a_c^2]/[8n_h(I_2-I_2')\exp(-k^2a_h^2)]$. The Bessel functions have the argument $k^2a_h^2$. By examining the approximate growth rate an alternative definition of the stability parameter K is found to be $K=\Theta_o/\Theta_s$, where Θ_o is the initial temperature of the cold component and Θ_s is the temperature necessary to stabilize the mode according to the linear criterion. During the nonlinear evolution the temperature increases past Θ_s.

A first integral of Eq. 32 is

$$\left(\frac{dZ}{dt}\right)^2 = \Omega^2A^2 = \gamma^2\left[\frac{Z^2 - 2K(\cosh Z-1)}{1 - K}\right] , \tag{33}$$

with the condition that $dZ/dt = 0$ when $Z = 0$. A typical numerical solution of this equation is shown in Fig. 2, for the plasma parameters of the simulation results[1] shown in Fig. 1. The amplitude $\Omega A = dZ/dt$ rises exponentially, is stabilized, oscillates once, and then damps away. The early stages of evolution are qualitatively the same as the simulation, but the simulation amplitude continues to oscillate, and the oscillation soon becomes noisy. Actually <u>any</u> mechanism which leads to a small <u>decrease</u> in wave energy will produce a continued oscillation.

Numerical results may be obtained without solving the differential equation. The wave amplitude reaches a maximum when $Z = K \sinh Z$; the amplitude is found directly from Eq. 33 and the temperature of the cold component from Eq. 28. A numerical comparison of the predicted and measured values of field energy and component temperature Θ is given in Table I. The ratio of field energy in the fundamental mode to the kinetic energy of the hot plasma is

$$\varepsilon_1 = [\int dx\, E^2/8\pi]/[N_h m\alpha^2] = (dZ/dt)^2/[2\omega_{ph}{}^2 k^2 a_h{}^2]. \qquad (34)$$

The values given are at the first maximum. For each of the three cases shown $n_c/n_h = 0.2884$ and $\omega_{ph}{}^2/\Omega^2 = 20$. Also the predicted and measured values of the oscillation period T of the fundamental wave amplitude are given. In each case the predictions are correct to within a factor of about 2.

Aamodt and Bodner[6] have developed a nonlinear theory describing the case where the desnity of hot plasma is small compared to the cold density. The theory is based upon a nonlinear smearing of the cyclotron resonances and is not directly applicable to the present cases. Byers and Grewal[7] report a modification of this theory by Bodner which predicts a field energy—hot ion energy ratio of $\varepsilon_1 = \Omega^4[2k^2a_h{}^2]^{1/2}/[8\pi\omega^2\omega_{ph}{}^2]$ in the notation used here. This formula predicts $\varepsilon_1 \cong 0.002$ for all the cases of Table 1, while ε_1 actually varies by a factor of 85.

The dependence of the field energy on the plasma parameters in the present theory is rather complicated. The growth rate is not the dominant factor, as can be seen by examining Table 1. The growth rate varies by a factor of 2, but the field energy changes by a factor of 85. The most important parameter is the stability parameter $K = \Theta_o/\Theta_s$, which is a measure of how much heating is needed for stabilization, but of course K is a function of the other plasma parameters. A parameter survey is in progress and will be reported later.

B. Harmonic Generation

Since spatial and temporal harmonics of the unstable mode have appeared in the charge density they must be included in the calculation of orbits. A general form for the electric field of these harmonics is

$$\delta E(x,t) = \sum_{p=2}^{\infty} E_p(t)\cos[pkx-p\omega t + \phi_p], \tag{35}$$

with amplitudes $E_p(t)$ and phases ϕ_p. The terms are treated as small perturbations in the analysis. The result of the calculation is a complicated set of coupled transcenental equations in which the phases ϕ_p must be time dependent. The largest terms in these equations are the contribution from the fundamental resonance ($p\omega=\Omega$) of the cold ions. A rough estimate of the harmonic amplitudes is found by using only these terms and the source terms from the charge density of Eq. 29.

This estimate is

$$\frac{\partial}{\partial x}\,[\delta E(x,t)][1 - \frac{\omega_{pc}^{2}}{(p\omega)^2-\Omega^2}] = 8\pi e n_c \sum_{p=2}^{\infty} \frac{1}{p!} (\frac{pkv_\theta}{2\Omega})^{2p} \sinh^p Z \cos(pkx-2p\Omega t).$$

The primitive result is that the electric field is modified by the inclusion of the dielectric function of the cold plasma, which is not changed by the heating. Now by defining the dimensionless amplitudes $A_p = ekE_p/2m\Omega^2$, setting $\phi_p = -\pi/2$, $\omega = 2\Omega$, and matching phases, the harmonic amplitudes are found to be

$$A_p(t) = \frac{\omega_{pc}^{2}}{\Omega^2}\left[1 - \frac{\omega_{pc}^{2}}{(2p\Omega)^2-\Omega^2}\right]^{-1} \frac{1}{pp!}\left(\frac{pkv_\theta}{2\Omega}\right)^{2p} \sinh^p Z. \qquad (36)$$

In Fig. 3 are plotted the field energies in the fundamental mode and the first two harmonics for the simulation of case 2 in Table 1. The numerical values at peak amplitude are shown in Table 1. Qualitatively the estimate is not very good. It predicts that the harmonic amplitudes continue to grow until A_1 goes to zero, while in the simulation A_1,A_2, and A_3 reach peak values at the same time. The simulation results also show that the phases ϕ_p are time dependent. However, the time dependence during the exponential growth phase is correctly predicted.

IV. DISCUSSION

The results summarized in Table I allow a critical examination of the approximations used in the theory. The first approximation of a single dominant wave is seen to be good for case 1 and 2, but not for case 3, where approximately a third of the field energy is in the harmonics at saturation. For this case the theoretical calculation of ε_1 is too large by a factor of three. Also the predicted energy in the third harmonic ε_3 is larger than the energy in the second harmonic ε_2, while this order is reversed in the simulation.

The resonant approximation, in which the frequency is set equal to twice the cyclotron frequency, is the worst for case 1, in which $\Delta = \omega - 2\Omega = 0.031\Omega$. The numerical agreement is not as good in this case as in case 2, with the smallest Δ. The overall agreement shows that this approximation, which is really basic to the theory, is sound.

The approximation used for hot ion orbits, that the velocity and phase are changing very slowly compared to the changes in wave amplitude, may be checked by comparing the period of the oscillations in velocity with the elapsed time for stabilization. According to the calculation in section III the shortest period, for the variable $\tau = \int A dt$, is approximately $10.5/\Omega$. In the simulation, saturation occurs at $\tau = 7.33/\Omega$ for case 3, and in a shorter time for the other cases. Clearly for this case the changes in the hot ion distribution should be taken into account. However, for this case the harmonic amplitudes have grown so large that the idealization of a single large wave is also invalid. This case represents a limit for the quantitative validity of the theory. It also represents a practical limit, in the sense that a hot-cold temperature ratio of 10,000 is unlikely to appear in a laboratory plasma.

The stabilization by the heating of the cold component seems to be well established by the theory and the simulation results. The results are generally encouraging, indicating that the energy in the electric field at saturation is smaller than previously predicted. The most important result of this analysis is the development and verification of methods useful in nonlinear plasma problems. A computer simulation can seldom be applied directly to describe a laboratory plasma, but analytic methods, such as those developed here, may be generalized when their utility has been established. Two such generalizations are being pursued. One is the development of the loss-cone instability with oblique propagation, which seems to be a straightforward extension of the analysis. The second is an application to ion cyclotron heating with external fields.

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TABLE 1

	Case 1		Case 2		Case 3	
	Theoretical	Simulation	Theoretical	Simulation	Theoretical	Simulation
ω/Ω	1.969	1.97	2.005	2.01	2.025	2.03
γ/Ω	0.098	0.103	0.202	0.208	0.232	0.236
$k^2 a_h^{\,2}$	7.50		8.55		9.20	
$k^2 a_c^{\,2}$	0.075		0.0214		0.00092	
K	0.855		0.242		0.0105	
Z	0.986		3.31		7.22	
Θ/Θ_o	1.53	1.9	13.7	15	684	570
A	0.0724		0.523		1.46	
$\Omega T/2\Pi$	5.52	4.4	2.02	3.4	1.10	1.0
ε_1	$1.75(10)^{-5}$	$3.0(10)^{-5}$	$8.01(10)^{-4}$	$5.7(10)^{-4}$	$5.77(10)^{-3}$	$1.9(10)^{-3}$
ε_2	$1.02(10)^{-6}$	$1.5(10)^{-6}$	$1.17(10)^{-4}$	$9.3(10)^{-5}$	$2.31(10)^{-3}$	$5.9(10)^{-4}$
ε_3	$2.67(10)^{-8}$	$3.3(10)^{-8}$	$3.50(10)^{-5}$	$2.0(10)^{-5}$	$3.18(10)^{-3}$	$1.4(10)^{-4}$

Caption for Table 1.

Comparison of Theoretical and Simulation Results. The frequencies ω/Ω, wavelength parameters $k^2 a_h{}^2$ and $k^2 a_c{}^2$, and the stability parameters K are initial values. The other quantities are measured at saturation. The dimensionless time parameter is Z; Θ/Θ_o is the ratio of cold component temperature at saturation to its initial value; ε_1 is the ratio of field energy in the fundamental mode to the hot ion energy; ε_2 and ε_3 are the energy ratios of the first two spatial harmonics; A is the amplitude of the fundamental and T is the period of the slow modulation of the fundamental mode.

FIGURE CAPTIONS

Figure 1. A linear plot of the amplitude of the fundamental unstable mode for case 2 of Table I. The amplitude plotted is $2\Omega^2 A/\omega_p^{\ 2} = A/12.88$. The time plotted is $\Omega t/2\Pi$.

Figure 2. A linear plot of the theoretical amplitude A of the fundamental mode for case 3.

Figure 3. A logarithmic plot of the ratio of field energy to hot ion energy for three harmonics. The top curve is for ε_1, the fundamental mode. The middle curve is for ε_2, the second harmonic. The bottom curve is for ε_3, the third harmonic.

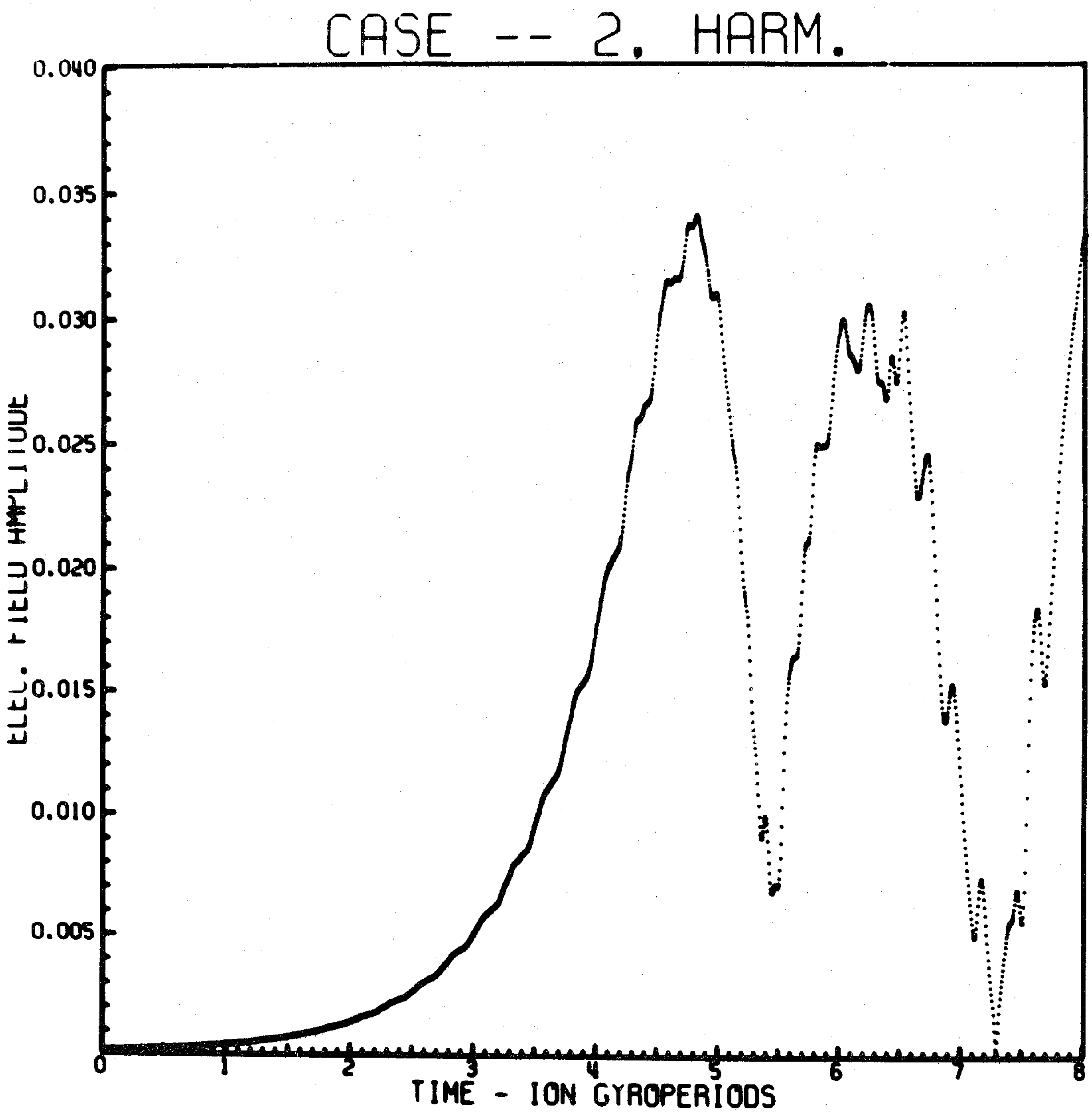
CASE -- 2. HARM.
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
ELEC. FIELD AMPLITUDE
0
1
2
3
4
5
6
7
8
TIME - ION GYROPERIODS

Fig. 1

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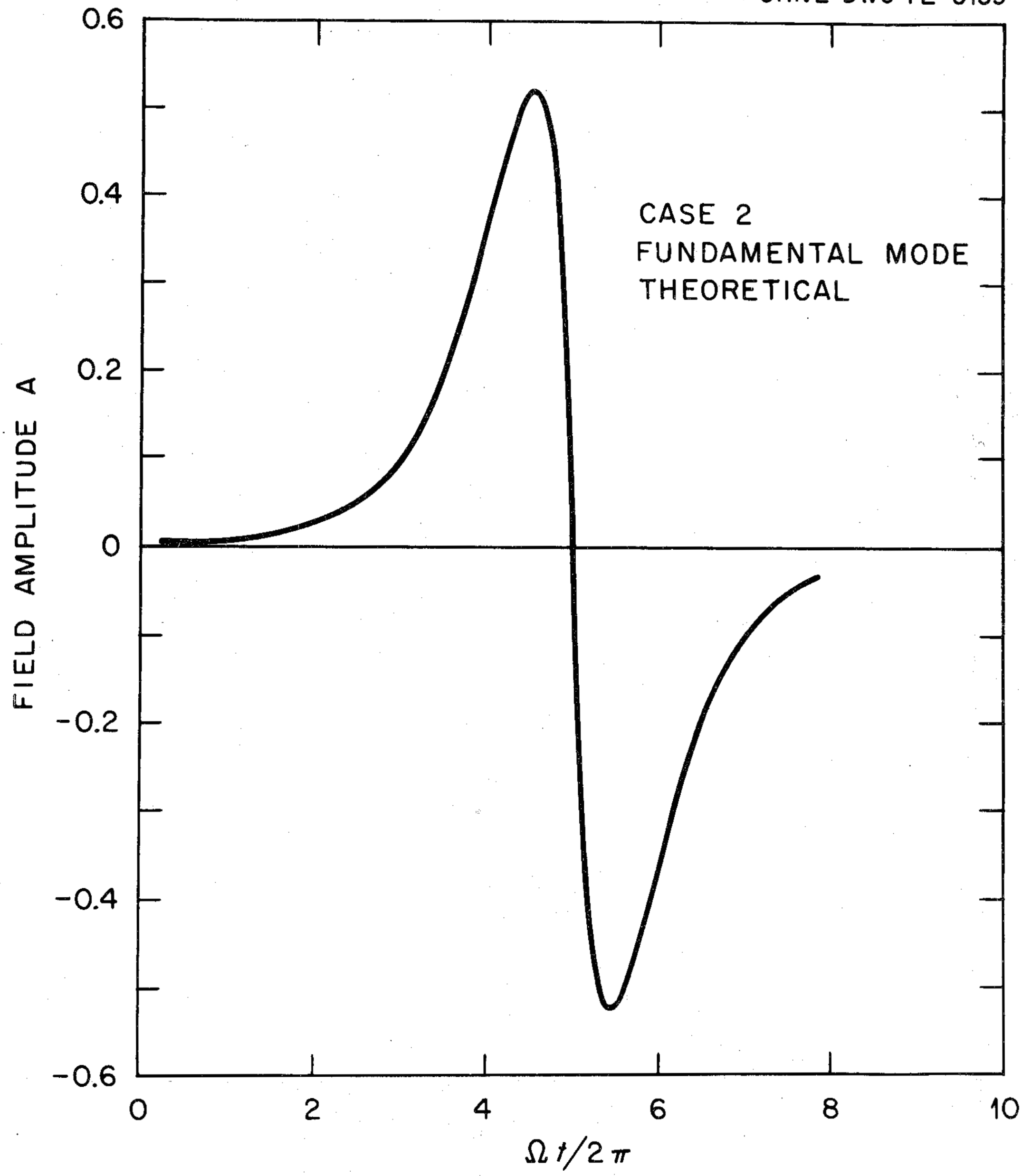

Fig. 2

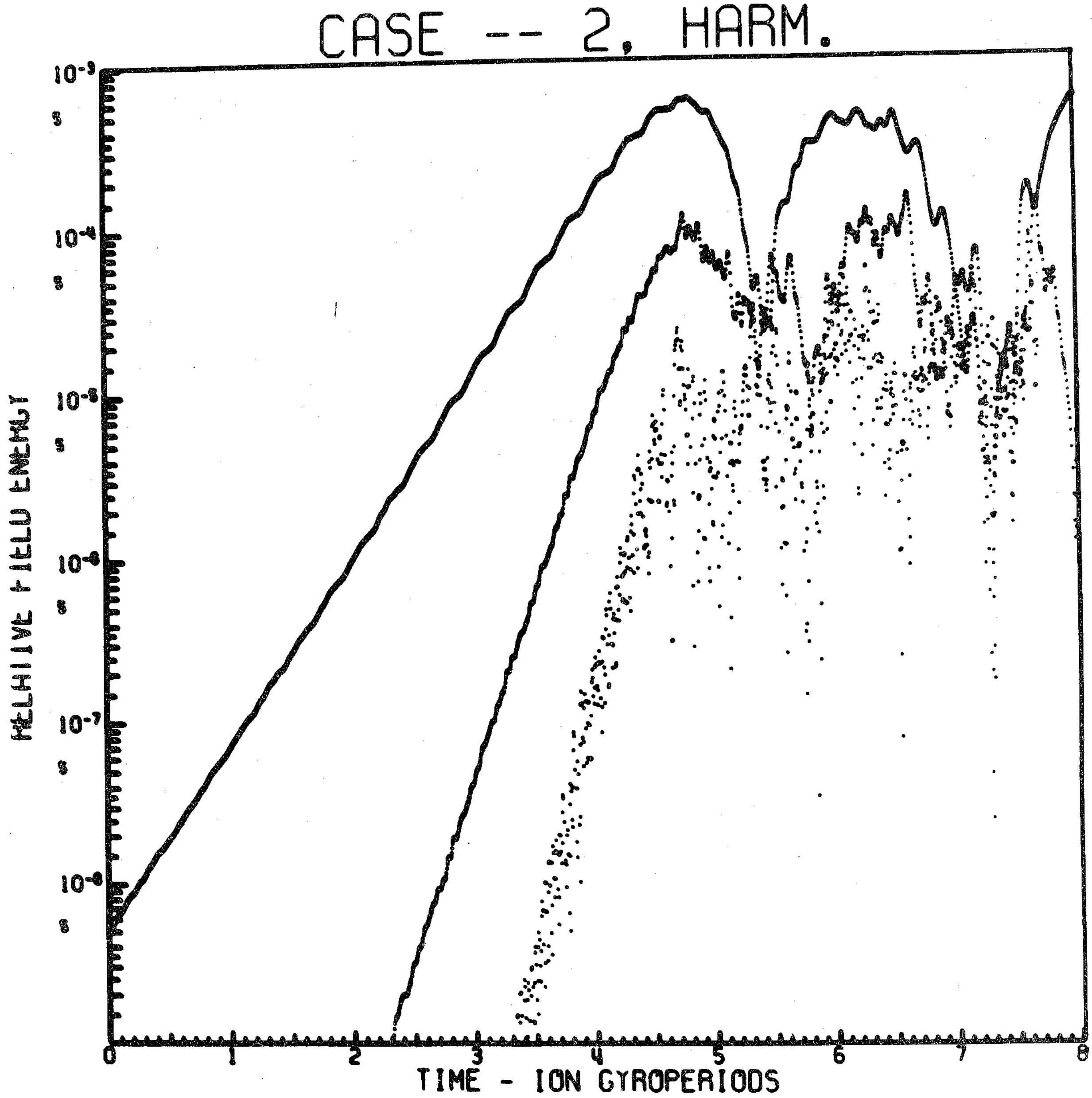
CASE -- 2, HARM.
RELATIVE FIELD ENERGY
TIME - ION GYROPERIODS
0
1
2
3
4
5
6
7
8

Fig. 3