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TOLERANCE FACTORS FOR THE TWO-PARAMETER EXPONENTIAL DISTRIBUTION

S. A. Patil
V. R. R. Uppuluri

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TOLERANCE FACTORS FOR THE TWO-PARAMETER EXPONENTIAL DISTRIBUTION

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AUGUST 1974

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Keywords: Two-parameter exponential distribution; Tolerance Factors (β-content and β-expectation); p.d.f. of weighted sum of chi-square variables.

TOLERANCE FACTORS FOR THE TWO-PARAMETER EXPONENTIAL DISTRIBUTION

1. INTRODUCTION AND SUMMARY

The tolerance limit factors based on the assumption of a Gaussian distribution of the underlying phenomenon, are often used in practice. For instance, in 1963, Burrows [1] used such tolerance factors in the context of finding tolerance limits for some data on ultimate yield strength of hafnium ingots. More recently, at the Oak Ridge National Laboratory, W. E. Lever and J. J. Beauchamp used Gaussian tolerance factors in connection with the crack length on the studs in a high flux isotope reactor. Though the assumption of normality may be questionable, several investigators use the tables of Gaussian tolerance factors since they are readily available [8].

As a matter of fact, not much is known about the tolerance factors for other distributions with location and scale parameters. In some situations the assumption that the underlying phenomenon has an exponential distribution may be more realistic than that of a Gaussian distribution. In the case of a two-parameter exponential distribution, the theoretical aspects of the β-content tolerance factors, motivated from 'best tests', were considered by Guenther [7]. In this paper we investigate this case, and obtain formulae for the β-content upper and lower tolerance factors which are amenable for computation. For comparison, we obtain the corresponding formulae for the β-expectation tolerance factors which were previously considered by Guttman [5].

2. β-CONTENT TOLERANCE LIMIT FACTORS

Let

$$f(x;\mu,\sigma) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}, \quad x \geq \mu, \quad -\infty < \mu < \infty, \quad \sigma > 0 \tag{2.1}$$

be the probability density function of a two-parameter exponential distribution. Based on a random sample $x_1, x_2, \ldots, x_n$ of size n from this population, it is of interest to find the β-content tolerance limit factors. As in the case of the Gaussian distribution [9,3], these factors will depend on the sufficient statistics of μ and σ. It is well known [2] that $t = \min(x_1, \ldots, x_n)$ and $s = \sum_{i=1}^{n} (x_i - t)$ are sufficient for μ and σ respectively.

Definitions:

For a given significance level $0 < \alpha < 1$, and given $0 < \beta < 1$:

(1) We say that λ_u is a β-content upper tolerance limit factor, whenever

$$P\left[\int_{-\infty}^{t + \lambda_u s} f(x;\mu,\sigma)dx \geq \beta\right] \geq \alpha. \tag{2.2}$$

(2) We say that λ_ℓ is a β-content lower tolerance limit factor, whenever

$$P\left[\int_{t + \lambda_\ell s}^{\infty} f(x;\mu,\sigma)\, dx \geq \beta\right] \geq \alpha. \tag{2.3}$$

In the case of an exponential distribution, the tolerance limits associated with the above definitions, agree with the β-content tolerance limits (upper and lower) discussed by Guenther [6].

Similarly, the β-expectation tolerance limit factors may be defined by

$$E\left[\int_{-\infty}^{t+\nu_u s} f(x;\mu,\sigma)\,dx\right] \geq \beta \tag{2.4}$$

$$E\left[\int_{t+\nu_\ell s}^{\infty} f(x;\mu,\sigma)\,dx\right] \geq \alpha \tag{2.5}$$

where E denotes the expectation and ν_u, ν_ℓ refer to the upper and lower factors respectively. Guttman [5] tabulated the values of ν_ℓ, and in a later section, we will give the expressions for ν_u and ν_ℓ.

3. MATHEMATICAL PRELIMINARIES

In this section, we obtain the analytical expressions for the determination of the upper (λ_u) and lower (λ_ℓ) β-content tolerance limit factors. In order to accomplish this, we need the distributions of

$$Z = \frac{t-\mu}{\sigma} + \lambda\frac{s}{\sigma} \text{ and } Y = \frac{t-\mu}{\sigma} - \frac{\lambda s}{\sigma} \text{ for } \lambda > 0. \tag{3.1}$$

It is well known [2] that $2n\lambda(t-\mu)/\sigma$ has a chi-square distribution with 2 degrees of freedom (abbreviated as $\chi^2(2)$), and $2s/\sigma$ has a chi-square distribution $\chi^2(2n - 2)$; and these are statistically independent. We obtain closed forms for the distribution of Z and Y useful for our purpose; these do not overlap with the known distribution theory [11] of weighted chi-square variables.

Lemma 1:

The cumulative distribution function of Z, denoted by F(z) is given by

$$F(z) = 1 - \frac{e^{-nz}}{(1-n\lambda)^{n-1}} - e^{-z/\lambda} \sum_{j=0}^{n-2} \frac{(z/\lambda)^j}{j!} \left\{1 - \frac{1}{(1-n\lambda)^{n-1-j}}\right\} \text{ for } n\lambda \neq 1$$

$$1 - e^{-nz} \sum_{j=0}^{n-1} \frac{(nz)^j}{j!} \text{ for } n\lambda = 1. \tag{3.2}$$

Proof:

Let $u = (t-\mu)/\sigma$ and $v = \lambda\ s/\sigma$. Using the distributions of t and s and their independence, we can write the joint probability density function (p.d.f.) of u and v as

$$f(u,v) = \frac{n}{\lambda^{n-1}(n-2)!}\ v^{n-2} \exp\left\{-\left(nu + \frac{v}{\lambda}\right)\right\},\ 0 < u,\ v < \infty. \tag{3.3}$$

Making the transformation $z = u + v$ and $v = v$ and integrating out v, we obtain the p.d.f. of z to be

$$f(z) = \frac{n\ e^{-nz}}{\lambda^{n-1}(n-2)!} \int_0^z e^{-(\frac{1}{\lambda} - n)v}\ v^{n-2} dv. \tag{3.4}$$

From this, we get the distribution function of Z as,

$$F(z) = \frac{1}{\lambda^{n-1}(n-2)!} \int_0^z e^{-(\frac{1}{\lambda} - n)v}\ v^{n-2}(e^{-nv} - e^{-nz})dv \tag{3.5}$$

and by integration by parts we obtain (3.2).

In the next lemma, we obtain the distribution of $Y = \frac{t-\mu}{\sigma} - \frac{\lambda s}{\sigma}$, for $\lambda > 0$.

Lemma 2:

$$P[Y \leq y] = \begin{cases} 1 - \dfrac{1}{(1 + n\lambda)^{n-1}} \; e^{-ny} \;, & y \geq 0. \\ e^{-y/\lambda} \displaystyle\sum_{j=0}^{n-2} \dfrac{(y/\lambda)^j}{j!} \left\{1 - \dfrac{1}{(1 + n\lambda)^{n-j-1}}\right\} \;, & y \leq 0. \end{cases} \tag{3.6}$$

Proof:

We will obtain this result, by the use of moment generating functions and the inversion formula. Using the moment generating functions of t and s, we can write the moment generating function of Y, as

$$M(t) = E[e^{tY}] = \frac{1}{(1 - \frac{t}{n})(1 + \lambda t)^{n-1}} \tag{3.7}$$

Using the partial fraction technique, we can write this as

$$M(t) = \frac{1}{(1 - \frac{t}{n})(1 + n\lambda)^{n-1}} + \sum_{k=1}^{n-1} \frac{n\lambda}{(1 + \lambda t)^k (1 + n\lambda)^{n-k}} \tag{3.8}$$

Using the inversion formula for moment generating functions, we can write the p.d.f. of Y as,

$$f(y) = \begin{cases} \displaystyle\sum_{k=1}^{n-1} \frac{n\lambda}{(1 + n\lambda)^{n-k}} \; \frac{(-y)^{k-1} \; e^{y/\lambda}}{(k - 1)! \; \lambda^k} \;, & y \leq 0 \\ \dfrac{n}{(1 + n\lambda)^{n-1}} \; e^{-ny} \;, & y \leq 0. \end{cases} \tag{3.9}$$

Using the definition of the cumulative distribution function, and interchanging the order of summation with integration, we obtain (3.6).

4. UPPER TOLERANCES

In this section we discuss the β-content and β-expectation upper tolerance limit factors. From definition (2.2) the β-content factor λ_u is given by

$$P\left[\int_0^{t+\lambda_u s} f(x;\mu,\sigma)\,dx \geq \beta\right] \geq \alpha.$$

For the two-parameter exponential distribution (2.1), this reduces to

$$P\left[1 - \exp\left\{-\frac{t-\mu}{\sigma} - \frac{\lambda_u s}{\sigma}\right\}\right. \geq \alpha$$

$$P\left[\frac{t-\mu}{\sigma} + \frac{\lambda_u s}{\sigma} \geq -\ln(1-\beta)\right] \geq \alpha\ . \tag{4.1}$$

We note that λ_u can be either positive or negative. First we show for the range of values of practical interest such as $\alpha = .90$, $.95$, and $\beta = .8$ to 1.0, that λ_u cannot be negative, for $n \geq 2$. This will be justified by contradiction. If λ_u is < 0, then (4.1) is equivalent to

$$P\left[Y \geq -\ln(1-\beta)\right] \geq \alpha \tag{4.2}$$

where Y was defined in (3.1). Since $-\ln(1-\beta)$ is always nonnegative (for $0 \leq \beta \leq 1$), and the distribution of Y is known from (3.6); we have

$$P\left[Y \geq -\ln(1-\beta)\right] = \frac{(1-\beta)^n}{(1-n\lambda)^{n-1}} \geq \alpha. \tag{4.3}$$

Now we can see for $n \geq 1$, and $\alpha \geq .90$ and $\beta \geq .8$, and $\lambda < 0$, (4.3) is incompatible. Hence we conclude that λ_u is positive in the range of values of interest of the three parameters n, β and α. Thus λ_u is determined by using the distribution of Z given by (3.2), as follows:

$$P\left[Z = \frac{t-\mu}{\sigma} + \lambda_u \frac{s}{\sigma} \geq -\ln(1-\beta)\right] \geq \alpha. \tag{4.4}$$

Explicitly, for given α and β, we find λ such that

$$\alpha = \left(\frac{1}{1-n\lambda}\right)^{n-1} (1-\beta)^n + (1-\beta)^{1/\lambda} \sum_{j=0}^{n-2} \left(\frac{-\ln(1-\beta)}{\lambda}\right)^j \frac{1}{j!} \left\{1 - \left(\frac{1}{1-n\lambda}\right)^{n-1-j}\right\}. \tag{4.5}$$

For $n = 2$, (4.5) reduces to

$$\alpha = \frac{2\lambda}{2\lambda-1} (1-\beta)^{1/\lambda} - \frac{1}{2\lambda-1} (1-\beta)^2. \tag{4.6}$$

(4.6) could also be seen from the distribution given by Geunther [6] on page 57, in equation (11.8). Even in this simple case $n = 2$, explicit solution of λ is not possible. This may be the reason why Guenther says that there are no techniques available for the evaluation of the tolerance factors in the β-content case. Mention was made about λ_u, by Hanson and Koopmans [4], and a particular value of λ_u obtained from D. B. Owen was used to illustrate the upper tolerance limit in the case of a two-parameter exponential distribution. In a personal communication Owen informed us that there is no record available of the values of λ_u. In tables 1a and 1b we give a short table of the values of λ_u, for $3 \leq n \leq 12$, $\beta = .80, .90, .95, .99, .999$ and $\alpha = .90$, and $\alpha = .95$.

For the sake of comparison we now briefly discuss the β-expectation upper tolerance factors, ν_u, defined by the expected value

$$E\left[\int_0^{t+\nu_u s} f(x;\mu,\sigma)dx\right] \geq \beta. \tag{4.7}$$

In the case of the two-parameter exponential distribution (2.1), this reduces to

$$E\left[1 - \exp\left\{-\frac{t-\mu}{\sigma} - \frac{\nu_u s}{\sigma}\right\}\right] \geq \beta. \tag{4.8}$$

The independence of $\frac{t-\mu}{\sigma}$ and $\frac{s}{\sigma}$ reduces the left side of (4.8) to

$$1 - E\left[\exp(-\frac{t-\mu}{\sigma})\right] E\left[\exp(-\frac{\nu_u s}{\sigma})\right] = 1 - \frac{n}{n+1}\frac{1}{(1+\nu_u)^{n-1}}. \tag{4.9}$$

Solving (4.8) for ν_u leads to

$$\nu_u = \left\{\frac{n}{(n+1)(1-\beta)}\right\}^{\frac{1}{(n-1)}} - 1, \tag{4.10}$$

which is comparable with equation (11.11) given by Guenther [6], after appropriate identifications. It is not clear why Guttman [5] did not given a short table of values of ν_u given by (4.10), though he worked on all allied problems. In particular, Guttman [5] focused attention on the lower β-expectation tolerance factors which will be discussed in the next section.

5. LOWER TOLERANCES

From definition (2.3) the β-content lower tolerance limit factor λ_ℓ is given by

$$P\left[\int_{t+\lambda_\ell s}^{\infty} f(x;\mu,\sigma)\, dx \geq \beta\right] \geq \alpha.$$

For the two parameter exponential distribution (2.1) this reduces to

$$P\left[\int_{(t-\mu+\lambda_\ell s)/\sigma}^{\infty} e^{-u}\, du \geq \beta\right] \geq \alpha. \tag{5.1}$$

Once again, we note that λ_ℓ can be either positive or negative. We will first discuss the situations when λ_ℓ will be negative, and where we need

to use the distribution of Y given by (3.6). Given $\lambda_\ell \leq 0$, the random variable Y could be either positive or negative, and the left side of (5.1) becomes

$$P\left[\int_{\max(0,Y)}^{\infty} e^{-u}\, du \geq \beta\right]$$

$$= P\left[\int_{\max(0,Y)}^{\infty} e^{-u}\, du;\ Y < 0\right]$$

$$+ P\left[\int_{\max(0,Y)}^{\infty} e^{-u}\, du;\ Y \geq 0\right]$$

$$= P\left[Y < 0\right] + P\left[0 \leq Y \leq -\ln\beta\right] \geq \alpha. \tag{5.2}$$

From the distribution of Y given by (3.6), we obtain

$$1 - \frac{1}{(1-n\lambda)^{n-1}} + \frac{1-\beta^n}{(1-n\lambda)^{n-1}} \geq \alpha \tag{5.3}$$

and λ_ℓ is given by

$$\lambda_\ell = \frac{1}{n}\left[1 - \left(\frac{\beta^n}{1-\alpha}\right)^{\left(\frac{1}{n-1}\right)}\right] \tag{5.4}$$

In the above discussion, we tacitly assumed that $\lambda_\ell \leq 0$; we note that $\lambda_\ell = 0$ if and only if $n = \frac{\ln(1-\alpha)}{\ln\beta}$, (5.5)
and for values of $n > \frac{\ln(1-\alpha)}{\ln\beta}$, λ_ℓ given by (5.4) yields inconsistent values. Hence, for values of $n > \frac{\ln(1-\alpha)}{\ln\beta}$, we need to use the distribution of Z given by (3.2). Using this, we obtain λ_ℓ to be the solution of the equation

$$(1-\alpha) = \frac{1}{(1-n\lambda)^{n-1}}\beta^n + \beta^{1/\lambda}\sum_{j=0}^{n-2}\left(-\frac{\ln\beta}{\lambda}\right)^j \frac{1}{j!}\left\{1 - \left(\frac{1}{1-n\lambda}\right)^{n-j-1}\right\}. \tag{5.6}$$

In tables 2a and 2b we indicate the values of $n(\approx \frac{\ln(1-\alpha)}{\ln\beta})$, for which $\lambda = 0$, for $\beta = .80, .90, .95, .99, .999$ and $\alpha = .90$ and $\alpha = .95$. For n greater than the value tabulated in 2a and 2b, we need to use the distribution of Z and equation (5.6).

There are no guidelines of the sign of λ_ℓ in the discussion by Guenther [6]. However, for the β-expectation lower tolerance factors (in the case of a two-parameter exponential distribution), both positive and negative values of ν_ℓ were tabulated by Guttman [5], in table III. In our notation, these tabulated values can be obtained from the following discussion. The β-expectation lower tolerance factor is obtained from

$$E\left[\int_{t+\nu_\ell s}^{\infty} e^{-\frac{x-\mu}{\sigma}} \frac{dx}{\sigma}\right] \geq \beta \quad . \tag{5.7}$$

When $\nu_\ell > 0$, we have $t + \nu_\ell s > 0$, and the left side of (5.7) reduces to

$$E\left[\exp\left\{-\frac{t-\mu}{\sigma} - \frac{\nu_\ell s}{\sigma}\right\}\right] = \frac{n}{n+1}\left(\frac{1}{1+\nu_\ell}\right)^{n-1} = \beta$$

and

$$\nu_\ell = \left\{\left[\frac{n}{(n+1)\beta}\right]^{\frac{1}{n-1}} - 1\right\} . \tag{5.8}$$

When $\nu_\ell < 0$, we need a finer analysis, and we have

$$E\left[\int_{\max(0,\ t+\nu s)}^{\infty} e^{-\frac{x-\mu}{\sigma}} \frac{dx}{\sigma}\right] = E\left[P\left[0 < \frac{X-\mu}{\sigma},\ \frac{t-\mu}{\sigma} + \frac{\nu s}{\sigma} < 0\right]\right]$$

$$+ P\left[0 < \frac{t-\mu}{\sigma} + \frac{\nu s}{\sigma} < \frac{X-\mu}{\sigma}\right]\Bigg]$$

$$= 1 - \frac{1}{(1-n\nu)^{n-1}} + \frac{n}{n+1}\ \frac{1}{(1-n\nu)^{n-1}} \geq \beta. \tag{5.9}$$

Finally, ν_ℓ is obtained from

$$\nu_\ell = \frac{1}{n}\left\{1 - \left[\frac{1}{(n+1)(1-\beta)}\right]^{\frac{1}{n-1}}\right\} . \tag{5.10}$$

The values obtained from (5.8) and (5.10) agree with the values given in table III by Guttman [5] after division by n - 1.

We present the values of the β-content lower tolerance limit factors in tables 3a and 3b, for n = 3 to 12, β = .8, .9, .95, .99, .999 and α = .90 and α = .95. These are based on expressions (5.4) and (5.6).

6. ILLUSTRATION

In this section, we use the data given by Proschan [10], on the time of successive failures of the air conditioning systems in jet airplanes, and obtain the tolerance factors. One set of data (for plane number 7907) on intervals between failures is given by 194, 15, 41, 29, 33 and 181. These are failure times after an overhaul; the plane is overhauled once in 2000 hours. For this data

$$\bar{x} = 82.16, \sqrt{\sum_{i=1}^{6}(x_i-\bar{x})^2/5} = 82.12, \text{ min(data)} = t = 15,$$

$$s = \sum_{i=1}^{6}(x_i-t) = 403.$$

From formula (2.2), the β-content upper tolerance limit factor (under the assumption of a two-parameter exponential distribution) is determined from

$$P\left[\frac{1}{\sigma}\int_0^{t+\lambda s} e^{-\frac{(x-\mu)}{\sigma}}\,dx \geq \beta\right] = \alpha.$$

For $\alpha = .90$, $\beta = .90$, from table [1a], we obtain $\lambda = .882$, and the upper tolerance limit is equal to $15 + (.882)(403) = 370.44$.

The β-content lower tolerance factor λ_ℓ is determined from

$$P\left[\frac{1}{\sigma}\int_{t+\lambda_\ell s}^{\infty} e^{-\frac{(x-\mu)}{\sigma}}\,dx \geq \beta\right] = \alpha.$$

For $\alpha = .90$, $\beta = .90$, from table [3a], we obtain $\lambda_\ell = -.066$; and the lower tolerance limit is equal to $15 - (.066)(403) = -11.60$.

For numerical comparison, we find the β-expectation tolerance factors discussed in sections 4 and 5. The β-expectation upper tolerance factor ν_u is given by

$$\nu_u = \left\{\frac{n}{(n+1)(1-\beta)}\right\}^{\frac{1}{(n-1)}} - 1,$$

and for $\beta = .90$ (and $n = 6$), is equal to $\nu_u = .537$, and the upper tolerance limit is $15 + (.537)(403) = 231.41$.

The lower β-expectation tolerance factor discussed in section 5 is given by

$$\nu_\ell = \left[\frac{n}{(n+1)\beta}\right]^{\frac{1}{n-1}} - 1 \text{ for } \nu_\ell > 0$$

and

$$\nu_\ell = \frac{1}{n}\left\{1 - \left[\frac{1}{(n+1)(1-\beta)}\right]^{\frac{1}{n-1}}\right\} \text{ for } \nu_\ell < 0.$$

Since $\beta = .90$ and $n = 6$, $\beta > \frac{n}{n+1}$ which implies ν_ℓ is negative and is given by $\nu_\ell = -.01232$, and the lower tolerance limit is equal to $15 + (-.01232)(403) = 10.04$.

7. REMARKS

In the past, for most problems Gaussian tolerance factors have been used. For instance, in the Context of Ultimate Yield Strengths on 30 Hafnium specimens, Burrows [1] used the assumption of normality to discuss the problem of tolerance limits. There may be several situations where the assumption of normality is not valid. We made an attempt to make available the β-content tolerance factors in the case of a two-parameter exponential distribution. There is a need to make a theoretical comparison of these β-content factors with the β-expectation tolerance factors.

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TABLE 1a. β-content Tolerance Factors (upper), λ_u, for a Two-parameter Exponential Distribution

$\alpha = .90$

$n \backslash \beta$	.8	.9	.95	.99	.999
3	2.450	3.733	5.028	8.045	12.370
4	1.254	1.876	2.501	3.958	6.045
5	.819	1.212	1.608	2.528	3.847
6	.599	.882	1.166	1.827	2.773
7	.469	.688	.907	1.417	2.147
8	.384	.561	.739	1.151	1.742
9	.324	.472	.621	.966	1.460
10	.279	.407	.534	.830	1.254
11	.245	.356	.468	.726	1.096
12	.218	.317	.415	.644	.972

TABLE 1b. β-content Tolerance Factors (Upper) λ_u for a Two-parameter Exponential Distribution

$\alpha = .95$

$n \backslash \beta$	.8	.9	.95	.99	.999
3	3.681	5.598	7.533	12.046	18.517
4	1.696	2.533	3.375	5.337	8.149
5	1.048	1.550	2.055	3.230	4.913
6	.742	1.091	1.441	2.256	3.424
7	.567	.831	1.095	1.710	2.590
8	.456	.666	.876	1.365	2.066
9	.379	.553	.726	1.130	1.708
10	.324	.471	.618	.960	1.450
11	.282	.409	.536	.833	1.257
12	.249	.361	.473	.733	1.107

TABLE 2a. Values of $n \approx \frac{\ln(1-\alpha)}{\ln \beta}$

$\alpha = .90$

β	.8	.9	.95	.99	.999
n	10.31	21.85	44.89	229.10	2301.43

TABLE 2b. Values of $n \approx \frac{\ln(1-\alpha)}{\ln \beta}$

$\alpha = .95$

β	.8	.9	.95	.99	.999
n	13.42	28.43	58.40	298.07	2994.23

TABLE 3a. β-content Tolerance Factors (lower), λ_ℓ, for a Two-parameter Exponential Distribution

$\alpha = .90$

n \ β	.8	.9	.95	.99	0.999
3	-0.421	-0.567	-0.643	-0.705	-0.719
4	-0.150	-0.218	-0.253	-0.281	-0.288
5	- .069	- .112	- .134	-0.151	-0.155
6	- .035	- .066	- .082	-0.094	-0.097
7	- .019	- .043	- .055	-0.064	-0.067
8	- .010	-0.028	- .039	-0.047	-0.048
9	- .004	-0.020	- .029	-0.035	-0.037
10	- .000	-0.015	- .022	-0.028	-0.029
11		-0.011	- .017	-0.022	-0.023
12		-0.008	- .014	-0.018	-0.019

TABLE 3b. β-content Tolerance Factors (lower), λ_ℓ, for a Two-parameter Exponential Distribution

$\alpha = .95$

n \ β	0.8	.9	.95	.99	.999
3	-.733	-.939	-1.047	-1.135	-1.155
4	-.254	-.340	- .384	- .420	- .428
5	-.120	-.171	- .197	- .218	- .222
6	-.065	-.101	- .119	- .133	- .136
7	-.039	-.065	- .079	- .090	- .092
8	-.024	-.045	- .056	- .065	- .067
9	-.015	-.032	- .041	- .049	- .050
10	-.009	-.024	- .032	- .038	- .039
11	-.005	-.018	- .025	- .025	- .032
12	-.002	-.014	- .020	- .021	- .026

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