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EFFECT OF CHARGE EXCHANGE REACTIONS ON NEOCLASSICAL TRANSPORT

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D. J. Sigmar* and J. F. Clarke†

ABSTRACT

A collision operator for the reaction H^+ (hot) + H^o (cold) $\rightleftharpoons$ H^o (hot) + H^+ (cold) is derived from the Boltzmann integral and incorporated in the proton drift kinetic equation for a toroidally confined plasma in the banana regime. In addition to the proton diffusion, the relaxation of the radial electric field and the parallel flow velocity is calculated and shown to occur in a few charge exchange times, much faster than via perpendicular ion viscosity.

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I. INTRODUCTION

Results from the ORMAK experiment indicate that the ion dynamics in plasmas of the TOKAMAK type may be described by the "banana-plateau" theory[1] provided that the effect of charge exchange collisions of the type

$$H^{+}(\text{hot}) + H^{o}(\text{cold}) \rightleftharpoons H^{+}(\text{cold}) + H^{o}(\text{hot}) \tag{1.1}$$

are included.[2] A physical discussion of the plasma behavior due to this reaction is given elsewhere.[3] In this paper, we incorporate charge exchange effects into the neoclassical banana regime theory and proceed to calculate certain modifications of the proton transport properties. To keep the analysis simple we shall not include the effect of impurity ions here.[4] The combination of impurity and charge exchange effects into a common theory is straightforward but cumbersome.

In Section II, a collision operator for the process (1.1) is derived from the Boltzmann integral and the classical momentum – and energy loss due to this operator is calculated. In Section III, the neoclassical versions of the proton-proton and the charge exchange collision operator are specified, to first order in $(r/R)^{1/2}$ (where R/r is the toroidal aspect ratio) and ν_{cx}/ν_{pp} (the ratio of charge-exchange to proton-proton collision frequency.) In Section IV we solve the drift kinetic equation for the protons. In Section V, we calculate the neoclassical momentum loss, the radial diffusion and the instantaneous flow velocity $u_{\parallel}$ of the protons parallel to the magnetic field. In Section VI we calculate the relaxation of the radial electric field and of $u_{\parallel}$, produced by the nonambipolar diffusion due to charge exchange. (We also allow an anomalous electron diffusion balanced by electron replenishment from ionization.) A summary and conclusions are given in the last section.

II. CHARGE EXCHANGE COLLISION OPERATOR, CLASSICAL FRICTION AND HEAT LOSS

Starting from Boltzmann's integral we have

$$-\frac{\partial}{\partial t}\bigg|_{cx} f_p(\underline{v}_p) = \int d^3v_n \int d\Omega\, \sigma_{cx}(v,\Omega)\,|\underline{v}_p - \underline{v}_n|$$

$$\times\,[f_p(\underline{v}_p)\,f_n(\underline{v}_n) - f_p(\underline{v}'_p)\,f_n(\underline{v}'_n)],$$

where p stands for protons and n for neutrals. The first term describes the loss of protons out of the velocity interval around $\underline{v}_p$, the second term describes the gain of protons into the velocity interval around $\underline{v}_p$, from the reaction with all protons and neutrals having $\underline{v}'_p$ and $\underline{v}'_n$ before the collision. While it is tedious to work out the collision-dynamics for Coulomb scattering it is trivial for the charge exchange process. We have simply

$$\underline{v}'_p = \underline{v}_n, \qquad \underline{v}'_n = \underline{v}_p.$$

Furthermore, the energy- and directional-dependence of σ_{cx} can be neglected in the integrals for proton temperatures less than 10 keV and we get

$$\left.\frac{\partial}{\partial t}\right|_{cx} f_p(\underline{v}) \equiv C_{cx}(f_p)$$

$$= \sigma_{cx}\left[f_n(\underline{v}) \int d^3v' \,|\underline{v}-\underline{v}'|\, f_p(\underline{v}') - f_p(\underline{v}) \int d^3v' \,|\underline{v}-\underline{v}'|\, f_n(\underline{v}')\right]. \tag{2.1}$$

Note that although C_{cx} conserves particle number it will not in general annihilate a Maxwellian proton distribution function. We will refer to the velocity integrals as the Rosenbluth potentials of f_p and f_n.[5] To calculate them, f_p and f_n are needed. The deviations of f_n from a Maxwellian have been calculated elsewhere[6] and are neglected throughout this paper. For the protons, anticipating a perturbation expansion of f_p in powers of ν_{cx} it will suffice to calculate

$$\mathcal{G}(f_p) \equiv \int d^3v' \,|\underline{v}-\underline{v}'|\, f_p(\underline{v}', \nu_{cx} = 0). \tag{2.2}$$

The most general expression for f_p describing a plasma flow parallel to the magnetic field[7] with velocity $u_{\|} < (T_p/m_p)^{1/2}$ is given by the displaced Maxwellian

$$f_p \sim f_{po}\left(1 + \frac{u_{\|}v_{\|}}{v_{Tp}^2}\right) \tag{2.3}$$

Here, $v_{Tp}^2 = T_p/m_p$ and

$$f_{po} = n_p \frac{\exp(-v^2/\alpha_p^2)}{\pi^{3/2}\alpha_p^3} \qquad \text{with} \quad \frac{m\alpha_p^2}{2} = T_p, \tag{2.4}$$

is a local Maxwellian for the protons. We assume the same shape, but a different density and temperature, for the neutrals.

Defining the errorfunction

$$\phi(x) \equiv \frac{2}{(\pi)^{1/2}} \int_0^x \exp(-t^2)\, dt,$$

one finds

$$\int d^3v' \frac{f_o}{n} \frac{|\underline{v}-\underline{v}'|}{\alpha} = x\left[(\phi - x\phi')\left(1+\frac{1}{2x^2}\right) + \frac{1+x^2}{x}\,\frac{2}{(\pi)^{1/2}}\exp(-x^2)\right]$$

$$\equiv x\gamma_o(x), \qquad (2.5)$$

where $x \equiv v/\alpha$, $\phi' = d\phi/dx$, and

$$\int d^3v' \frac{f_o}{n} \frac{|\underline{v}-\underline{v}'|\, v'_{\parallel}}{\alpha^2} = \frac{v_{\parallel}}{\alpha}\, x\gamma_1(x), \qquad (2.6a)$$

where

$$\gamma_1(x) = \frac{1}{15}\left[\frac{1}{x^4}\,\frac{4}{(\pi)^{1/2}}\int_0^x \exp(-t^2)\, t^6\, dt + x\phi'\right] + \frac{1}{3}x\phi'$$

$$-\left(\frac{\phi - x\phi'}{2x^2}\right) - \frac{1}{3}\left(\frac{1+x^2}{x}\right)\frac{2}{(\pi)^{1/2}}\exp(-x^2). \qquad (2.6b)$$

The function $(\phi - x\phi')/2x^2$ is tabulated in Spitzer,[8] the first term in (2.6b) is small for small and large values of x and could be neglected without much error. Using (2.3) for the protons and taking a Maxwellian for the neutrals we obtain

$$C_{cx}[f_p(\underline{v})] = \nu^o_{cx} n_p \left\{ \frac{f_{no}(\underline{v})}{n_o} \left[x_p \gamma_o(x_p) + \frac{u_{\parallel} v_{\parallel}}{v^2_{Tp}} x_p \gamma_1(x_p) \right] - \frac{f_p(\underline{v})}{n_p} [x_p \gamma_o(x_n)] \right\} \tag{2.7a}$$

where

$$\nu^o_{cx} \equiv \sigma_{cx} n_o a_p . \tag{2.7b}$$

For a Maxwellian proton distribution function, $u_{\parallel} = 0$, whereupon (2.7) is similar to a Krook-operator, having an energy dependent collision frequency. It is straightforward to prove particle conservation, and to calculate the parallel momentum and heat loss, defined as

$$R_{\parallel cx} \equiv \int d^3v \; m_p v_{\parallel} C_{cx}(f_p),$$

$$Q_{cx} \equiv \int d^3v \; \frac{m_p v^2}{2} \; C_{cx}(f_p),$$

respectively. One finds, expanding in powers of $T_n/T_p < 1$,

$$R_{\parallel cx} = -m_p n_p \nu^o_{cx} u_{\parallel p} \left(\frac{8}{3(\pi)^{1/2}} \right) \left[1 + \frac{1}{2} \frac{T_n}{T_p} + O\left(\frac{T_n^2}{T_p^2} \right) \right] \tag{2.8}$$

$$Q_{cx} = -T_p n_p \nu^o_{cx} \frac{4}{(\pi)^{1/2}} \left[1 - \frac{1}{2} \frac{T_n}{T_p} + O\left(\frac{T_n^2}{T_p^2} \right) \right] . \tag{2.9}$$

(For details, see Appendix A.) An exact calculation yields $Q_{cx} \propto -\nu^{o}_{cx} n_p (T_p - T_n)$, as expected,[9] having the same asymptotic expansion as in 2.9).) This charge exchange heat loss is in addition to (and may exceed) the proton heat loss through ion heat conductivity. In a Tokamak, the proton temperature is balanced against these losses by Joule heating of the electrons and subsequent electron-ion collisions. The neutral temperature is maintained through the constant influx of charge exchange neutrals.[2] The charge exchange friction will be worked out in the banana regime, subsequently.

III. NEOCLASSICAL COLLISION OPERATORS

We adopt the banana regime ordering for the protons and assume the weak charge exchange limit, so that

$$\hat{\omega}_b \gg \nu_{pp} \gg \nu_{cx} \tag{3.1}$$

where $\hat{\omega}_b$ is the average bounce frequency. (For typical parameters, such as those occurring in the ORMAK experiment,[3] $\tau_b \sim 10^{-5}$ s, $\tau_{pp} \sim 10^{-3}$ s, and $\tau_{cx} \sim 10^{-2}$ s.)

In steady state on the banana diffusion time scale, the proton distribution function $f = f_o + f_1$ (we now drop the index p for the protons, but keep the index n for the neutrals) solves[7]

$$v_{\parallel} \nabla_{\parallel} f_1 + \frac{\partial f_o}{\partial r} v_{Dr} = C_{pp}(f) + C_{cx}(f). \tag{3.2}$$

As usual, the expansion of f is in powers of the gyroradius and in addition we will expand f_1 in the collision frequencies, according to (3.1).

For the proton-proton collision operator we take Kovrizhnyk's[10,7] model operator

$$C_{pp}(f) = \nu_{pp}\overline{\mathscr{L}}(f_1) + \nu_{pp}\frac{v_{\parallel}p}{v_{Tp}^2} f_o \qquad (3.3a)$$

where, using the notation of Ref. 7, $\overline{\mathscr{L}} = 2hq\frac{\partial}{\partial\lambda}\lambda q\frac{\partial}{\partial\lambda}$, $q = \frac{|v_{\parallel}|}{v} = \left(1-\frac{\lambda}{h}\right)^{1/2}$, $\lambda = \frac{\mu B_o}{E}$, $h = 1 + \frac{r}{R}\cos\theta$,

$$\frac{\nu_{pp}}{\nu_{po}} = \frac{1}{2x^5}\left[x\phi' + (2x^2-1)\phi\right], \qquad \nu_{po} = \frac{n_p}{a_p^3}\,\frac{4\pi e^4 \ln\Lambda}{m_p^2}.$$

The velocity p in (3.3) is determined a posteriori from momentum conservation, i.e.,

$$\int d^3v\, mv_{\parallel}C_{pp}(f) = 0,$$

or, using (3.3a) and the identity

$$\int d^3v\, A(v)\,\overline{\mathscr{L}}(f) = -\int d^3v\, A(v)\, f; \qquad f, A(v) \ldots \text{ arbitrary}$$

$$p = \int d^3v\, \nu_{pp}v_{\parallel}f_1 \Big/ n\nu_*, \qquad n\nu_* \equiv \int d^3v\, \nu_{pp}\frac{v_{\parallel}^2}{v_T^2} f_o. \qquad (3.3b)$$

$C_{cx}(f)$ is given in (2.1) leading to an integral equation for f. However, because of the ordering (3.1) one can expand f in powers of ν_{cx} and use $f(\nu_{cx}=0)$ (see Eq. (2.2)) in the Rosenbluth potential, as mentioned. Thus

$$C_{cx}(f_p) = \sigma_{cx}\{f_n(\underline{v})\, \mathscr{G}(f_{po}+f_{p1}(\nu_{cx}=0)) - f_p \mathscr{G}(f_{no})\}. \tag{3.4}$$

For $\nu_{cx} = 0$ we write the standard solution[7] of (3.2) as

$$f = f_o + F + Hg_c^o \tag{3.5a}$$

where f_o is Maxwellian,

$$F = -\frac{v_{\parallel}}{\Omega_\theta}\frac{\partial f_o}{\partial r}, \tag{3.5b}$$

with $\Omega_\theta = |e|B_\theta/m$ and B_θ the poloidal magnetic field. H is the step-function (equal to 1 for circulating particles) and

$$g_c^o = f_o S_o \frac{\sigma v}{2} \int_\lambda^{\lambda_c} \frac{d\lambda'}{\langle q \rangle}, \tag{3.5c}$$

with

$$S_o = \frac{1}{\Omega_\theta}\frac{f'_o}{f_o} + \frac{u_{\parallel}^{RHH}}{v_{Tp}^2}. \tag{3.5d}$$

Here, $f'_o = \partial f_o/\partial r$, $\sigma = v_{\parallel}/|v_{\parallel}|$ and $\langle \ldots \rangle = \int_0^{2\pi}(d\theta/2\pi)\left(1+\frac{r}{R}\cos\theta\right)\ldots$ is the flux surface average. On g_c^o, the subscript stands for circulating particles, the superscript for the order in the ν_{cx}-expansion. $u_{\parallel}^{RHH}$ is the parallel flow velocity of the proton-electron problem,[7] omitting $O[(r/R)^{1/2}]$ terms:

$$u_{\|}^{RHH} = -\frac{T_p}{|e|B_\theta}\left(\frac{n_p'}{n_p} + \frac{|e|\phi'}{T_p} - .17\,\frac{T_p'}{T_p}\right). \tag{3.6}$$

$\phi' = \partial\phi/\partial r$ is the radial plasma potential.

At this point we mention that the "standard" solution (3.5) is incomplete: The neoclassical distortion f_p^* owing to friction between circulating and trapped protons has been omitted.[11] This friction scales as $\sqrt{r/R}\ \nu_{pp}$, compared with the charge exchange friction scaling as ν_{cx}. Since the ordering used in this paper is

$$\nu_{cx} \lesssim \sqrt{r/R}\ \nu_{pp}, \qquad \sqrt{r/R}\ \nu_{cx} \lesssim (r/R)\ \nu_{pp} \ll 1,$$

one should retain the distortion f_p^* for a complete theory. Whereas the corresponding f_p^* for the electron distribution function has been worked out[12, 13] this has not been the case for f_p^*.

Neglecting temperature gradients, Eqs. (3.5) combine to give simply

$$\left.\begin{aligned} f &= f_o + F \ldots && \text{for trapped particles} \\ f &= f_o\left(1 + \frac{u_{\|}^{RHH} v_{\|}}{v_{Tp}^2}\right) && \text{for circulating particles} \end{aligned}\right\} \tag{3.7}$$

Since $\langle \int d^3v\, v_\parallel F \rangle \sim (r/R)^{3/2}$, the calculation of $u_\parallel \equiv \langle \int d^3v\, v_\parallel f \rangle / n_p$ to lowest order in $(r/R)^{1/2}$ from (3.7) will automatically result in (3.6). We have calculated the Rosenbluth potential of distributions such as (3.7) in Section II.

Keeping the radial temperature gradient one finds the Rosenbluth potential

$$\mathcal{G}(F+Hg_c^o) = n_p \frac{v_\parallel}{v}\, 2x^2 \gamma_1(x) \left(u_\parallel^{RHH} + u_{\parallel T} \right) \tag{3.8a}$$

(for details, see Appendix B.) Here,

$$u_{\parallel T} = -\hat{n}_T \frac{T'}{|e| B_\theta} \left(\frac{\gamma_2(x)}{\gamma_1(x)} - 1.33 \right), \tag{3.8b}$$

where

$$2x^2 \gamma_2(x) \equiv 5x^2 \gamma_1(x) - x \frac{\partial}{\partial x} (x^2 \gamma_1), \qquad x = \frac{v}{a_p} \tag{3.9}$$

and γ_1 has been defined in (2.6). The fraction of trapped particles[14] $\hat{n}_T = 1.46\,(r/R)^{1/2}$. Note that $u_{\parallel T}$ depends on v and $(r/R)^{1/2}$. The appearance of $\hat{n}_T$ indicates a trapped particle effect.

Using (3.8) and (2.5) we find from Eq. (3.4) for the charge exchange operator to first order in ν^o_{cx} and $(r/R)^{1/2}$

$$C_{cx}(f_p) = \nu^o_{cx} x_p \left\{ \frac{n_p}{n_o} f_{no}(v) \left[\gamma_o(x_p) + \gamma_1(x_p) \frac{v_\parallel \left(u_\parallel^{RHH} + u_{\parallel T}\right)}{v^2_{Tp}} \right] - f_p(v)\, \gamma_o(x_n) \right\}, \tag{3.10}$$

where $f_p = f_{po} + f_{p1}$. Since $\left(u_\parallel^{RHH} + u_{\parallel T}\right)$ and f_{p1} are of first order in the gyroradius we write (3.10) as

$$C_{cx}(f_p) = (C_{cx} f_p)_o + (C_{cx} f_p)_1, \tag{3.11a}$$

where

$$(C_{cx} f_p)_o = \nu_{cx} \left[\frac{n_p}{n_o} f_{no}(\underline{v}) \frac{\gamma_o(x_p)}{\gamma_o(x_n)} - f_{po}(\underline{v}) \right], \tag{3.11b}$$

and

$$(C_{cx} f_p)_1 = \nu_{cx} \left[\frac{n_p}{n_o} f_{no} \frac{\gamma_1(x_p)}{\gamma_o(x_n)} \frac{v_\parallel \left(u_\parallel^{RHH} + u_{\parallel T}\right)}{v^2_{Tp}} - f_{p1} \right]. \tag{3.11c}$$

Here we have defined the velocity dependent collision frequency

$$\nu_{cx} \equiv \nu^{o}_{cx} x_p \gamma_o(x_n). \tag{3.11d}$$

Note that $(C_{cx}f_p)_o$ is even in σ and $(C_{cx}f_p)_1$ is odd. $(C_{cx}f_p)_o$ contains only known functions, $(C_{cx}f_p)_1$ resembles a Krook-operator.

This completes the specification of the collision operators.

IV. SOLUTION OF THE PROTON KINETIC EQUATION

We return to Eq. (3.2) with C_{pp} given by (3.3) and C_{cx} by (3.10), (3.11). In the banana regime ordering (3.1) the solution is[7]

$$f_1 = F + g_c + g_T,$$

with F given by (3.5b) and $\frac{\partial}{\partial\theta} g_{c,T} \equiv 0$. (C, T ... for circulating, trapped protons.) Expanding in ν/ω_b we find the usual constraint equations[7] determining $g_{c,T}$.

$$O = \left\langle \frac{\overline{\mathscr{L}}}{v_{\parallel}} (g_c+F) + f_o \frac{p}{v_{Tp}^2} + \frac{(C_{cx}f_p)_o}{\nu_{pp}v_{\parallel}} + \frac{(C_{cx}f_p)_1}{\nu_{pp}v_{\parallel}} \right\rangle \tag{4.1}$$

and

$$O = \int_{\theta_1(\lambda)}^{\theta_2(\lambda)} \frac{d\theta}{|v_{\parallel}|} \left[\overline{\mathscr{L}} g_T + \frac{(C_{cx}f_p)_o}{\nu_{pp}} \right], \tag{4.2}$$

where $\theta_{1,2}$ are the turning points of the banana orbits, satisfying $v_{\parallel} = 0$. In the derivation of (4.2) we used the even parity in σ of $(C_{cx}f_p)_1/v_{\parallel}$ vs the odd parity of $(C_{cx}f_p)_o/v_{\parallel}$. Note also, that $(C_{cx}f_p)_1$ is first order in m/e while $(C_{cx}f_p)_o$ is zero order in m/e. However, since we assume $\nu_{cx}/\nu_{pp} < 1$ we order $(C_{cx}f_p)_o$ as shown in (4.1), (4.2). For the circulating particles, the response g_{co} to the driving term $(C_{cx}f_p)_o$ is easy to calculate from (4.1). However, since $(C_{cx}f_p)_o/\nu_{pp}v_{\parallel}$ is odd in σ, g_{co} will be even in σ and can therefore not contribute to $u_{\parallel}$ or Γ. (The contribution of g_{co} to Γ, i.e., $\langle \int d^3v\, v_{Dr} g_{co} \rangle$ has the right symmetry in σ but the wrong symmetry in the poloidal angle θ, since $\partial g_{co}/\partial\theta \equiv 0$.) Consequently, we will ignore g_{co} henceforth. Similarly, for the trapped particles, the even parity of $(C_{cx}f_p)_o$ in (4.2) produces a nonzero response g_{To} which is even in σ and θ, so that again g_{To} cannot contribute to $u_{\parallel}$ and Γ and we neglect g_{To} also.

It remains to solve for g_{c1}, the response to $(C_{cx}f_p)_1$, from the equation

$$-\left\langle \frac{\bar{\mathscr{L}}}{v_{\parallel}} \right\rangle g_{c1} = \left\langle \frac{\bar{\mathscr{L}}}{v_{\parallel}} F + f_{op} \frac{p}{v_{Tp}^2} + \frac{(C_{cx}f_p)_1}{\nu_{pp} v_{\parallel}} \right\rangle , \tag{4.3}$$

with the last term given by (3.11c), where $f_{p1} = F + g_{c1}$. (We drop the subscript on g_{c1}, from now on.)

Defining the smallness parameter $\hat{\nu} \equiv \nu_{cx}/\nu_{pp}$, $q \equiv |v_{\parallel}|/v$, $\langle qh \rangle \equiv Q$, $\hat{g} \equiv g/f_{op}$, and using the identity $\langle v/v_{\parallel} \rangle = -2\sigma\, (\partial Q)/\partial\lambda$, we get

$$\frac{\partial}{\partial\lambda} \lambda \frac{\partial\hat{g}}{\partial\lambda} + \frac{Q'}{Q} \lambda \frac{\partial\hat{g}}{\partial\lambda} + \hat{\nu} \frac{Q'}{Q} \hat{g} = -\left(\frac{\sigma v}{2}\right) \frac{S_o + \hat{\nu} S_1}{Q}, \tag{4.4}$$

where

$$S_o = \frac{1}{\Omega_\theta}\frac{f'_o}{f_o} + \frac{p}{v_{Tp}^2} \qquad (4.5a)$$

$$S_1 = \frac{1}{\Omega_\theta}\frac{f'_o}{f_o} + \frac{n_p}{n_o}\frac{f_{no}}{f_{po}}\frac{\gamma_{1p}}{\gamma_{on}}\frac{u_{\|}^{RHH} + u_{\|T}}{v_{Tp}^2}\,, \qquad (4.5b)$$

and, for brevity, we wrote

$$\frac{\gamma_1(x_p)}{\gamma_o(x_n)} \equiv \frac{\gamma_{1p}}{\gamma_{on}}\,.$$

Equation (4.4) is to be solved in the subspace of circulating particles. The boundary conditions are (i) $\lambda \left.\frac{\partial\hat{g}}{\partial\lambda}\right|_{\lambda=0} = 0$ and (ii) $\hat{g}(\lambda=\lambda_c) = \hat{g}_T(\lambda_c)$ at the transition layer. In the charge exchange free case $g_T \equiv 0$.[7] As can be concluded from (4.2), g_T is of $O[(\nu_{cx}/\nu_{pp})\cdot(r/R)^{1/2}]$ (the factor $(r/R)^{1/2}$ deriving from the magnitude of the support region, i.e., the trapped particle subspace.) Short of solving the kinetic equation in the transition layer we neglect $\hat{g}_T(\lambda_c)$ and use the boundary condition

$$\hat{g}(\lambda_c) = 0. \qquad (4.6)$$

This limits our theory to neglecting $O[\nu_{cx}(r/R)^{1/2}]$ everywhere.

We solve (4.4) by iteration. For $\hat{\nu} = 0$, Eq. (4.4) reverts to the standard proton-electron problem with solution

$$\hat{g}^o = \frac{\sigma v}{2}\, S_o \int_\lambda^{\lambda_c} \frac{d\lambda'}{Q}\,. \qquad (4.7)$$

We note that S_o as given in (4.5) contains the selfconsistent value of the momentum restoring-velocity p contained in the proton-proton operator C_{pp}. In the absence of charge exchange, $p = u_{\|}^{RHH}$ results from condition (3.3b). However, for $\nu_{cx} > 0$, one cannot expand p in a power series in $\hat{\nu}$ as will become apparent. (Recalling that p is determined from $\int d^3v\, v_{\|} C_{pp}(f_p) = 0$, all processes concerning the proton momentum balance matter equally for p, magnetic particle trapping being only one. Thus, a proper expansion would be in powers of $\hat{\nu}/(r/R)^{1/2}$ rather than in powers of $\hat{\nu}$.) Thus p is treated as a free parameter, to be determined later.

Inserting $\hat{g}^o$ from Eq. (4.7) into the term $\hat{\nu}(Q'/Q)\hat{g}$ in (4.4) produces a solution

$$\hat{g} = \hat{g}^o + \hat{g}'$$

and one finds

$$\lambda \frac{\partial \hat{g}'}{\partial \lambda} = -\hat{\nu}\, \frac{\sigma v}{2} \left[S_o \int_{\lambda}^{\lambda_c} \frac{d\lambda'}{Q} - \frac{S_o}{Q} \int_0^{\lambda_c} \frac{d\lambda}{Q} + \frac{\lambda}{Q}\,(S_o + S_1) \right]. \tag{4.8}$$

For the velocity moments needed later, a knowledge of $\lambda(\partial \hat{g}/\partial \lambda)$ is sufficient. In fact, one needs only

$$\int_0^{\lambda_c} d\lambda\, \lambda\, \frac{\partial \hat{g}'}{\partial \lambda} = -\hat{\nu}\, \frac{2\sigma v}{3} \left[(S_1 + 2S_o)\left(1 - 1.46\,(r/R)^{1/2}\right) - 3S_o\left(1 - 1.95\,(r/R)^{1/2}\right) \right]$$

$$+ O\left(\frac{r}{R}\right), \tag{4.9a}$$

where the elliptic integrals of Eq. (4.8) are worked out in Appendix C. For value of completeness we list also the well known result for $\hat{g}^o$

$$\int_0^{\lambda_c} d\lambda\ \lambda \frac{\partial \hat{g}^o}{\partial \lambda} = -\frac{2\sigma v}{3} S_o\left(1 - 1.46\ (r/R)^{1/2}\right). \tag{4.9b}$$

V. CALCULATION OF $u_{\parallel}$, and $R_{\parallel cx}$

We now calculate the quantities

$$\begin{Bmatrix} u_{\parallel} \\ p \end{Bmatrix} = \left\langle \int d^3v\ v_{\parallel} \frac{f_{po}}{n_p} \begin{Bmatrix} 1 \\ \dfrac{\nu_{pp}}{\nu_*} \end{Bmatrix} (\hat{F} + H\hat{g}^o + H\hat{g}') \right\rangle, \tag{5.1}$$

using

$$d^3v = \sum_{\sigma} \frac{2\pi}{h} \frac{E\ dE\ d\lambda}{m^2 |v_{\parallel}|}$$

and

$$\int_0^{\lambda_c} d\lambda\ \hat{g} = -\int_0^{\lambda_c} \lambda \frac{\partial \hat{g}}{\partial \lambda} d\lambda$$

where we employed the boundary condition (4.6). The last integral has been worked out in (4.9).

After a lengthy calculation one finds from Eq. (5.1)

$$p = u_{\parallel}^{RHH} \frac{\hat{n}_T + \frac{\nu_{cx}^o}{\nu_{po}} K_4}{\hat{n}_T + \frac{\nu_{cx}^o}{\nu_{po}} K_3} \tag{5.2}$$

where $\hat{n}_T = 1.46\,(r/R)^{1/2}$,

$$K_3 = \frac{8}{3(\pi)^{1/2}} \int_0^{\infty} dx_p\, x_p^4 \exp(-x_p^2)\, \frac{5}{2}\, \frac{\nu_{cx}}{\nu_{po}}$$

and

$$K_4 = \frac{8}{3(\pi)^{1/2}} \int_0^{\infty} dx_p\, x_p^4 \exp(-x_n^2)\, \frac{5}{2}\, \frac{\nu_{cx}}{\nu_{po}}\, \frac{a_p^3\, \gamma_1(x_p)}{a_n^3\, \gamma_o(x_n)} .$$

For $\nu_{cx}^o = 0$, one recovers the standard result, but notice that one must keep to $O[(r/R)^{1/2}]$ although p is of $O(1)$ in the aspect ratio expansion. For $\nu_{cx}^o > 0$, (5.2) shows an expansion of p is possible only if an ordering in terms of $\Delta \equiv \nu_{cx}^o/\nu_{po}\, \hat{n}_T$ has been assumed. It is reasonable to assume that $\Delta < 1$ over most of the radial extent of the plasma, in the banana regime, and in this limit (5.2) takes on the form

$$p = u_{\parallel}^{RHH} \left[1 + \frac{\nu_{cx}^o}{\nu_{po}} (R/r)^{1/2} \frac{(K_4 - K_3)}{1.46} + \ldots \right] . \tag{5.3}$$

Finally, $u_{\parallel}$ follows from (5.1) as

$$u_{\parallel} = p\left(1 - \hat{n}_T - \frac{\nu^o_{cx}}{\nu_{po}} K_1\right) + u_{\parallel}^{RHH} \frac{\nu^o_{cx}}{\nu_{po}} K_2$$

$$- \hat{n}_T \frac{v^2_{Tp}}{\Omega_\theta}\left(A_1 + \frac{5}{2} A_2\right) \quad . \tag{5.4a}$$

where as usual[7]

$$A_1 = \frac{n'_p}{n_p} + \frac{|e|\phi'}{T_p} - \frac{3}{2}\frac{T'_p}{T_p}, \qquad A_2 = \frac{T'_p}{T_p}. \tag{5.4b}$$

For brevity, we also define

$$\overline{A}_2 = \frac{v^2_{Tp}}{\Omega_\theta} A_2 \tag{5.4c}$$

Furthermore,

$$K_1 = \frac{8}{3(\pi)^{1/2}} \int_0^\infty dx_p \; x_p^4 \; \exp(-x_p^2) \; \frac{\nu_{cx}}{\nu_{pp}}$$

$$K_2 = \frac{8}{3(\pi)^{1/2}} \int_0^\infty dx_p \; x_p^4 \; \exp(-x_n^2) \; \frac{\nu_{cx}}{\nu_{pp}} \frac{a_p^3}{a_n^3} \frac{\gamma_1(x_p)}{\gamma_o(x_n)} \quad .$$

The integrals $K_{1,2,3,4}$ are evaluated in Appendix D, as a function of T_n/T_p. Again, for $\nu^o_{cx}/\nu_{po} \ll \hat{n}_T$,

$$u_{\parallel} = u_{\parallel}^{RHH} - 1.7\,(r/R)^{1/2} \frac{T'_p}{|e|B_\theta}$$

$$- u_{\parallel}^{RHH} \frac{\nu^o_{cx}}{\nu_{po}} (K_1 - K_2 - K_3 + K_4) + O(\Delta). \tag{5.5a}$$

Compared to Eq. (148) of Ref. 7, the second term in Eq. (5.5a) has been omitted there. The third term shows the modification of the parallel flow velocity due to charge exchange. As shown in Appendix D,

$$K_1 - K_2 - K_3 + K_4 \approx 3.01 \left[1 - 0.29 \frac{T_n}{T_p} + O\left(\frac{T_n^2}{T_p^2}\right)\right]. \tag{5.5b}$$

We can now write down the proton momentum loss $R_{\parallel cx}$ due to charge exchange collisions, correct to first order in ν_{cx} and $(r/R)^{1/2}$. $R_{\parallel cx}$ consists of the classical piece derived in Section II plus two neoclassical pieces proportional to $(r/R)^{1/2}$ apparent in (3.10). Using $f_{p1} = F + Hg^o$ and Eq. (5.3) for p, it is straightforward to calculate

$$R_{\parallel cx} = \left\langle \int d^3v\, m_p v_\parallel C_{cx}(f_p) \right\rangle$$

$$= - \frac{8}{3(\pi)^{1/2}} \nu_{cx}^o m_p n_p \left\{ u_\parallel^{RHH} \left(1 + \frac{1}{2}\frac{T_n}{T_p}\right) - 1.46\,(r/R)^{1/2} \frac{T_p'}{|e|\,B_\theta} \times \left(1.67 + 0.33\,\frac{T_n}{T_p}\right)\right\}. \tag{5.6}$$

(Here, as in Section II we have evaluated the velocity integrals involving ν_{cx} only in the limit $(T_n/T_p) < 1$, see Appendix A.)

The first term in (5.6) corresponds closely to the classical result in (2.8). The second term in (5.6) will give rise to an additional proton-banana friction due to charge exchange collisions. As can be seen from the neoclassical momentum balance equation[7]

$$O = |e|\, B_\theta \Gamma_{cx} + R_{\parallel cx}.$$

both terms give rise to a charge exchange driven proton diffusion, Γ_{cx}. Comparing the diffusion due to the first term in (5.6) with the proton diffusion in the absence of charge exchange and impurity ion effects (cf. Ref. 7) we find the scaling

$$\left(\frac{\Gamma_{cx}}{\Gamma}\right)_{proton} \sim \frac{\nu^{o}_{cx}}{\nu_{pp}} (R/r)^{1/2} \, (m_i/m_e)^{1/2}, \tag{5.7}$$

a number which may exceed unity for the usual ordering $(\nu^{o}_{cx}/\nu_{pp}) < 1$.

VI. RELAXATION OF PROTON PARALLEL FLOW VELOCITY AND OF RADIAL ELECTRIC FIELD

As Equation (5.5a) has shown, the radial gradients give rise to a toroidal bulk flow of the plasma. Equation (5.5) has been derived omitting the relaxation term $(\partial f/\partial t)$ in the kinetic equation, thereby implicitly assuming that such a relaxation occurs on a time scale slower than the effective collision time in the banana regime. As can be seen from the neoclassical moment equations (cf. Ref. 7), the gradients decay on the diffusion time scale

$$\tau_{ei} \left[\frac{r_n}{\rho_{e\theta}}\right]^2 (R/r)^{1/2}$$

(r_n ... plasma radius, $\rho_{e\theta}$ ... electron gyroradius in the poloidal field, τ_{ei} ... 90° scattering time). However, due to the reflux of charge exchange neutrals and ionization on one hand and Ohmic heating on the other, the density and temperature gradients are maintained in a quasi-steady state and the decay of $u_{\parallel}$ will be determined by other mechanisms. Rosenbluth et al[15] have treated the relaxation of the parallel flow velocity due to perpendicular ion viscosity for an electron-proton plasma and found a very slow decay rate scaling as χ/r_p^2 where the viscosity $\chi \sim 0.1\ \rho_i^2 \nu_{ii}^2/\iota^2$ with ρ_i the ion gyroradius in the total field, $\nu_{ii} \sim (m_e/m_i)^{1/2}/\tau_{ei}$ and ι the rotational transform. One expects this decay to be somewhat enhanced in the presence of impurity ions, and/or nonaxisymmetric magnetic field variations[16] but here we wish to point out a much more rapid mechanism for this decay process, namely nonambipolar diffusion such as charge exchange driven proton diffusion.

We start from Poisson's equation for the radial electric field E_r,

$$\mathrm{div}\ \frac{\partial E_r}{\partial t} = -4\pi |e|\ \mathrm{div}\ (\Gamma_i - \Gamma_e),$$

or, with $\mathrm{div} = \frac{1}{r}\frac{\partial}{\partial r}\, r$ and a natural boundary condition at $r = 0$,

$$\frac{\frac{\partial E_r}{\partial t}}{B_\theta} + \frac{4\pi |e|}{B_\theta}\, (\Gamma_i - \Gamma_e) = 0, \tag{6.1}$$

showing that only nonambipolar diffusion can affect E_r. As shown in Ref. 15, for each species the radial diffusion Γ in its most general form is given by

$$\Gamma = -\frac{mr}{eB_\theta} \left\langle \int d^3v\, v_\parallel h \left[Cf - \frac{\partial f}{\partial t} - \frac{e}{m}\frac{\partial \phi}{\partial t}\frac{\partial f}{\partial \epsilon} \right] \right\rangle + O\left(\frac{m^2}{e^2}\right) \qquad (6.2)$$

where

$$E_r = -\frac{\partial \phi}{\partial r} \quad \text{and} \quad \epsilon = \frac{v^2}{2}.$$

The first term describes collisional diffusion which will be ambipolar to the extent that momentum is conserved in the collision processes under consideration. The second term is driven by $mn(\partial u_\parallel/\partial t)$, clearly much larger for the ions than for the electrons. The last term is driven by the decay of the radial plasma potential. Since, however,

$$f = f_o + f_1,$$

where f_o is even in σ and $f_1 \sim v_\parallel f_o$ vanishes for $v = 0$ and $v = \infty$, this term cannot contribute to Γ. We obtain from (6.2)

$$|e|\, B_\theta(\Gamma_i - \Gamma_e) = -\sum R_i - \sum R_e + m_i n_i \frac{\partial u_{\parallel i}}{\partial t} \qquad (6.3)$$

where in principle the sum over the friction terms $R \equiv \langle \int d^3v v_\parallel hCf \rangle m$ includes all collision processes. However, since ordinary Coulomb collision terms such as $R_{ei} + R_{ie}$ cancel, (6.3) contains only the nonambipolar momentum loss mechanisms. In this paper we single out $R_{\parallel cx}$ for the protons, neglect any further nonambipolar losses for the protons and lump all possible electron processes (e.g., pseudo-classical diffusion) into R_e^a, where the superscript stands for "anomalous." We will further assume that R_e^a does not depend explicitly on the radial electric field, so that we can treat R_e^a as an external driving term.

We combine Eqs. (6.1) and (6.3) to get

$$\frac{\dot{E}_r}{B_\theta} + \frac{\omega_{pi}^2}{\Omega_{\theta i}^2}\left(\dot{u}_{\parallel} - \frac{R_{\parallel cx} + R_e^a}{m_p n_p}\right) = 0, \tag{6.4}$$

where the dot stands for $(\partial/\partial t)$. This equation is valid on each flux surface separately. In contrast, for the problem of Ref. 15, $u_{\parallel}$ is determined by a radial diffusion equation. For Tokamak plasmas, $(\omega_{pi}^2/\Omega_{\theta i}^2) \gg 1$, showing that a small amount of nonambipolar diffusion will produce a large relaxation rate for E_r. Generally, (6.4) reveals that the relaxation process for E_r and $u_{\parallel}$ will continue until the source of nonambipolar diffusion, namely $(R_{\parallel cx} + R_a^e)$, vanishes. Specifically, we write Eq. (5.5a) for $u_{\parallel}$ as

$$u_{\parallel} = u_{\parallel}^{RHH}(1-\nu_o) - 1.17\ \hat{n}_T \bar{A}_2,$$

where $\bar{A}_2$ has been defined in (5.4c) and

$$\nu_o \equiv \frac{\nu_{cx}^o}{\nu_{po}}(K_1 - K_2 - K_3 + K_4)$$

$$\approx \frac{\nu_{cx}^o}{\nu_{po}}\, 3.01\left(1 - 0.29\frac{T_n}{T_p}\right). \tag{6.5}$$

From (5.6),

$$R_{\parallel cx} = -m_p n_p \nu_1 u_{\parallel}^{RHH} + m_p n_p \nu_2 \hat{n}_T \bar{A}_2$$

where

$$\nu_1 \equiv \frac{8}{3(\pi)^{1/2}} \nu_{cx}^o \left(1 + \frac{1}{2}\frac{T_n}{T_p}\right), \qquad \nu_2 = \frac{8}{3(\pi)^{1/2}} \nu_{cx}^o \left(1.67 + 0.33 \frac{T_n}{T_p}\right).$$

We recall that

$$u_{\parallel}^{RHH} = \frac{E_r}{B_\theta} - \frac{T_p}{|e|\, B_\theta}\left(\frac{n_p'}{n_p} - 0.17 \frac{T_p'}{T_p}\right) \equiv \frac{E_r}{B_\theta} + u_g,$$

where we have defined u_g as the gradient driven part of $u_{\parallel}^{RHH}$, for brevity.

Assuming the density and temperature gradients and the poloidal field B_θ are constant on the charge exchange time scale ν_{cx}^{-1},

$$\dot{u}_{\parallel} = \frac{\dot{E}_r}{B_\theta}(1-\nu_o)$$

and (6.4) becomes

$$\frac{\dot{E}_r}{B_\theta}(1-\nu_o) + \nu_1 \frac{E_r}{B_\theta} = -\nu_1 u_g + \nu_2 \hat{n}_T \bar{A}_2 + \frac{R_e^a}{m_p n_p} \qquad (6.6)$$

showing a fast relaxation of the radial electric field and a time asymptotic solution

$$\left.\frac{E_r}{B_\theta}\right|_\infty = -u_g + \frac{\nu_2}{\nu_1}\hat{n}_T\bar{A}_2 + \frac{R_e^a}{m_p n_p \nu_1} \tag{6.6}$$

valid after a few charge exchange times. Equation (6.6) exhibits a complete decay of $u_\parallel^{RHH} = (E_r/B_\theta) + u_g$ to zero order in the $(r/R)^{1/2}$ expansion. To first order, however, there remains an effect $\propto \bar{A}_2$, and the anomalous electron momentum loss. Inserting (6.6) in (5.5), we obtain for $t \gg (v_{cx}^o)^{-1}$

$$\left.u_\parallel\right|_\infty \approx 1.46\,(r/R)^{1/2}\,\frac{T_p'}{|e|\,B_\theta}\left[-1.17 + (1-\nu_o)\,\frac{1.67 + 0.33\,\dfrac{T_n}{T_p}}{1 + 0.5\,\dfrac{T_n}{T_p}}\right]$$

$$+ (1-\nu_o)R_e^a \Big/ m_p n_p \nu_{cx}^o\,\frac{8}{3(\pi)^{1/2}}\left(1 + \frac{1}{2}\,\frac{T_n}{T_p}\right), \tag{6.7}$$

where ν_o has been given in (6.5). Thus, in the absence of a temperature gradient, Eq. (6.7) predicts a value for $u_\parallel$ such that the proton momentum loss due to charge exchange is balanced by the anomalous electron momentum loss. Note that nonambipolar diffusion cannot relax the toroidal flow driven by $(r/R)^{1/2}$ $(\partial T_p'/\partial r)$, a "banana" term related to thermal friction. Experimental evidence of a plasma flow consistent with this term has recently been observed.[17] The analogous $O[(r/R)^{1/2}]$-term has not been kept in the analysis of Ref. 15. Moreover, comparing the present result (6.7) with the time asymptotic value for $u_\parallel$ due to relaxation by ion viscosity, one concludes that the relaxation rate due to charge exchange is fast enough to be observable in present experiments while the relaxation due to ion viscosity is not.

VII. SUMMARY AND CONCLUSIONS

We have added to the standard neoclassical transport theory in the banana regime the effect of proton charge exchange collisions with a Maxwellian population of neutral hydrogen produced by multiple charge exchange, in the limit of small charge exchange frequency to proton-proton collision frequency. A collision operator C_{cx} is derived from the Boltzmann integral, including $O[(r/R)^{1/2}]$-terms, i.e., magnetic particle trapping. Charge exchange collisions produce significant momentum and heat loss for the protons. Compared to the standard banana regime results, the corresponding particle diffusion scales as

$$\frac{\nu_{cx}^{o}}{\nu_{pp}} (R/r)^{1/2} (m_i/m_e)^{1/2}$$

and the heat loss as

$$\frac{\nu_{cx}^{o}}{\nu_{pp}} (R/r)^{1/2} \left(\frac{T_p - T_n}{T_p}\right) \frac{r_T^2}{\rho_{\theta i}^2},$$

where $T_{p,n}$ is the proton/neutral temperature, $r_T \sim [(1/T_p)(dT_p/dr)]^{-1}$ and $\rho_{\theta i}$ is the poloidal proton gyroradius. C_{cx} has an even and an odd piece in σ $(= v_{\|}/|v_{\|}|)$. The even piece produces even distortions to both the trapped and the circulating proton distribution f_p, which, however, do not contribute to odd moments of f_p such as radial diffusion Γ or parallel flow velocity $u_{\|}$, because these moments vanish on the flux surface average. The odd piece of C_{cx} produces distortions with non-vanishing flux averaged moments, thus contributing to Γ and $u_{\|}$.

Keeping terms of $O[(r/R)^{1/2}]$ in the Rosenbluth potential for C_{cx}, Γ and $u_{\parallel}$ take on "banana terms" of $O[(r/R)^{1/2}]$ besides their more easily predicted O(1) terms following from the simple estimate $R_{cx} \equiv \int d^3v\, m_p v_{\parallel} C_{cx}(f_p) \sim -m_p n_p \nu^o_{cx} u_{\parallel}$.

The nonambipolar momentum loss R_{cx} relaxes the radial electric field E_r of the toroidal plasma much more rapidly than the perpendicular ion viscosity previously invoked. On a time scale of $O[(\nu^o_{cx})^{-1}]$, E_r and $u_{\parallel}$ adjust themselves so as to annihilate the sum of $R_{\parallel cx}$ and other nonmomentum conserving friction terms such as the anomalous electron friction R^a_e. Keeping to $O[(r/R)^{1/2}]$, the time asymptotic value of $u_{\parallel}$ does not vanish but remains at a finite value $\propto (r/R)^{1/2}$, given by the first term of (6.7). Concerning R^a_e and the corresponding anomalous electron diffusion Γ^a_e, an upper limit for Γ^a_e may be determined by the steady state requirement that the anomalous loss be balanced by ionization.

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Appendix A

To find the functions γ_o and γ_1 (Eq. (2.5), (2.6)) expand $|\underline{v}-\underline{v}'|$ in spherical harmonics.[18] Using polar coordinates (v, ϕ, θ) one obtains

$$\int_0^{2\pi} d\phi' \, |\underline{v}-\underline{v}'| = 2\pi \begin{cases} v'\left[a_o\left(\frac{v}{v'}\right) + a_1\left(\frac{v}{v'}\right) \cos\theta \cos\theta' + \ldots\right], & v < v' \\ v\left[a_o\left(\frac{v'}{v}\right) + a_1\left(\frac{v'}{v}\right) \cos\theta \cos\theta' + \ldots\right], & v > v' \end{cases}$$

where

$$a_n(x) = x^n \left(\frac{x^2}{2n+3} - \frac{1}{2n-1}\right).$$

Exploiting the experimental fact $T_n/T_p \ll 1$ we expand γ_o, γ_1 in the ranges $x \to 0$, $x \to \infty$, finding

$$x \to 0 \quad x\gamma_o(x) = \frac{2}{\sqrt{\pi}}\left(1 + \frac{x^2}{3} + \ldots\right), \qquad x\gamma_1(x) = -\frac{2}{3\sqrt{\pi}}\left(1 - \frac{x^2}{5} + \ldots\right)$$

$$x \to \infty \quad x\gamma_o(x) = x\left(1 + \frac{1}{2x^2} + \ldots\right), \qquad x\gamma_1(x) = -\frac{1}{2x}\left(1 - \frac{1}{2x^2} + \ldots\right)$$

These limits can be used for the energy averages occurring in the expressions for $R_{\|cx}$ and Q_{cx}. Specifically, we use the small argument limit for $x_p\gamma_1(x_p)$ in $\mathcal{J}_n[x_p\gamma_1(x_p)]$, and the large argument limit for $x_n\gamma_o(x_n)$ in $\mathcal{J}_p[x_n\gamma_o(x_n)]$, where

$$\mathcal{J}_s \equiv \frac{8}{3\sqrt{\pi}} \int_0^\infty dx_s \, x_s^4 \, e^{-x_s^2}; \qquad s = p, n$$

Using integral tables[19] the integrals can be calculated for arbitrary values of T_n/T_p.

Appendix B

The calculation of $\mathscr{G}(F)$ proceeds along the lines of Appendix A. The integral underlying $\gamma_2(x)$ defined in (3.9) follows from differentiation with respect to the temperature of the integral underlying $\gamma_1(x)$.

The result is

$$\mathscr{G}(F) = -n_p \frac{v_\parallel}{v} \frac{v_{Tp}^2}{\Omega_\theta} 2x^2[\gamma_1(x) A_1 + \gamma_2(x) A_2],$$

where $A_{1,2}$ is defined in (5.4b).

To calculate $\mathscr{G}(Hg^o)$ we transform to the pitch angle variable[7] $\lambda = \mu B_o/E = h \sin^2 \theta$.

The stepfunction H restricts λ to $0 \leqslant \lambda \leqslant 1 - r/R$. The λ-integration leads to elliptic integrals listed in Appendix C. The energy integration requires

$$\frac{8\pi}{3} \int_0^\infty \frac{dv' \, v'^4 f_o}{a_n^2} \left\{ \begin{matrix} a_1\left(\frac{v}{v'}\right) \\ \frac{v}{v'} \, a_1\left(\frac{v'}{v}\right) \end{matrix} \right\} = 2x^2 \gamma_1(x) .$$

and

$$\int_0^\infty \frac{dv' \, v'^6 f_o}{a^4 n} \left\{ \begin{matrix} a_1\left(\frac{v}{v'}\right) \\ \frac{v}{v'} \, b_1\left(\frac{v'}{v}\right) \end{matrix} \right\} = 5[x^2 \gamma_1(x)] - x \frac{\partial}{\partial x} [x^2 \gamma_1(x)]$$

where $x = v/a$. The total result is given in (3.8a).

Appendix C

Define $I_1 \equiv \int_0^{\lambda_c} \frac{d\lambda}{\langle qh \rangle}$. As shown in Ref. 7, Eq. (80),

$$\left\langle \int_0^h \frac{d\lambda}{qh} \right\rangle - \int_0^{1-\epsilon} \frac{d\lambda}{\langle qh \rangle} = 1.95\sqrt{\epsilon} + O(\epsilon),$$

where $\epsilon = r/R$. The first integral gives $[2 + O(\epsilon)]$, where upon

$$I_1 = 2[1 - 0.975\sqrt{\epsilon}].$$

Define $I_2 \equiv \int_0^{\lambda_c} \frac{d\lambda\lambda}{\langle qh \rangle} = -\int_0^{\lambda_c} d\lambda \int_{\lambda_c}^{\lambda} \frac{d\lambda'}{\langle qh \rangle}$, where $\lambda_c = 1 - \epsilon$. With $\langle q^{-1} \rangle = -2 \frac{\partial}{\partial\lambda} \langle qh \rangle$ and $\langle q(\lambda_c) \rangle = \frac{2}{\pi}\sqrt{2\epsilon}$

$$\frac{1}{2}\int_{\lambda}^{\lambda_c} \frac{d\lambda'}{\langle qh \rangle} = \langle q(\lambda) \rangle - \langle q(\lambda_c) \rangle - \sqrt{\epsilon/2}\int_1^{M(\lambda)} dm \left[\frac{1}{\frac{2}{\pi} E(m)} - \frac{2}{\pi} K(m) \right] / m^{3/2}.$$

Here, $M(\lambda) = 2\lambda\epsilon/[1 - \lambda + \lambda\epsilon - \epsilon^2]$.

The elliptic integral has been calculated in Ref. 7 and we obtain

$$I_2 = \frac{4}{3}[1 - 1.46\sqrt{\epsilon}].$$

Appendix D

Using (3.11d), and the techniques of Appendix A,

$$K_3 = \frac{5}{2}\,\frac{8}{3\sqrt{\pi}}\int_0^\infty dx_p\, x_p^5\, e^{-x_p^2}\,\gamma_o(x_n)$$

$$= \frac{20}{3\sqrt{\pi}}\left[1 + \frac{1}{4}\frac{T_n}{T_p} + O\left(\frac{T_n^2}{T_p^2}\right)\right],$$

$$K_4 = \left(\frac{a_n}{a_p}\right)^3 \frac{20}{3\sqrt{\pi}}\int_0^\infty dx_n\, x_n^5\, e^{-x_n^2}\,\gamma_1\left(x_n\,\frac{a_n}{a_p}\right)$$

$$= -\frac{T_n}{T_p}\,\frac{5}{3\sqrt{\pi}} + O\left(\frac{T_n^2}{T_p^2}\right).$$

K_2 is of $O\left(\frac{T_n^2}{T_p^2}\right)$, which we neglect.

$$K_1 = \frac{8}{3\sqrt{\pi}}\int_0^\infty dx_p\, x_p^5\, e^{-x_p^2}\,\frac{\gamma_o(x_n)}{\hat{\nu}_{pp}(x_p)}$$

where $\hat{\nu}_{pp}(x) \equiv \dfrac{\nu_{pp}(x)}{\nu_{po}} = \dfrac{\phi - G}{x^3}$, a function we gave subsequent to Eq. (3.3a) and which is tabulated in Ref. 8. From Appendix A, $\gamma_o(x_n) \approx 1 + \left(\frac{T_n}{T_p}\right)/2x_p^2$. One finds numerically

$$c_1 = \int_0^\infty dx\, x^5\, e^{-x^2}/\hat{\nu}_{pp}(x) = 12/\pi^{1/2} = 6.77$$

$$c_2 = \int_0^\infty dx\, x^3\, e^{-x^2}/2\hat{\nu}_{pp}(x) = 1.02,$$

and $K_1 = c_1 + \dfrac{T_n}{T_p}\,c_2 + O\left(T_n^2/T_p^2\right).$

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