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REPRESENTATION OF NONLINEAR HEREDITARY MECHANICAL BEHAVIOR

J. A. Geary

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Printed in the United States of America. Available from
the U.S. Atomic Energy Commission, Technical Information Center
P.O. Box 62, Oak Ridge, Tennessee 37830
Price: Printed Copy $4.00 ; Microfiche $1.45

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ORNL-TM-4525
(ORNL-Sub-3863-1)
UC-79, -79e, -79h, -79k

Contract No. W-7405-eng-26

Reactor Division

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J. A. Geary E. T. Onat
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JUNE 1974

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REPRESENTATION OF NONLINEAR HEREDITARY MECHANICAL BEHAVIOR*

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ABSTRACT

A general theory of state variable representation of hereditary mechanical behavior is developed. The theory is appropriate to the study of finite isothermal deformations and enables one to construct, in principle, a state variable representation from observed phenomenological behavior.

1. INTRODUCTION

In classical continuum mechanics the problem of determining deformations and stresses that occur within a body subject to external stimuli rests on two separate foundations. The first is the conservation laws, which govern the space-time behavior of the *fields* of interest and are applicable to all materials.[1] The second, which is necessary to close the problem, consists of local statements (constitutive equations) about the mechanical behavior of the particular material of which the body is composed. In the simpler case of isothermal deformations the constitutive equations *specify* the relationship F that exists between the local deformation D and the local stress σ.

If the material of interest is elastic, the stress σ depends only on the deformation D existing at the time at which the stress is measured. However, if the material is not elastic, the current stress $\sigma(t)$ depends in some manner and degree on the history of local deformation [i.e., $D(\tau)$ on $\tau \leqslant t$]. Constitutive equations for an inelastic (hereditary) material must specify the precise nature of this dependence on the past.

From a mathematical point of view, a hereditary material may be regarded as an operator F that maps the set X of deformation histories of interest onto the set Y of stress responses.

It is easy to define a class of idealized materials by specifying X and Y and by assigning certain general properties to F. The class C of materials considered in this report is defined as follows (cf. Section 2): X is composed of continuous and bounded histories of the deformation gradient tensor $D(\tau)$ on $[0,\infty)$. The stress histories produced by the elements of X are assumed to be continuous and uniformly bounded. The causal operator F is time invariant and uniformly continuous, and it satisfies the principle of objectivity.[2] It will be seen that the class C contains most of the hereditary solids of interest.

The mathematical representation of the mechanical behavior of a particular material in the class C is more difficult. The task of representation involves the specification of a sequence of mathematical operations (an algorithm) that would enable one to compute the stress response of the material to any deformation history of interest.

The purpose of this report is to show that the mechanical behavior of a material in class C admits a *state variable representation* and to establish the main features of this representation. This mode of representation [cf. Eq. (1.3)] employs a system of differential equations that governs the evolution of the *state and orientation* of the material in the reduced state and orientation space as a function of the deformation applied to it. The stress is then determined as a function of the state and the orientation. The representation developed here can be constructed, in principle, with the aid of phenomenological experiments.

It should be pointed out that the subject of representation of hereditary behavior has a long history associated with the names of Maxwell, Volterra, Boltzman, and Fréchet. However, fundamental questions

*Research performed under Subcontract No. 3863 for the Oak Ridge National Laboratory, operated by Union Carbide Corporation under contract with the U.S. Atomic Energy Commission.

concerning *finite* deformations of hereditary materials have been answered only recently in a satisfactory manner. We refer here to the work of Green and Rivlin,[3] Green, Rivlin, and Spencer,[4] Noll,[5] and Coleman and Noll.[6] Although the above-mentioned work is of basic importance, the explicit representations contained herein are of the integral form. More recent developments have shown that integral representations suffer, in general, from serious drawbacks.[7-9] In contrast, representations based on state variables do not suffer from these drawbacks and offer additional conceptual and computational advantages. Section 11 presents a review and comparison of various modes of representation, together with references to past work.

Motivation for the present and previous related work[7-11] comes from the fact that experimental evidence on hereditary mechanical behavior is often found in the form of pairs of histories of deformation and stress. It is therefore natural to raise the often asked question: Can one construct a state variable representation of hereditary behavior from a study of such "input-output" pairs? In this report we provide an affirmative answer to this question in the ideal case where experimental evidence of the above type is available in unbounded quantities and in "noiseless" form. We hope that present results could be helpful in the construction of representations in realistic situations where the data are finite and "noisy." For an assessment of this expectation, see Ref. 8.

We now give a summary of the ideas presented in this report. Section 3 defines the notion of state and orientation as in Refs. 7 and 10 in terms of observable histories of deformation and stress. This definition induces an equivalence relation on the set of deformation histories on the time interval $[0,T]$ and thereby creates a quotient set. We refer to this set as the state and orientation set Σ. The set Σ is then made a metric space[10] by introducing a notion of distance based on a comparison of future responses (Section 4). It is then shown that Σ is separable, and thus it is possible to speak of the dimension of Σ. We next argue that it is reasonable to expect that Σ is finite dimensional or that it can be approximated by a finite dimensional space.

Sections 5 and 6 discuss the effect of superimposed rigid rotations. To summarize the results obtained in these sections, consider an element which is in the state and orientation $S \in \Sigma$. Rotate this element "suddenly" by an amount Q, where Q is an element of the group $O^+(3)$* of rigid-body rotations. This rotation moves the state and orientation point S to a point $P_Q S \in \Sigma$ and thereby creates a transformation P_Q on Σ. The set of all P_Q, $Q \in O^+(3)$ is a transformation group G on Σ. Section 6 shows that under certain circumstances Σ can be embedded in a finite dimensional euclidean space R_n in such a way that the image of G in R_n is a group of *linear* orthogonal transformations. This result enables us to use the theory of group representations to show that the state and orientation S of an element can be represented by a sum of irreducible *tensors of even rank:*

$$S = q_1 + \ldots + q_m \,, \tag{1.1}$$

where each q_i defines through its components an element in an invariant subspace of R_n.

A sudden rigid-body rotation Q of the element causes the state and orientation $(q_1, \ldots, q_m)$ to move to $(P_Q q_1, \ldots, P_Q q_m)$, where $P_Q q_i$ has the meaning of ordinary tensor transformation appropriate to the rank of tensor q_i(cf. Section 7).

The above results constitute a main contribution of this report. Many workers in the field assume that the internal or structural variables are tensors.[12,13] The present work provides a justification for this assumption without recourse to physical considerations related to internal structure.

*The notation $O^+(3)$ represents the group of proper orthogonal transformations in three-dimensional space, whereas O(3) represents the full orthogonal group.

In Section 8 we consider the evolution of the state and orientation of an element as a function of the applied deformation. We show that, under certain assumptions of smoothness, the rate of change of state and orientation of an element $\dot{S}(t)$ depends on its present state and orientation $S(t)$ and on the present rates of deformation and rotation (the tensors V and Ω respectively) in the following manner:

$$\dot{S}(t) = g[S(t), V] + T(\Omega)\, S\,, \tag{1.2}$$

where $T(\Omega)$ is a linear operator on R_n, whose properties are discussed in Section 8.

The complete state variable representation of mechanical behavior is composed of the following two basic statements:

$$\sigma(t) = f[S(t)]\,, \tag{1.3a}$$

$$\dot{S}(t) = g[S(t), V] + T(\Omega)\, S(t)\,. \tag{1.3b}$$

Equation (1.3*a*) asserts that the present stress is a function of the present state and orientation. Equation (1.3*b*) provides a system of differential equations for determining the state and orientation of an element when appropriate initial conditions are given. The functions f and g are subject to the invariance requirements

$$\begin{aligned} f(P_Q S) &= Q\, f(S)\, Q^T \\ g(P_Q S, QVQ^T) &= P_Q\, g(S, V) \end{aligned} \tag{1.4}$$

for all $Q \in O^+(3)$. We observe that these invariance requirements result from the principle of objectivity and are in no way a consequence of any symmetries that the material may possess.

The symmetries that an element in the state and orientation S may possess are determined by asking whether the equation

$$P_Q S = S$$

has any solutions for $Q \in O^+(3)$. In particular, if the material is isotropic in its initial state S_0, we have

$$P_Q S_0 = S_0 \tag{1.5}$$

for all $Q \in O^+(3)$.

It is interesting to observe that in the state variable representation, initial isotropy of the material plays a role only in the initial condition for Eq. (1.3*b*). This point is further discussed in Section 11.

In Section 10 we consider as an example the class of materials for which $g(S, V)$ is linear both in S and V and f is linear in S. We give an explicit representation for this material and show that linear dependence on S and V implies isotropy in a limited sense.

2. SPECIFICATION OF THE CLASS OF MATERIALS CONSIDERED

We define the class of materials of interest in terms of the behavior they exhibit under the following set of "thought" experiments. We start at time $\tau = 0$ with a supply of identical, homogeneous test specimens that are identically oriented with respect to a given fixed coordinate frame.

We shall say that these specimens are in the virgin state and orientation S_0. We imagine that a typical experiment consists of the application of a homogeneous* time-dependent isothermal deformation to a specimen and the observation of the resulting stress. A material acts, through such experiments, as an operator F that maps the deformation histories of interest into stress histories.

The applied homogeneous deformation is measured in terms of its gradients $D_{ij}(\tau)$. Thus

$$x_i = D_{ij}(\tau)\, X_j \qquad i, j = 1, 2, 3\ , \tau \in [0,\infty)\ , \tag{2.1}$$

where X_j and x_i are the initial and the current coordinates of the material point X_j in the *fixed* rectangular cartesian coordinate frame f and repeated indices imply summation. We denote by $\sigma_{ij}(\tau)$, $\tau \in [0,\infty)$ the stress components observed in the same coordinate frame. Note that σ_{ij} is a symmetric second-rank tensor, whereas D_{ij} need not be symmetric.

We introduce the short-hand notation,

$$D(\tau) \text{ and } \sigma(\tau) \qquad \tau \in [0,\infty)$$

for the tensor-valued histories $D_{ij}(\tau)$ and $\sigma_{ij}(\tau)$, $\tau \in [0,\infty)$. In the language of systems theory $D(\tau)$, $\tau \in [0,\infty)$ is an input and $\sigma(\tau)$, $\tau \in [0,\infty)$ is the corresponding output. For simplicity we shall often denote an input simply by D and an output by σ.

The input set X (i.e., the set of deformation gradient histories of interest) is composed of tensor-valued functions D defined on $[0,\infty)$. These functions are assumed to be *continuous* and *bounded* in their components. Note that, as Eq. (2.1) implies, if $D \in X$, then

$$D(0) = I \tag{2.2}$$

and

$$\det D(\tau) > 0 \qquad \tau \in [0,\infty)\ . \tag{2.3}$$

Now let L denote the linear space of all second-rank tensor histories on $[0, \infty)$ whose components are continuous and bounded, addition and scalar multiplication of elements of L being defined component wise and in the usual manner. We shall use† the following norm for L

$$\|D\| = \sup_{\tau}\, [D_{ij}(\tau)\, D_{ij}(\tau)]^{1/2} \qquad \tau \in [0,\infty)\ , \tag{2.4}$$

where $D \in L$.

---

*The classes of material considered here are "simple," so that in the study of mechanical behavior it is sufficient to apply only homogeneous deformations and to observe the resulting stresses.[1]

†Sometimes we focus the attention on deformations on $[0,T]$. For such a case we use the same norm with τ restricted on $[0,T]$.

It follows from the definition of X [cf. Eqs. 2.2 and (2.3)] that X is a subset of L.

We further assume that the output space Y is a bounded subset of L. This means that the stress response σ of the material is subject to the inequality

$$\|\sigma\| < M , \tag{2.5}$$

where M is independent of $\sigma \in Y$.

From a mathematical point of view a material in the class of interest acts (by means of tests described above) as a mapping F from X to Y:

$$\sigma = F(D) \qquad D \in X, \sigma \in Y . \tag{2.6}$$

It may be pointed out that the deformation histories that can be applied to a material element will in fact lie within a bounded subset of X. This follows from the restrictions that physically realizable experimental procedures place on the magnitude of $D(\tau)$ and on the magnitude of $\dot{D}(\tau)$, where $\dot{D}(\tau)$ is the right-hand derivative[14] of $D(\tau)$. Note also that the duration of a test will be finite.

Nevertheless the development of mathematical ideas presented in this report is considerably simplified if one works with X rather than with $\tilde{X}$, which is a restriction of a bounded subset of X to a finite time interval. One must of course make sure that the properties one assigns to the system (X, Y, F) produce a physically acceptable class of materials when the input set is restricted to $\tilde{X}$.

We complete the definition of the class of materials considered in this report by assigning the following properties to F.

Causality

As usual, we assume that the mapping F is causal. This means that the stress $\sigma(t)$ at any given time $t > 0$ depends only on the current and past values of the displacement gradients and *not* on any future value of D. Thus at a *given time* t the mapping F creates a mapping of the set $X[0,t]$ of input segments D on $[0,t]$ to the space of symmetric second-rank tensors such that

$$\sigma(t) = f_t(\underset{[0,t]}{D}) , \tag{2.7}$$

where $\underset{[0,t]}{D}$ denotes an input segment on $[0,t]$, and $\sigma(t)$ is the value of the stress at time t.

Principle of Objectivity

Next we consider the principle of objectivity[2] and its implications for F. The principle of objectivity requires that rigid-body rotations superimposed on a deforming element rotate the surface tranctions acting on the element by the same amount. In order to offer a more formal statement of the principle, consider a deformation history D which gives rise to σ. Next superimpose a time-dependent rigid-body rotation*

*For a given time τ, $Q(\tau)$ is an element of the *group* $O^+(3)$ of proper orthogonal transformations in R_3. Thus $Q(\tau)\,Q^T(\tau) = I$ and $\det Q(\tau) = +1$. We assume that the components of $Q(\tau)$, $\tau \in [0,\infty]$ are continuous and that $Q(0) = I$. Therefore $Q(\tau)$, $\tau \in [0,\infty)$ belongs to X.

$Q(\tau) \in X$ on $D(\tau)$ to obtain the composite deformation

$$Q(\tau)\, D(\tau) \qquad \tau \in [0,\infty) .$$

The principle of objectivity states that the composite deformation QD must give rise to the stress history $Q(\tau)\, \sigma(\tau)\, Q^T(\tau)$ on $\tau \in [0,\infty)$. Therefore the principle of objectivity places the following restriction on F:

$$F(QD) = Q\, F(D)\, Q^T \tag{2.8}$$

for any $D \in X$ and for any $Q \in X$ with $Q(\tau) \in O^+(3)$, $\tau \in [0,\infty)$.

Note that in terms of the functional f_t introduced by Eq. (2.7) the principle of objectivity takes the form

$$f_t(\underset{[0,t]}{QD}) = Q(t)\, f_t(\underset{[0,t]}{D})\, Q^T(t) , \tag{2.9}$$

which shows that only the value of Q at time t affects the stress at time t.

Time Invariance or Lack of Aging

We assume that the material is such that $\sigma(0) = 0$ and, moreover, in the absence of deformations the stress remains zero. Thus

$$D(\tau) = I, \quad \tau \in [0,\infty) \rightarrow \sigma(\tau) = 0, \quad \tau \in [0,\infty) . \tag{2.10}$$

We next assume, for simplicity, that the material is *nonaging*. This means that if $D \in X$ produces the stress response $\sigma \in Y$, then

$$D_1(\tau) = \begin{cases} I & \tau \in [0,t] \\ D(\tau - t) & t \leqslant \tau \end{cases} \tag{2.11}$$

must produce the output stress

$$\sigma_1(\tau) = \begin{cases} 0 & \tau \in [0,t] \\ \sigma(\tau - t) & t \leqslant \tau \end{cases}$$

for any positive t.

Smoothness of F

Next we discuss properties of smoothness that we wish to assign to the mapping F.

We shall assume that $F : X \rightarrow Y$ is *uniformly* continuous[15] in the norm (2.4); namely, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|D_1 - D_2\| < \delta \rightarrow \|F(D_1) - F(D_2)\| < \epsilon \,, \tag{2.12}$$

where D_1 and D_2 are any two elements of X.

When studying the behavior of a material (i.e., the triple X, Y, F), it is convenient to discuss separately the influence of the past and future segments of the deformation on the future behavior of the material. The reason for this division is that for many materials the future behavior may not depend "strongly" on every detail of the past deformation. We shall discuss this point in some detail here to prepare the groundwork for the definition of state introduced in the next section.

To this end, we let t denote the present time; we define the future time as

$$\xi = \tau - t, \qquad \tau \geqslant t$$

and the future relative deformation $D^*(\xi)$ by the relation

$$D(t + \xi) = D^*(\xi)\, D(t) \,, \tag{2.13}$$

where D^*D denotes the products of the matrices D^* and D. Thus D^* represents the deformation gradients measured with respect to the configuration of the material at time t. It follows from Eq. (2.13) that $D^*(\xi)$, $\xi \in [0,\infty)$ or simply D^* is an element of X.

Next we introduce, for a fixed t, the future stress

$$\sigma^*(\xi) = \sigma(t + \xi) \qquad \xi \in [0,\infty) \,. \tag{2.14}$$

We observe that $\sigma^*(0) = \sigma(t)$, so that the future stress σ^* is not necessarily an element of Y but belongs to a bounded subset Y^* of L.

We now let the material be subjected to a given deformation D on $[0,t]$. In the next section we will consider the future response of the material to all extensions of this deformation into the future. Thus we shall consider the relationship that exists between $\{D^*\}$ and $\{\sigma^*\}$ for a given $\underset{[0,t]}{D}$.

For a given past input $\underset{[0,t]}{D}$ the system mapping $F : X \rightarrow Y$ induces a mapping $F^* : X \rightarrow Y^*$ given by

$$\sigma^* = F^*(\underset{[0,t]}{D} \,; D^*) \,. \tag{2.15}$$

Thus a material element which has been subjected to the past deformation $\underset{[0,t]}{D}$ produces the future stress $F^*(\underset{[0,t]}{D} \,; D^*)$ under the future deformation D^*.

We observe that the uniform continuity of F implies that for a given $D^* \in X$ and $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|\underset{[0,t]}{D_1} - \underset{[0,t]}{D_2}\| < \delta \rightarrow \|F^*(\underset{[0,t]}{D_1} \,; D^*) - F^*(\underset{[0,t]}{D_2} \,; D^*)\| < \epsilon \tag{2.16}$$

for any $D_1, D_2 \in X[0,t]$.

We wish to assume here (for the sake of later developments) that a stronger version of Eq. (2.16) holds for F^*. Namely, for a given $\epsilon > 0$ there exists a $\delta > 0$ such that Eq. (2.16) holds for any $D^* \in X$ and for

any $D_1, D_2 \in X[0,t]$. We point out that this stronger version of Eq. (2.16) would automatically hold if X were taken to be a bounded subset of L.

It should be pointed out that the properties we assign to X, Y and F and F* define a broad class of materials. Most nonlinear viscoelastic solids would belong to this class. However, the continuity of F would exclude materials of the Voigt type. For such solids the ideas introduced below must be modified in an appropriate manner.

3. THE NOTION OF STATE AND ORIENTATION

We are now ready to define the notion of state and orientation. We shall say that two classes of specimens (which have been, respectively, subjected to deformations D_1 and D_2 on $[0,t]$) are in the same state and orientation at time t if they produce identical future stresses under identical future deformations. A more formal statement of this definition is obtained with the help of the mapping [Eq. (2.15)]:

Definition: Two deformations D_1 and D_2 on $[0,t]$, $t \geqslant 0$ are said to give rise to the same state and orientation (or belong to the same state and orientation) if and only if

$$F^*(\underset{[0,t]}{D_1}; D^*) = F^*(\underset{[0,t]}{D_2}; D^*) \tag{3.1}$$

for all future deformations $D^* \in X$.

The above definition* induces an equivalence relation on $X[0,t]$ (the set of all deformations of interest on $[0,t]$) and thereby a partition of this set into equivalence classes. We shall use the letter S to denote such a class. Thus deformations on $[0,t]$ that belong to this class S give rise to the state and orientation S.

We now let Σ^t denote the set of all *distinct* state and orientations produced at time t by deformations belonging to $X[0,t]$. Clearly Σ^t is the quotient set† of $X[0,t]$ under the equivalence relation [Eq. (3.1)]. Thus the set Σ^t is, in general, smaller than $X[0,t]$. Indeed the expectation that Σ^t will be much smaller than $X[0,t]$ makes the notion of state and orientation a potentially useful tool of representation.

Note that when t = 0, Eq. (3.1) implies that all virgin identically oriented specimens are in the same state and orientation S_0 and evidently Σ^0 is composed of only S_0.

We shall now show that the time invariance of the material (X, Y, F) enables us to work with a single state and orientation set (instead of $\cup\Sigma^t$, $t \in [0,T]$) when studying the behavior of the material over a given time interval $[0,T]$. For this purpose consider all deformations that belong to the state and orientation S in Σ^t:

$$\{\underset{[0,t]}{D}\} = S \in \Sigma^t . \tag{3.2}$$

Now consider the set of deformations $\{\underset{[0,t]}{\bar{D}}\}$, $t \leqslant T$ where each $\bar{D}$ is derived from an element of the set (3.2) by a right translation:

$$\bar{D}(\tau) = \begin{cases} I & \text{on } [0,T-t] \\ D(\tau + t - T) & \text{on } [T-t,T] \end{cases} . \tag{3.3}$$

*According to this definition, if D_1 and D_2 give rise to the same state and orientation at time t, then it is not necessary that $D_1(t) = D_2(t)$.

†Σ^t may therefore be referred to as the *reduced* state and orientation set.

It follows from the time invariance of the material [cf. Eq. (2.11)] that

$$\{ \underset{[0,T]}{\bar{D}} \} \subset \bar{S} \in \Sigma^T . \tag{3.4}$$

This observation shows that a state and orientation S produced at some time $t \in [0,T]$ can be mapped naturally *into* the state and orientation set Σ^T by a right translation of an input segment belonging to S and by the identification of the resulting state and orientation in Σ^T.

In this report we shall be interested in the representation of the mechanical behavior over the time interval $[0,T]$, where T is a fixed time which can be chosen as large as we please. For convenience we shall refer to the state and orientation set at time T as simply Σ. The states produced on $[0,T]$ will be represented by points of Σ following the procedure discussed above [cf. Eqs. (3.2) and (3.3)].

We note for future convenience that the definition [Eq. (3.1)] establishes a quotient mapping of $X[0,T]$ onto Σ

$$\mathcal{S} : X[0,T] \to \Sigma . \tag{3.5}$$

Note also that the mapping [Eq. (2.15)], the definition [Eq. (3.1)], and the above remarks concerning Σ create for a given S a mapping $F^*: X \to Y^*$ given by

$$\sigma^* = F^*(S; D^*) . \tag{3.6}$$

Thus a material element which is in the state and orientation S produces the future stress $F^*(S; D^*)$ under the future deformation $D^* \in X$.

Note for future reference that

$$F^*(\underset{[0,t]}{D} ; D^*) = F^*(S; D^*) \quad \text{if} \quad \underset{[0,t]}{D} \in S . \tag{3.7}$$

We now let S(t) denote the state and orientation of an element at time t. It follows from the definitions [and also from Eq. (3.6)] that the stress $\sigma(t)$ carried by this element at time t depends only on S(t):

$$\sigma(t) = f[S(t)], \qquad f : \Sigma \to R_6 . \tag{3.8}$$

It is also a consequence of the definitions and causality that the state and orientation of an element at time $t + \Delta t \in [0,T]$ is a function of its state and orientation at time t and of the future deformation D^* that is applied to it on the interval $[0,\Delta t]$. We may therefore write

$$S(t + \Delta t) = G[S(t) ; \underset{[0,\Delta t]}{D^*}] , \tag{3.9}$$

where $G : \Sigma \times X[0,\Delta t] \to \Sigma$.

One might say that Eqs. (3.8) and (3.9) constitute a primitive representation of mechanical behavior, but we aim to introduce coordinates for S and to replace (3.9) by a system of differential equations. We begin this task by introducing a metric for Σ.

4. INTRODUCTION OF A METRIC FOR Σ

The state and orientation set Σ defined in the previous section is as yet without a structure. For instance, no notion of distance exists between distinct points of Σ. Here we introduce a notion of distance in Σ by comparing the future behavior of test specimens.[10]

We choose as metric for Σ the function $\rho : \Sigma \times \Sigma \to R^+$, defined by

$$\rho(S_1, S_2) = \sup_{D^* \in X} \|F(S_1; D^*) - F(S_2; D^*)\| . \tag{4.1}$$

In order to see the meaning of $\rho(S_1, S_2)$ we note from (3.6) that

$$F(S_1; D^*) - F(S_2; D^*) = \sigma_1{}^*(\xi) - \sigma_2{}^*(\xi), \ \xi \in [0,\infty) ,$$

where $\sigma_1{}^*$ and $\sigma_2{}^*$ are the future stresses in specimens in S_1 and S_2, respectively, that result from the application of the same future deformation D^*. Thus $\rho(S_1, S_2)$ is the largest possible "difference" between future stress responses. It follows from the boundedness of Y and hence Y^* that the metric space Σ is bounded.

It is of course necessary to show that $\rho(S_1, S_2)$ satisfies the metric axioms.[15] That the first two axioms are satisfied follows readily from the definitions. The proof of the triangle inequality is straightforward but a little cumbersome and will not be given here.

Σ *is separable.* We first observe that the mapping

$$\mathcal{S} : X[0,T] \to \Sigma$$

is uniformly continuous (note that Σ is now a metric space). This follows almost immediately from the property [Eq. (2.16)] that we assigned to F and from the definition of state and orientation and Eqs. (3.7) and (4.1).

Now note that $X[0,T]$, the set of deformation histories on $[0,T]$ is separable. This is a consequence of the choice of norm for $X[0,T]$ [cf. Eq. (2.4)] and of the separability of $C[0,T]$.[16] Therefore Σ is a continuous image of a separable metric space and hence is separable.

It is known that the separability of Σ enables one to embed it in a normed linear space (cf. Section 6).

Finally, we wish to note for future reference that the image of a compact subset of $X[0,T]$ in Σ is compact.

5. IMAGES IN Σ OF RIGID-BODY ROTATIONS OF MATERIAL ELEMENTS

Consider a deformation history $D \in X[0,T]$ which gives rise at time T to a state and orientation $S \in \Sigma$. Next consider the time-dependent rigid-body rotation history (cf. Section 2) $Q \in X[0,T]$. Let

$$Q(T) = R . \tag{5.1}$$

Consider now the composite deformation

$$Q(\tau)\, D(\tau) \qquad \tau \in [0,T] \tag{5.2}$$

which is obtained by the superposition of Q on D.

Let S' denote the state and orientation produced at time T by the composite deformation [Eq. (5.2)].

We now assert that the principle of objectivity [cf. Eq. (2.8)] and the definition of state and orientation imply (*a*) S' does not depend on the values of $Q(\tau)$ on $[0,T)$ but only on $Q(T) = R$ [cf. Eq. (5.1)] and (*b*) if $D_1 \in X[0,T]$ is another deformation which gives rise to S, then the composite deformation

$$Q(\tau)\, D_1(\tau), \qquad Q(T) = R, \qquad \tau \in [0,T]$$

also gives rise to S'.

The proofs of these assertions are easy consequences of the definitions and will not be given here. The assertions (*a*) and (*b*) imply that S' depends only on S and $R = Q(T)$.

This observation shows that the superimposed rigid-body rotations Q subject to Eq. (5.1) applied to all $D \in X[0,T]$ create a mapping $P_R : \Sigma \to \Sigma$ given by

$$S' = P_R S\,. \tag{5.3}$$

It is convenient to think intuitively of the action of P_R on S as being induced by the application of an instantaneous rigid-body rotation R to a specimen in S, although such rotations do not belong to $X[0,T]$ (cf. Section 2).

Next we wish to show that P_R is an isometry; namely,

$$\rho(S_1, S_2) = \rho(P_R S_1, P_R S_2)\,. \tag{5.4}$$

The proof of Eq. (5.4) rests on the observation that by the principle of objectivity [cf. Eq. (2.8)] and the definition of P_R, we have for a constant $R \in O^+(3)$:

$$F^*(P_R S; RD^*R^T) = RF^*(S; D^*)R^T\,. \tag{5.5}$$

Therefore

$$\begin{aligned} \|F^*(P_R S_1; RD^*R^T) - F^*(P_R S_2; RD^*R^T)\| &= \|R[F^*(S_1; D^*) - F^*(S_2; D^*)]\, R^T\| \\ &= \|R(\sigma_1{}^* - \sigma_2{}^*)R^T\|\,, \end{aligned} \tag{5.6}$$

where $\sigma_1{}^*$ and $\sigma_2{}^*$ are the future stresses caused by D^* in the elements in S_1 and S_2 respectively [cf. Eq. (3.8)]. But by definition of the norm [Eq. (2.4)],

$$\|R(\sigma_1{}^* - \sigma_2{}^*)R^T\| = \|\sigma_1{}^* - \sigma_2{}^*\|\,, \tag{5.7}$$

(5.4) then follows from the definition (4.1) and from (5.6) and (5.7).

We wish to note that the set $\{P_R\}$ with $R \in O^+(3)$ is a group G, where the group multiplication is the product

$$P_R P_Q \tag{5.8}$$

of the transformations P_Q and P_R. It follows from the definition of P_R that

$$P_R P_Q = P_{RQ} , \tag{5.9}$$

which shows that $\{P_R\}$ is closed under the operation (5.8). The associative property of group multiplication follows from that of $O^+(3)$ and Eq. (5.9). The identity element of G is simply P_I, where I is the identity in $O^+(3)$. The inverse of P_Q is P_{Q^T} [Eq. (5.9)].

Since P_Q is an isometry and since its inverse exists, it follows that P_Q is a homeomorphism of Σ onto itself.

The mapping $\kappa : O^+(3) \to G$ given by

$$\kappa(Q) = P_Q \tag{5.10}$$

is, in view of Eq. (5.9), a *homomorphism.* We shall assume that for the class of materials considered, the homomorphism κ is one-to-one. (This means that κ is an *isomorphism.*)

It is desirable to explore the physical meaning of the last assumption.* Observe that if κ is not one-to-one, there would exist two different rigid-body rotations Q_1 and Q_2 such that

$$P_{Q_1} = P_{Q_2} .$$

This would imply that

$$S = P_{Q_1^T Q_2} S \quad \text{for any } S \in \Sigma . \tag{5.11}$$

If the stress $\sigma = f(S)$ associated with some $S \in \Sigma$ has *three distinct principal* values, then one could show, using the considerations associated with Eqs. (6.2), (6.3), and (6.4), that a point on the orbit of S [namely, on $\{P_Q S: Q \in O^+(3)\}$] would violate Eq. (5.11). Thus we conclude that if the state and orientation space Σ has a point S such that $\sigma = f(S)$ has three distinct principal values, then the mapping κ must be one-to-one. It is now clear that the above assumption is equivalent to the very mild assumption that the material must carry at some state of its deformation a stress tensor with three distinct principal values.

We note for future reference that since κ is an isomorphism and since $O^+(3)$ is a compact Lie group,[17] G is also a compact Lie group.

We now introduce an alternative interpretation of P_R which should serve to increase our understanding of P_R, and will enable us to extend the domain of definition of κ [cf. (5.10)] to O(3), the *full* group of orthogonal transformations on R_3. This extension will enable us to assert later that the state variables must be *even*-rank tensors. This extension could not be done without the alternative interpretation since a continuum cannot be subjected to an improper orthogonal transformation.

To achieve the extension we let a future deformation $D^* \in X$, applied to an element in S, produce the future stress $\sigma^* \in Y^*$. Call this test (*a*). Let R be a constant element of $O^+(3)$; then RD^*R^T applied to an element in $P_R S$ will produce the future stress $R\sigma^*R^T$ [cf. (5.5)]. Call this test (*b*).

---

*This assumption may seem to leave elastic fluids outside the scope of this report. (For such materials, $P_Q S = S$ for all $Q \in O^+(3)$ and for all $S \in \Sigma$.) However, it is easily seen that the case where the state variables q_i are all scalars (cf. Section 7) corresponds to elastic fluids.

Consider now a new rectangular cartesian coordinate frame f_R such that if x'_i and x_i are the coordinates of a given point in f_R and f respectively, then

$$x'_i = R_{ij}x_j .$$

Since D^* and σ^* are histories of second-rank tensors, test (*a*), when observed in f_R, would be seen as the application of RD^*R^T and the production of $R\sigma^*R^T$. Thus test (*a*) observed in f_R will be indistinguishable from test (*b*) observed in f.

Therefore, an observer in f_R would decide that test (*a*) represents the behavior of an element in $P_RS \in \Sigma$.

Hence we have the following alternative interpretation of P_R: If a material specimen is judged to be in the state and orientation S by an observer in f, then an observer in f_R would judge the same specimen to be in P_RS.

This alternative interpretation enables us to define P_R in the case where R is an *improper* orthogonal transformation ($RR^T = I$, $\det R = -1$). Clearly for such an R, $-R \in O^+(3)$. Since

$$(-R)D^*(-R^T) = RD^*R^T \text{ and } (-R)\sigma^*(-R^T) = R\sigma^*R^T ,$$

it follows from the above interpretation of P_R that

$$P_R = P_{-R}, \qquad R \in O(3) . \tag{5.12}$$

Symmetries of a Deforming Element

Let S_0 denote the state and orientation of a *virgin* element. If a material is isotropic in its initial state, then*

$$P_Q S_0 = S_0 \quad \text{for all} \quad Q \in O^+(3) . \tag{5.13}$$

This observation is a consequence of the usual definition of isotropy.[1] Equation (5.13) implies that there are no preferred directions (initially) within an isotropic material.

If a specimen is not fully isotropic in its virgin state,

$$P_Q S_0 = S_0, \qquad Q \in g_{S_0} \subset O^+(3) ,$$

where g_{S_0} is a subgroup of $O^+(3)$ and is said to be the isotropy subgroup of the specimen in the state and orientation S_0. (Note that for some materials g_{S_0} consists only of the identity transformation I.)

We can, in a similar manner, define the isotropy group of an arbitrary state and orientation $S \in \Sigma$. Thus, if for $S \in \Sigma$,

$$P_Q S = S, \qquad Q \in g_S \subset O^+(3) ,$$

we say the element in S has the isotropy subgroup g_S.

---

*In view of (5.12), (5.13) implies $P_Q S_0 = S_0$, $Q \in O(3)$.

We expect that during the course of deformation, an initially isotropic specimen will, in general, lose some or all of its symmetry. If this happens, g_S will become smaller.

It will be of interest to compare symmetries of material elements in different state and orientations. To this end, let S_1 and S_2 have isotropy subgroups g_{S_1} and g_{S_2} respectively.

It is said that these are conjugate subgroups if there exists $H \in O^+(3)$ such that

$$H g_{S_1} H^T = g_{S_2} \, . \tag{5.14}$$

This implies that

$$g_{P_H S_1} = g_{S_2} \, ,$$

which shows that if Eq. (5.14) holds, S_1 and S_2 have the same symmetries when S_1 is rotated by an amount H.

It is of interest to know the number of state and orientations which have nonconjugate isotropy subgroups. We shall see that if the number of such isotropy subgroups is finite, a powerful theorem due to Mostow[18] will enable us to embed Σ equivariantly in a finite dimensional euclidean space when Σ is finite dimensional.

It follows from Ref. 19 that if Σ is a compact manifold, the number of nonconjugate isotropy subgroups is finite in view of the fact that G is a compact Lie group of homeomorphisms.

6. THE EQUIVARIANT EMBEDDING OF Σ IN EUCLIDEAN SPACE

Before we discuss the details of the embedding of Σ in a normed linear space, it is desirable to offer a brief discussion of the motivation behind such a move. The basic aims of the remainder of this report are (1) to introduce coordinates for the state and orientation S and (2) to derive a system of differential equations that govern the evolution of S.

As we shall see, these aims can be achieved if the state and orientation space Σ can be mapped into a normed linear space N in an appropriate manner. Obviously, this mapping should be invertible and should be smooth enough so that, for instance, the images of curves under this mapping remain curves.

Clearly we want to use a homeomorphism ϕ (i.e., a one-to-one and bicontinuous mapping of Σ onto $\hat{\Sigma} \subset N$). Furthermore, the study of the laws of evolution of S requires that the image $\theta(P_R)$ of P_R (cf. Section 7) in $\hat{\Sigma}$ must be specified explicitly. The transformation $\theta(P_R)$ on $\hat{\Sigma}$ takes the points $\phi(S)$ to $\phi(P_R S)$:

$$\phi(P_R S) = \theta(P_R)\, \phi(S) \, . \tag{6.1}$$

It is easily seen that the set of transformations $\{\theta(P_R)\}$ as P_R ranges over G is a group which is *isomorphic* to G.

If ϕ is such that $\theta(P_R)$; $P_R \in G$, $R \in O^+(3)$ is the restriction to $\hat{\Sigma}$ of a linear transformation on N, then the theory of group representations can be employed to describe explicitly the action of $\theta(P_R)$ in $\hat{\Sigma}$ (cf. Sections 7 and 8). An embedding which has the above mentioned property is called an equivariant embedding. Thus an equivariant embedding is a pair (ϕ,θ), where ϕ is a homeomorphism of Σ into N and θ is an isomorphism of $G = \{P_R\}$ into the linear group on N such that Eq. (6.1) holds.

It is desirable to know circumstances under which an equivariant embedding (ϕ,θ) will exist for Σ and G as defined in this report.

Two theorems due to Mostow[18,19] are central to the question of the existence of equivariant embeddings.

Theorem 1. Let G be a compact Lie group operating faithfully on a separable metric finite dimensional space Σ. If Σ has at most a finite number of nonconjugate isotropy subgroups under G, there is a homeomorphism ϕ of Σ into R_n and an isomorphism θ of G into the orthogonal group* on R_n such that ϕ is equivariant with respect to θ [i.e., Eq. (6.1) holds].

Theorem 2. Let G be a compact Lie group operating on a compact manifold M. Then M has at most a finite number of nonconjugate isotropy subgroups under G.

In the present case G is a compact Lie group, and Σ is a separable metric space (cf. Sections 4 and 5). However, in order to apply theorem 1 it is necessary that two further conditions be satisfied: namely, that Σ be finite dimensional and that Σ have at most a finite number of nonconjugate isotropy subgroups under G.

It might be argued that the finite dimensionality of Σ might reasonably be assumed because the number of particles in a material specimen is finite and each has only a finite number of degrees of freedom.

In any event, in the present work we shall assume that Σ is finite dimensional.†

To determine whether Σ has a finite number of nonconjugate isotropy subgroups under G, we observe first that $O^+(3)$ has an infinite number of nonconjugate subgroups (e.g.. subgroups generated by rotations of $2\pi/n$ about a fixed axis, $n = 1, 2, \ldots$). Therefore since G is isomorphic to $O^+(3)$, it is possible that Σ could have an infinite number of nonconjugate isotropy subgroups under G.

It is clear that whether Σ has a finite number of nonconjugate isotropy subgroups will depend on the material. For instance, if the material is completely anisotropic in its initial state S_0 (i.e., $P_Q S_0 = S_0$ only when $Q = 1$) and if it remains completely anisotropic during its deformation, there is only one (trivial) isotropy subgroup for Σ.

However, we often study materials that exhibit symmetries both in their initial and deformed states. The question therefore arises as to whether we can create (by subjecting material specimens to appropriate deformations) state and orientations that have an infinite number of nonconjugate isotropy subgroups.

To investigate this question, it is helpful to consider the stress σ carried by a deforming material element. We have already observed (cf. Section 3) that

$$\sigma = f(S) , \tag{6.2}$$

where the tensor-valued function f is subject to the invariance requirement‡

$$f(P_Q S) = Q\, f(S)\, Q^T \qquad Q \in O^+(3) . \tag{6.3}$$

Evidently if $Q \in g_S$, the isotropy subgroup of the state and orientation S, then $P_Q S = S$ and consequently, in view of Eq. (6.3),

$$\sigma = Q\sigma Q^T .$$

---

*Note that the theorem asserts that $\theta(P_R)$ is not only linear but also orthogonal.

†Even if Σ is infinite dimensional for a given material, it might be possible to represent the mechanical behavior of that material with sufficient accuracy by a finite dimensional state and orientation space.

‡This requirement follows immediately from the principle of objectivity and the definition of state and orientation.

It follows therefore that

$$g_S \subset g_\sigma \,, \tag{6.4}$$

where

$$g_\sigma = \{Q : Q \in O^+(3), Q\sigma Q^T = \sigma\} \,.$$

The group g_σ depends on the principal values of σ. If the three principal values of σ are distinct, then g_σ is the group (of order 4) generated by 180° rotations about the principal axes of σ. Thus if we let Σ_1 denote the set of points in Σ whose associated stresses have three distinct principal values, then in view of Eq. (6.4) we can make the following observation. Namely, if

$$S \in \Sigma_1 \,,$$

then g_S is conjugate to one of the following three groups: identity, 180° rotations about an axis, and 180° rotations about three mutually orthogonal axes. This means, of course, that the points in Σ_1 have a finite number of nonconjugate subgroups. Hence we can apply Mostow's theorem 1 to Σ_1 (or $\Sigma_1 \cup S_0$, where S_0 is the initial state and orientation of the material*) when Σ is finite dimensional. Therefore, it is clear that if we restrict attention to deformation histories in $X[0,T]$ that give rise to stresses having three distinct principal values for $\tau > 0$, the corresponding state and orientations can be embedded equivariantly in R_n if Σ is finite dimensional.

We must now discuss the points in $\Sigma - \Sigma_1$, where each point will have an associated stress with at least two equal principal values. In this case g_σ will be either $O^+(3)$ or will be conjugate to $O^+(2)$, the group of rotations about an axis. Since $O^+(3)$ and $O^+(2)$ have an infinite number of nonconjugate subgroups, we cannot conclude on the basis of Eq. (6.4) that the points in $\Sigma - \Sigma_1$, and hence in Σ, have a finite number of nonconjugate isotropy subgroups.

One could of course add to Σ_1 all other state and orientations that have a finite number of nonconjugate isotropy subgroups and then apply the embedding theorem 1. It may be that for a given class of materials such additions would enable one to use the results of Sections 7 and 8 for all deformation histories of practical interest.

There may exist also a class of materials for which the state and orientation space Σ is a *compact manifold.* We have seen that for the class of materials considered here, Σ is a bounded separable metric space, but we cannot assert that Σ is a manifold. If such were the case, theorems 2 and 1 would show that an equivariant embedding of Σ in euclidean space is possible.

It is desirable to know the properties that must be assigned to the mapping F and the input and output spaces X and Y which define the class of materials of interest such that the resulting state and orientation space is a compact manifold. Although we have considered some aspects of this problem we cannot report any real progress in this subject. However, it may be useful to discuss a particular case. It is easily seen that one can choose a subset M of $X[0,T]$ such that M is compact and right translations† of $D \in M$ are also in M.

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*Note that the initial state is stress free and therefore does not belong to Σ_1.

†This requirement assures us that the states created on $[0,T]$ by a $D \in M$ lie on the image Σ_M of M under $\mathcal{E}$.

An example of such a set is the deformation histories D in $X[0,T]$ that satisfy

$$|D_{ij}(\tau_1) - D_{ij}(\tau_2)| \leqslant m\,|\tau_1 - \tau_2| \qquad i,j = 1,2,3\,,$$

where m is a *fixed* positive constant. It may be noted that this set would contain all deformation histories that can be employed in actual tests.

Now as noted before, the mapping $\mathcal{S} : X[0,T] \to \Sigma$ is continuous and therefore Σ_M, the image of M under $\mathcal{S}$, is compact. We have therefore shown that tests of practical interest give rise to a *compact* state and orientation space Σ_M. The question as to whether this compact set is a manifold must now be discussed.* Considerations based on the construction of Σ show that we could expect Σ_M to be at most a manifold with a *boundary.*[20] Further discussion of this case would require, among other considerations, whether theorem 2 is applicable to a compact manifold with boundary.

7. REPRESENTATION OF STATE AND ORIENTATION BY IRREDUCIBLE TENSORS

We shall assume that Σ has been equivariantly embedded in R_n. For reasons of economy in notation we shall use the same symbols to denote the embedded entities, so that Σ will stand for $\phi(\Sigma)$ and P_Q for $\theta(P_Q)$ (cf. Section 6). Note that in view of Mostow's theorem 1, P_Q is now a linear *orthogonal* transformation on the entire space R_n. Therefore $G = \{P_Q; Q \in O^+(3)\}$ constitutes a linear representation of $O^+(3)$ and the following observations hold:[21]

1. R_n is the direct sum of mutually orthogonal subspaces $M_1, \ldots, M_m$.
2. Each M_i is of odd dimension.
3. Each M_i is invariant under the action of G, namely, if $S \in M_i$, then $P_Q S \in M_i$, $P_Q \in G$.

These remarks imply that each $S \in R_n$ may be expressed uniquely in the form

$$S = q_1 + q_2 + \ldots + q_m,\ q_i \in M_i$$

and that (7.1)

$$P_Q S = P_Q q_1 + \ldots + P_Q q_m,\ P_Q q_i \in M_i.$$

Here the M_i values are chosen in such a way that they have no *proper* subspaces that are invariant under G. Thus the restriction of G to each M_i is an *irreducible* representation of $O^+(3)$ and therefore each M_i is of odd dimension as noted above.

The next task is to specify the action of G on each M_i. This can be done clearly and economically with the aid of *irreducible tensors.*[17] We illustrate this point by the following example.

Suppose that M_1 is five dimensional. Consider now the nine-dimensional space T_2 of all second-rank tensors r_{ij} with i,j = 1, 2, 3. An orthogonal transformation P_Q on T_2 has the form

$$(P_Q r)_{ij} = Q_{ip} Q_{jq} r_{pq},\ Q \in O^+(3)\,. \tag{7.2}$$

---

*Note that $P_Q \Sigma_M$, $Q \in O^+(3)$ is also *compact.*

The space of symmetric traceless second-rank tensors $T_2{}^S$ is a five-dimensional subspace of T_2. If $r \in T_2{}^S$, then $P_Q r$ is also given by Eq. (7.2) and is again in $T_2{}^S$. When (7.2) is restricted to $T_2{}^S$, it provides an irreducible five-dimensional representation of $O^+(3)$. This representation of $O^+(3)$ is isomorphic to that given by G on M_1 (cf. Ref. 17). Therefore, we may take M_1 to be $T_2{}^S$. This procedure, which defines the action of P_Q on M_1 by (7.2), may be generalized to invariant subspace of dimensions 1, 3, 7, 9, ...

In particular, one-dimensional representations are given by scalars and three-dimensional representations by antisymmetric second-rank tensors. Irreducible representations of dimension seven and nine can be obtained from fourth-rank tensors which satisfy certain conditions of symmetry.[17,21] Representations of still higher dimension are given by tensors of higher rank and appropriate symmetry. Such tensors are called irreducible tensors.[17] In view of these remarks we will take each M_i to be a space of irreducible tensors, and the action of the P_Q on M_i will then be described by a tensor transformation appropriate to the rank.

In Section 5 we introduced an alternative interpretation for P_Q and used this alternative interpretation to extend P_Q to the full orthogonal group O(3). As a result of this interpretation we have

$$P_{-Q} = P_Q \, . \tag{7.3}$$

If (7.2) is to hold for tensor representations of actions of P_Q in M_i, then it is necessary that the tensors involved be of *even rank.* It may be useful to summarize the above ideas in a less formal manner.

We have shown that the state and orientation of a material element can be represented* by a number of even-rank irreducible tensors $q_1, \ldots, q_m$. A sudden rigid-body rotation Q of the element causes the state and orientation $(q_1, \ldots, q_m)$ to move to $(P_Q q_1, \ldots, P_Q q_m)$, where $P_Q q_i$ has the meaning of ordinary tensor transformation appropriate to the rank of the tensor q_i.

It may be pointed out that many workers[12,13] in the field assume that the internal variables are tensors. This assumption is often introduced on "physical" grounds. In this report we arrived at the above summarized results by the mathematical structure developed in Sections 2 to 6.

8. GROWTH LAWS

We are finally in a position to discuss the laws of evolution (or growth) of the state and orientation of an element as a function of the applied deformation.

We recall from Section 3 that if S(t) and $S(t + \Delta t)$ denote the state and orientation of a material specimen at times t and $t + \Delta t$, respectively, then $S(t + \Delta t)$ can depend only on S(t) and on the future deformation D* on $[0,\Delta t]$. We therefore wrote

$$S(t + \Delta t) = G[S(t); \underset{[0,\Delta t]}{D^*}] \, , \tag{8.1}$$

where $G : \Sigma \times X[0,T] \to \Sigma$. This mapping may be called a primitive growth law for the state and orientation.

We now investigate the possibility of replacing Eq. (8.1) by a system of differential equations. We recall that as a result of the equivariant embedding discussed in Section 6, the state and orientation space Σ is a subset of R_n, and consequently S(t) and $S(t + \Delta t)$ are vectors in R_n. It is therefore meaningful to talk about $S(t + \Delta t) - S(t)$ and about $\dot{S}(t)$, the time derivative of S(t).

---

*If Σ can be equivariantly embedded into a finite dimensional euclidean space.

Observe that if Δt is sufficiently small and if $D^*(\xi)$ is sufficiently smooth, we can write

$$D^*(\xi) = I + \dot{D}^*(0)\,\xi + \ldots \text{ on } [0,\Delta t]\ , \tag{8.2}$$

where $\dot{D}^*(0)$ is the right derivative of $D^*(\xi)$ at $\xi = 0$.

Observe from Eq. (2.13) that

$$\dot{D}^*(0) = \text{grad } v\ , \tag{8.3}$$

where v denotes the velocity field of the deformation at time t. The symmetric part V of grad v is called the rate of deformation tensor and the antisymmetric part Ω is called the rate of rotation tensor.[22] Thus

$$\dot{D}^*(0) = V + \Omega\ . \tag{8.4}$$

In view of Eqs. (8.1), (8.2), and (8.4), it would be reasonable to expect that for small Δt, $S(t + \Delta t)$ will be a function of $S(t)$ and of $V + \Omega$. In this case $\dot{S}(t)$, the right-hand derivative of $S(t)$, would depend on $S(t)$ and on $V + \Omega$:

$$\dot{S}(t) = g(S(t); V + \Omega). \tag{8.5}$$

Indeed it can be shown that if the mapping $\mathcal{S} : X[0,T] \to \Sigma$ is Lipschitz continuous, and if $\dot{S}(t)$ exists, then Eq. (8.5) must hold.[22] Reference 23 also asserts that if $\mathcal{S}$ is differentiable, then $\dot{S}(t)$ exists.

We now show that as a consequence of the principle of objectivity the differential growth law [Eq. (8.5)] must be of the form (1.2), namely,

$$\dot{S}(t) = g(S(t), V) + T(\Omega)\, S(t)\ , \tag{8.6}$$

where g is a form invariant function of its arguments and T is a linear operator on R_n that depends on the rate of rotation Ω at time t in the manner described below.

We first consider the derivative with respect to τ of $P_{Q(\tau)}S$ at time t. Here S is a given element of Σ and

$$Q(\tau) \in O^+(3) \ \text{ with } \ Q(t) = I. \tag{8.7}$$

This derivative depends upon $\dot{Q}(t)$ and S in the following manner:

$$[P_{Q(\tau)}S]^{\cdot} = T(\dot{Q})S\ , \tag{8.8}$$

where $T(\dot{Q})$ is a *linear* operator on R_n that depends on $\dot{Q}$. Furthermore $T : R_n \to R_n$ has the following properties:

1. T is skew symmetric: $T + T^T = 0$.
2. The subspaces M_i that are invariant under $P_Q \in G$ are also invariant under T.
3. T depends linearly on $\dot{Q}$.
4. $P_R\, T(\dot{Q}) = T(R\dot{Q}R^T)P_R$.

The proof of Eq. (8.7) and assertions 1 to 4 rest on the tensor representation of S [cf. Eq. (7.1)] :

$$P_{Q(\tau)} = \sum_{i=1}^{m} P_{Q(\tau)} q_i \,, \tag{8.9}$$

where $P_{Q(\tau)} q_i$ are appropriate tensor transformations induced by $Q(\tau)$.

The time derivative of the right-hand side of Eq. (8.8) can be computed easily to arrive at Eq. (8.7) and the assertions listed above. As an example of such a computation, consider $P_{Q(\tau)} r$, where r is a second-rank symmetric tensor:

$$P_{Q(\tau)} r = Q(\tau)\, r\, Q^T(\tau) \tag{8.10}$$

and therefore

$$(P_{Q(\tau)} r)^{\cdot} = \dot{Q} r - r \dot{Q} \,, \tag{8.11}$$

where we used the facts that $Q(t) = I$ (cf. 8.6) and $\dot{Q}$ is antisymmetric.

Now observe that the right-hand side of Eq. (8.10) defines a *linear* operator on the space of symmetric second-rank tensors. Moreover, this operator has the properties 1, 3, and 4 listed above. Similar considerations for higher rank tensors lead to Eq. (8.7) and properties 1 to 4.

We are now ready to prove that Eq. (8.5) must be of the form (8.6). Let an element in the state and orientation S(t) be subjected to the future deformation D* to produce the state and orientation $S(\tau)$:

$$D^* \rightarrow S(\tau) \quad \tau \geqslant t \,. \tag{8.12}$$

Now consider the application of the future deformation $R(\tau)\, D^*(\tau - t)$, $\tau > t$, with $R(t) = I$ to a specimen in S(t). By the principle of objectivity the state and orientation produced by this composite deformation will be $P_{R(\tau)}\, S(\tau)$:

$$RD^* \rightarrow P_{R(\tau)}\, S(\tau), \tau \geqslant t; \;\; R(t) = I \,. \tag{8.13}$$

On the other hand, at $\tau = t$

$$\dot{D}^* = V + \Omega$$

and (8.14)

$$(RD^*)^{\cdot} = V + \Omega + \dot{R}(t) \,.$$

Now, applying (8.5) to deformations associated with (8.12) and (8.13), we obtain

$$\dot{S}(t) = g[S(t), V + \Omega]$$

and (8.15)

$$(P_R S)^{\cdot}_{\tau=t} = g[S(t), V + \Omega + \dot{R}] \,.$$

But

$$(P_R S)^{\cdot}_{\tau=t} = \frac{d}{d\tau}\,[P_{R(\tau)}\,S(t)]_{\tau=t} + \frac{d}{d\tau}\,[P_{R(t)}\,S(\tau)]_{\tau=t}\,. \tag{8.16}$$

In view of Eq. (8.18), and since $R(t) = I$, Eqs. (8.15) and (8.16) yield

$$g[S(t), V + \Omega + \dot{R}] = T(\dot{R})\,S(t) + g[S(t), V + \Omega]\,. \tag{8.17}$$

Since $\dot{R}(t)$ is an arbitrary antisymmetric second-rank tensor, we can choose it to be $-\Omega$, in which case (8.16) produces the desired result:*

$$g[S(t), V + \Omega] = g[S(t), V] + T(\Omega)\,S(t)\,. \tag{8.18}$$

Next we must show that $g(S,V)$ is form invariant:

$$g(P_Q S, QVQ^T) = P_Q g(S, V) \qquad Q \in O^+(3)\,. \tag{8.19}$$

For this purpose we observe again from the principle of objectivity that if the rate of deformation $(V + \Omega)$ gives rise to $\dot{S}(t)$ when applied to an element in $S(t)$, then the rate of deformation $Q(V + \Omega)\,Q^T$ gives rise to $P_Q\,\dot{S}(t)$ when applied to an element in $P_Q\,S(t)$. Thus in view of Eq. (8.17)

$$\dot{S}(t) = g[S(t), V] + T(\Omega)\,S(t)$$

and

$$P_Q\,\dot{S}(t) = g(P_Q\,S(t), QVQ^T) + T(Q\,\Omega\,Q^T)P_Q\,S(t)\,.$$

Now using the property 4 of T and comparing the above equations we obtain Eq. (8.19).

We have thus constructed the state variable representation of mechanical behavior for the class of materials considered here. It may be useful to summarize this representation.

The state and orientation S of an element is a vector in R_n, and it can be represented as a sum of *even-rank irreducible* tensors q_i:

$$S = q_1 + \ldots + q_m\,.$$

The stress σ carried by an element is a function only of its state and orientation S:

$$\sigma = f(S)\,.$$

---

*Observe that $T(-\Omega) = -T(\Omega)$ in view of property 3.

The rate of change of state and orientation, $\dot{S}$, depends in the following manner on the present state and orientation and on the rate of deformation and rotation tensors V and Ω, respectively, that are being applied at present:

$$\dot{S} = g(S, V) + T(\Omega)\, S\ .$$

The functions f and g are form invariant [cf. (6.3), (8.19)], and the linear operator T on R_n is described above.

9. DISCUSSION OF THE GROWTH LAW

In this section we offer a discussion of a simpler form of the growth law [Eq. (8.6)].

We first observe that the linear dependence in (8.6) of $\dot{S}$ on Ω is a consequence of the principle of objectivity. It is natural to inquire whether a linear dependence on the rate of deformation tensor V may be expected. It is known that there are important materials for which the dependence of $\dot{S}$ on V is not linear. An elastic-plastic solid is an example of such a material where the dependence on V is only piecewise linear.[24] For materials which exhibit "smoother" behavior a linear dependence of $\dot{S}$ on V is a possibility.

Indeed it can be shown that if the mapping $\mathcal{S} : X[0,T] \to \Sigma \subset L$ is Fréchet differentiable, then the function g(S, V) in (8.6) must be of the following form:

$$g(S, V) = g(S) + h(S)\, V\ , \tag{9.1}$$

where $g : \Sigma \to R_n$ and, for each S in Σ, h(S) is a linear mapping from R_6 to R_n.

In view of (8.18) we have also the invariance requirements*

$$g(P_Q S) = P_Q\, g(S);\ h(P_Q S) p_Q = P_Q h(S) \tag{9.2}$$

where p_Q represents the action* of Q in R_6.

The complete growth law for this class of materials has the form

$$\dot{S} = g(S) + h(S)\, V + T(\Omega)\, S\ . \tag{9.3}$$

Note that g(S) is the term which controls the change in state in the absence of external stimuli (i.e., when V and Ω vanish), and thus is a measure of the internal relaxation which takes place within the material. Note also that once the structure of the state space (i.e., the number and dimension of the invariant subspaces M_i) is known, the operator T is known explicitly. Therefore, in order to single out a material of the above type one must specify the functions f(S) [cf. Eq. (6.2)], g(S), and the operator h(S).

For comments on the form of f, g, and h for metals subjected to elevated temperatures and stresses, the reader is referred to Refs. 8 and 25. It is shown in this work that the problem of classification and identification of a broad class of materials reduces to the specification of only g(S) in Eq. (9.3). It is also shown in Ref. 25 that thermodynamic considerations place further restrictions on the mapping g(S). A study of these restrictions in the present context will be given elsewhere.

---

*Thus $p_Q V = QVQ^T$.

We now focus attention on initially isotropic materials, so that the initial state and orientation S_0 has the property

$$P_Q S_0 = S_0 \quad \text{for} \quad Q \in O^+(3) . \tag{9.4}$$

This means that the components of S_0 in the invariant subspaces M_i are zero unless *dim* $M_i = 1$ (cf. Section 7).

Now we wish to show that if the growth law is of the form (9.1) and if the material is initially isotropic, then $\dot{S}$ at $\tau = 0$ can have components only in those subspaces M_i that correspond to scalars and to traceless symmetric second-rank tensors. In other words, in the initial deformation of an isotropic virgin material the only state variables that can be created are scalars and second-rank symmetric traceless tensors. This observation might explain the tendency by the workers in the field to take the state variables as either scalars or symmetric traceless second-rank tensors.

In proving the above assertion, first observe from Eqs. (2.10) and (2.11) that in the absence of deformation a nonaging material maintains its initial state S_0. Thus in Eq. (9.3) we must have

$$g(S_0) = 0 . \tag{9.5}$$

On the other hand the invariance requirement (9.2) on $H(S_0)$ implies, in view of Schur's lemma [17] that

$$h(S_0)V = \Sigma C_{\alpha_i} V' + \Sigma\, d_{\beta_i} \operatorname{tr} V , \tag{9.6}$$

where $V' = V - 1/3\,(\operatorname{tr} V)I$ is the deviatoric part of V and $C_{\alpha_i} V'$ and $d_{\beta_i} \operatorname{tr} V$ are components of $h(S_0)V$ in the invariant subspaces M_{α_i} and M_{β_i}, respectively, where M_{α_i} is five dimensional and M_{β_i} is one dimensional.

On the other hand, since S_0 has nonzero components only in the one-dimensional invariant subspaces, the properties 1 and 2 of the operator T imply that

$$T(\Omega)\, S_0 = 0 \tag{9.7}$$

for any Ω.

Combining now (9.1) with (9.5) to (9.7), we obtain

$$\dot{S}(0) = \Sigma\, C_{\alpha_i} V' + \Sigma\, d_{\beta_i} \operatorname{tr} V ,$$

which shows that only those state variables that are scalars and second-rank traceless tensors can be created initially.

10. EXAMPLES OF REPRESENTATIONS

It is desirable to consider explicit examples of the general representation (1.3) that we have constructed in the previous sections.

The simplest nontrivial example is the one where f(S) and g(S,V) are linear functions of their arguments. Note that the term $T(\Omega)\, S$ in (1.3) precludes consideration of a totally linear material in the presence of finite deformations. The simple representation to be considered has the form

$$\sigma = fS ,$$

$$\dot{S} = AS + hV + T(\Omega)\, S\,, \tag{10.1}$$

where f is a linear mapping from $\Sigma \subset R_n$ to R_6, A is a linear mapping from $\Sigma \subset R_n$ to R_n, and h is a linear mapping from R_6 to R_n. It is assumed that the state set Σ is a bounded subset of R_n and that the orbits of points in Σ under the group $G = \{P_Q\}$ are contained in Σ. It is a consequence of Eqs (6.3), (9.2), and (10.1) that f, A, and h are subject to the invariance requirements

$$fP_Q = p_Q f,\ AP_Q = P_Q A \ \text{ and } \ hp_Q = P_Q h,\ Q \in O^+(3)\,. \tag{10.2}$$

It should be emphasized that these invariance requirements result from the principle of objectivity and the particular form of Eq. (10.1) and are in no way a consequence of any symmetries that the material may possess.

In Section 7 we discussed the action of the transformations $G = \{P_Q\}$ in R_n. We observed that the $\{P_Q : R_n \to R_n\}$ is composed of irreducible representations of $O^+(3)$, and this gave rise to the result that S is the sum of a number of tensors. We now let $r_i (i = 1, \ldots, i_1)$ denote the state variables that are scalars and $s_i (i = 1, \ldots, i_2)$ denote the state and orientation variables that are second-rank symmetric traceless tensors. State and orientation variables that are second-rank antisymmetric or of rank four or higher will be denoted by $\mathbf{t}_i$, $\mathbf{w}_i$, etc.

We wish now to study the implications of the invariance requirements (10.2). The main tool for this is Schur's lemma,[17] which when applied to the invariance requirement*

$$fP_Q = p_Q f,\ \ Q \in O^+(3)\,,$$

yields the following result: The stress can depend only on the state and orientation variables that are scalars or second-rank symmetric traceless tensors. Moreover, this dependence must be of the form

$$\sigma = \Sigma a_i s_i + \mathbf{I}\Sigma b_i r_i\,, \tag{10.3}$$

where we now use for emphasis boldface letters to describe tensors. Here **I** denotes (3 X 3) Kronecker delta and a_i and b_i are constants.

On the other hand, the invariance requirement $hp_Q = P_Q h$ implies that hV can have components only in the one- and five-dimensional invariant subspaces of R_n and that hV must be of the form (9.6). The invariance requirement on A implies, in view of Schur's lemma, that under A, scalars produce scalars, second-rank symmetric traceless tensors produce the same kind of tensors, etc.

Combining these observations with (10.1) and remembering that the linear operator $T(\Omega)$ has the properties (1) and (2) (cf. Section 8), we obtain

$$\begin{aligned} \dot{r}_i &= \Sigma b_{ij} r_j + d_i \operatorname{tr} \mathbf{V}\,, \\ \dot{s}_i &= \Sigma a_{ij} s_j + c_i \mathbf{V}' + T(\Omega)\, s_i\,, \end{aligned} \tag{10.4}$$

*We remember that $\{p_Q\}$ is the representation of $O^+(3)$ in R_6, and therefore it is composed of one- and five-dimensional irreducible representations.

and

$$\dot{\mathbf{t}}_i = \Sigma c_{ij}\, \mathbf{t}_j + T(\Omega)\, \mathbf{t}_i\,, \tag{10.5}$$

etc., where b_{ij}, a_{ij}, d_i, c_i, c_{ij}, . . . are constants and $\mathbf{V}'$ is the deviatoric part of $\mathbf{V}$.

The above observations show that state and orientation variables that are second-rank antisymmetric or of fourth or higher rank do not contribute to the description of mechanical behavior of the solid defined by Eq. (10.1) because of the lack of coupling between (10.5) and Eqs. (10.3) and (10.4). Therefore, these variables may be omitted and the representation (10.1) can be reduced to (10.3) and (10.4).

We now show that the consideration of the initial state $(s_1^0, \ldots, s_{i_2}^0\,;\, r_1^0, \ldots, r_{i_1}^0)$ introduces some mild restrictions on the coefficients in (10.3) and (10.4). Indeed, since $\sigma = 0$ at the initial state we must have

$$\Sigma a_i s_i^0 = 0 \quad \text{and} \quad \Sigma b_i r_i^0 = 0\,. \tag{10.6}$$

Moreover, in the absence of deformations the initial state remains unchanged [cf. (2.11)], so that

$$\sum_1^{i_2} a_{ij} s_j^0 = 0 \qquad i = 1, \ldots, i_2$$

and (10.7)

$$\sum_1^{i_1} b_{ij} r_j^0 = 0 \qquad i = 1, \ldots, i_1\,.$$

In the case where $s_i^0 \neq \mathbf{0}$ for some i the material defined by (10.3), (10.4), (10.6), and (10.7) is *not* isotropic in its initial state. However, the structure of (10.3) and (10.4) is such that the material of interest can be said to be "*isotropic*" in a limited sense even when $s_i^0 \neq \mathbf{0}$. Indeed, if one considers deformation histories where $\Omega = \mathbf{0}$ at every instant of interest, then in (10.4) the term $T(\Omega)\, s_i$ vanishes and for this class of deformation histories (10.3) and (10.4) become form invariant under orthogonal coordinate transformations. Thus an observer who works only with the class of irrotational deformation histories would judge that the material defined by (10.3) and (10.4) is *isotropic*.

We have therefore reached the rather surprising result that linearity [of f(S) and g(S, V)] implies isotropy in a limited sense. This result may lead one to ask if the existence of linear anisotropic solids in the case of infinitesimal deformations contradict the above result. It can easily be seen that there is no contradiction. Indeed, in order to arrive at a representation appropriate for infinitesimal deformations, one could linearize the general representation in the neighborhood of the initial state S_0. The linearized representation would look very much like (10.1), but it would not be valid in the neighborhood of the *entire* orbit of S_0 if the material is not isotropic. Therefore the linearized representation may not be subject to the invariance requirements (10.2), and hence one cannot assert that (10.3) and (10.4) must hold. A more detailed study of this case leads one to materials which are linear and anisotropic in the sense of theory of linear viscoelasticity.

We end this section by pointing out that other explicit examples of representations are available in Ref. 8. We also point out that Ref. 24 deals with the application of the ideas developed in this report to the case of elastic-plastic materials.

11. COMPARISON OF REPRESENTATIONS

It is desirable to compare the present work with earlier work on representations.

To this end we recall [cf. Eq. (2.7)] that the mapping $F : X \to Y$ induces, in view of causality of F, a mapping $f_T : X[0,T] \to R_6$ given by

$$\sigma(T) = f_T(\underset{[0,T]}{D}) , \tag{11.1}$$

where $\underset{[0,T]}{D}$ denotes a deformation gradient history on [0,T] and $\sigma(T)$ is the stress produced by this deformation at time T. Equation (11.1) contains the tacit assumptions that the material is in the virgin state at $\tau = 0$ and that all elements are identically oriented at $\tau = 0$ with respect to the fixed coordinate frame in which the stress and the deformation are measured. If the material is nonaging, as is assumed here, then the functional (11.1) represents mechanical behavior over the time interval [0,T] by means of right translations of D [cf. (3.2) and (3.3)].

It was shown by Green and Rivlin[3] and Noll[5] that (11.1) must be of the form

$$\sigma(T) = D(T)\, \mathsf{F}[\underset{[0,T]}{g(\tau)}]\, D^T(T) , \tag{11.2}$$

where

$$g = D^T D$$

is Green's deformation tensor and F is a tensor-valued (symmetric second-rank) functional of the history of g on [0,T]. This result follows from the principle of objectivity.

In view of (11.2) the problem of representation of mechanical behavior reduces to the representation of the functional F.

Much effort has been devoted to the construction of integral representations for F (cf. Ref. 26). It is known from Fréchet's work[27] and Ref. 3 that when F is "continuous" and the domain of F is restricted to a compact set and the material is nonaging, then F can be represented to any desired degree of accuracy by a sum of multiple integrals:

$$\mathsf{F}_{ij}[\underset{[0,T]}{g}] = \int_0^T K_{ijk\ell}(T-\tau)\, g_{k\ell}(\tau)\, d\tau + \int_0^T \int_0^T K_{ijk\ell mn}(T-\tau_1, T-\tau_2)\, g_{k\ell}(\tau_1)\, g_{mn}(\tau_2)\, d\tau_1\, d\tau_2 + \dots . \tag{11.3}$$

Various modifications of (11.3) are available. It can be shown[28] that the above sum may be replaced by a polynomial of linear functionals*

$$\int_0^T C_{ijk\ell}(T-\tau)\, dg_{k\ell}(\tau)\,. \tag{11.4}$$

Another form of representation is obtained by replacing strain components in (11.3) by nonlinear functions of these components.[29]

If the material is isotropic in its initial state, then the sum (11.3) gains the additional structure† discussed by Green and Rivlin[30] and Green, Rivlin, and Spencer.[4]

Examples of applications of integral representations may be found, for instance, in Ward and Onat,[31] Onaran and Findley,[32] Lockett,[33] Pipkin and Rogers,[29] and Wang and Onat.[34]

In the course of work on the applications of integral representations, it has become clear that these representations suffer from serious drawbacks. It was shown by Onat[7] (see also Ref. 33) that the number of terms needed in Eq. (11.3) for a reasonably accurate description of elastic-plastic (or creep) behavior of metals is prohibitively large. This drawback emanates from the fact that (11.3) is based on the Weierstrass approximation of continuous functions by polynomials. As is well known, polynomials are an awkward tool of representation for functions that are not smooth enough.[35] In the representation of elastic-plastic behavior, functions that are not smooth enough are encountered and the above-mentioned drawback ensues.

Another difficulty with (11.3) (or its various modifications) is the lack of *effective* experimental procedures for the determination of the number and the nature of multiple integrals and the related problem of extreme sensitivity of the kernels in (11.3) to scatter in experimental data.[9]

The representations based on state variables do not suffer from these drawbacks. Elastic-plastic behavior admits a compact and relatively simple state variable representation.[23] On the other hand, one can plan, on the basis of the mathematical structure constructed in this paper, phenomenological experiments to obtain information on the number of state variables and their tensorial nature and on the growth law.[8,24] Moreover, the knowledge of the internal structure of the material may suggest that certain physical quantities are "natural" state variables and these variables obey already known growth laws (for a discussion of this point see Ref. 36).

Another advantage of the state variable representation is that in the *computer* analysis of structures composed of hereditary materials, the state variable representation places a lighter burden on the memory of the computer.

We may also point out conceptual advantages of the state variable representation (1.3), which defines vector fields in the space of state and orientation. Global properties of these fields may be studied with the methods of the theory of dynamical systems. A simple example of such a study is given in Ref. 25, where various aspects of mechanical behavior such as creep, rate sensitivity, and plasticity are discussed in a unified manner and in geometrical terms.

Finally, it must be remembered that representations based on state variables are widely used in mechanics, in thermodynamics, and in physics. Most constitutive equations (proposed or in current use) are

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*For a related result, see Ref. 6.

†It is interesting to observe that isotropy modifies the representation (11.3) in a remarkable way, whereas the only role played by isotropy in the state variable representation (1.3) is in the initial condition for the growth law.

of this type (cf. constitutive equations for elastic-plastic solids,[37] various theories of creep,[38] and constitutive equations proposed by Chu,[12] Shapery,[39] Biot,[40] and Rice).[41]

In this report we introduced a general theory of state variable representations that is appropriate to the study of isothermal finite deformations. The theory enables one to construct, in principle, state variable representations from observed phenomenological behavior.

ACKNOWLEDGMENT

The authors would like to acknowledge the helpful discussions with Professors R. P. Langlands and G. D. Mostow.

REFERENCES

1. C. Truesdell and W. Noll, "The Non-Linear Field Theories of Mechanics," *Handbuch der Physik,* III/3, Springer-Verlag, 1965.

2. W. Noll, "On the Continuity of the Solid Fluid States," *J. Rational Mech. Anal.* **4**, 3–81 (1955).

3. A. E. Green and R. S. Rivlin, "The Mechanics of Non-Linear Materials with Memory," *Arch. Rational Mech. Anal.* **1**, 1–34 (1957).

4. A. E. Green, R. S. Rivlin, and A. J. M. Spencer, "The Mechanics of Nonlinear Materials with Memory," *Arch. Rational Mech. Anal.* **3**, 82–90 (1959).

5. W. Noll, "A Mathematical Theory of the Mechanical Behavior of Continuous Media," *Arch. Rational Mech. Anal.* **2**, 197–226 (1958).

6. B. D. Coleman and W. Noll, "Foundations of Linear Viscoelasticity," *Rev. Mod. Phys.* **33**, 239–49 (1961).

7. E. T. Onat, "Description of the Mechanical Behavior of Inelastic Solids," pp. 421–34 in *Proc. Fifth U.S. National Congress of Applied Mathematics and Mechanics, Minnesota, 1966.*

8. E. T. Onat and F. Fardshisheh, *Representation of Creep of Metals.* ORNL-4783 (1972).

9. M. A. Gradowczyk, "On the Accuracy of the Green-Rivlin Representation for Viscoelastic Materials," *Int. J. Solids Struct.* **5**, 873–77 (1969).

10. E. T. Onat, "The Notion of State and Its Implications in Thermodynamics of Inelastic Solids," pp. 292–314 in *Proc. IUTAM Symposium, Vienna, 1966,* Springer, 1968.

11. E. T. Onat, "Representation of Inelastic Behavior by Means of State Variables," pp. 213–25 in *Proc. IUTAM Symposium, East Kilbride 1968,* Springer, 1970.

12. B. T. Chu, "Thermodynamics of Deformation of Solids," pp. 299–343 in *A Critical Review of Thermodynamics,* Ed. by E. B. Stuart, B. Galor, and A. J. Brainard, Mono Book Corp. Baltimore, Md., 1970.

13. E. Kröner, "Dislocations: A New Concept in the Continuum Theory of Plasticity," *J. Math. Phys.* **42**, 22–37 (1963).

14. T. M. Apostol, *Mathematical Analysis,* p. 90, Addison-Wesley, 1957.

15. G. F. Simmons, *Introduction to Topology and Modern Analysis,* McGraw-Hill, New York, 1963.

16. L. A. Lusternik and V. J. Sobolev, *Elements of Functional Analysis,* Gordon and Breach, New York, 1968.

17. M. Hamermesh, *Group Theory and Its Application to Physical Problems,* Addison-Wesley, Reading, Mass., 1964.

18. G. D. Mostow, "Equivariant Embeddings in Euclidean Space," *Ann. Math.* **65**(3), 432–46 (1957).

19. G. D. Mostow, "On a Conjecture of Montgomery," *Ann. Math.* **65**(3), 513–16 (1957).

20. M. Spivak, "Differential Geometry," Lecture Notes, Brandeis University, 1970.

21. I. M. Gel'fand, R. A. Minlos, and Z. Ya Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications,* Pergamon, Oxford, 1963.

22. W. Prager, *Introduction to Mechanics of Continua,* Ginn, Boston, Mass., 1961.

23. J. A. Geary and E. T. Onat, "Representation of Non-Linear Dynamical Systems," to be published.

24. E. T. Onat and F. Fardshisheh, "Representation of Elastoplastic Behavior by Means of State Variables," to appear in Proc. Int. Symp. on Plasticity, Warsaw, 1972.

25. E. T. Onat and F. Fardshisheh, "Representation of Creep, Rate Sensitivity and Plasticity," *SIAM J. Appl. Math.* **25**, 522–38 (1973).

26. R. M. Christensen, *Theory of Viscoelasticity,* Academic, New York, 1971.

27. M. Fréchet, "Sur les fonctionelles continues," *Ann. Si. de l'Ecole Norm. Sup. (3)* **27**, 193–216 (1910).

28. R. V. S. Chacon and R. S. Rivlin, "Representation Theorems in the Mechanics of Materials with Memory," *Z. Angew. Math. Phys.* **15**, 444–47 (1964).

29. A. C. Pipkin and T. G. Rogers, "A Non-Linear Integral Representation for Viscoelastic Behavior," *J. Mech. Phys. Solids* **16**, 59–72 (1968).

30. A. E. Green and R. S. Rivlin, "The Mechanics of Non-Linear Materials with Memory III," *Arch. Rational Mech. Anal.* **4**, 387–404 (1960).

31. I. M. Ward and E. T. Onat, "Nonlinear Mechanical Behavior of Oriented Polypropylene," *J. Mech. Phys. Solids* **11**, 217–29 (1963).

32. K. Onaran and W. N. Findley, "Combined Stress-Creep Experiments on a Nonlinear Viscoelastic Material to Determine the Kernel Functions for a Multiple Integral Representation of Creep," *Trans. Soc. Rheol.* **9**(2), 299–327 (1965).

33. F. J. Lockett, "Creep and Stress-Relaxation Experiments for Nonlinear Materials, *Int. J. Eng. Sci.* **3**, 59–73 (1965).

34. T. T. Wang and E. T. Onat, "Nonlinear Mechanical Behavior of 1100 Aluminum at 300°F," *Acta Mech.* **5**, 54–70 (1968).

35. S. Bernstein, *Lecons sur les proprietes et la Meilleure Approximation des Functions Analytiques d'une Variable Reele,* Gauthier Villars, p. 35, 1927.

36. J. J. Gilman, *Micromechanics of Flow in Solids,* McGraw-Hill, New York, 1969.

37. A. E. Green and P. Naghdi, "A General Theory of an Elastic-Plastic Continuum," *Arch. Rational Mech. Anal.* **18**, 251 (1965).

38. Yu. N. Rabotnov, *Creep Problems in Structural Members,* North Holland, 1969.

39. R. A. Schapery, "On a Thermodynamic Constitutive Theory and Its Application to Various Nonlinear Materials," *Proc. IUTAM Symposium, East Kilbride, 1968,* Springer, 1970.

40. M. A. Biot, "Linear Thermodynamics and the Mechanics of Solids," pp. 1–18 in *Proc. 3rd U.S. National Congress of Applied Mechanics, 1958.*

41. J. R. Rice, *J. Mech. Phys. Solids* **19**, 433 (1971).

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