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ON THE STABILITY OF MAGNETICALLY-CONFINED, HIGH-BETA PLASMA*

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ABSTRACT

Plasma waves propagating perpendicular to the magnetic field in a high-beta, magnetically-confined plasma can be unstable in configurations for which the guiding-center drifts are in the "favorable" direction. We illustrate the basic instability mechanism and give the conditions for growth using the Maxwell and Vlasov equations in a slab model of the plasma.

Variational analyses of ideal MHD and guiding-center plasma models have been used by many workers[1,2,3] to show that high-beta plasmas in various magnetic fields can be unstable to ballooning modes if the curvature of the magnetic lines of force is unfavorable. This result is not affected by the plasma currents, which may reverse the sign of the gradient of the magnetic field, together with the sign of the guiding-center drift velocity, without significantly changing the curvature of the field lines. Based on this result it is often concluded that a plasma cannot "dig its own well"; i.e., improve its stability by virtue of its own diamagnetic current.

The same analysis can give very different results when applied to interchange perturbations,[4] showing a strong stabilizing effect

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associated with reversing the sense of the buiding-center drifts. Here we seek to elucidate the properties of the unstable ballooning modes predicted by the variational analyses, using the more microscopic Vlasov-Maxwell plasma model in a simple slab geometry. The Vlasov kinetic equation permits the straightforward inclusion of finite gyroradius effects, missing from the MHD and guiding-center theories, and offers additional insight into the limits of validity of the fluid models. The applicability of the fluid theories is particularly questionable for hot-electron plasmas, exemplified by the ELMO series of experiments,[5] in which the characteristic drift frequencies may easily exceed the ion gyrofrequency.

We consider a hot plasma whose density is uniform in the y- and z-directions but decreases monotonically in the x-direction. It is immersed in a z-directed magnetic field which is uniform in the absence of the plasma. A diamagnetic current flows in the -y direction, balancing the force due to the plasma pressure gradient and causing the magnetic field to increase in x : $2d\,\ln B/dx = -\beta d\,\ln P/dx$, where $\beta \equiv 2\mu_0 P/B^2$. We investigate the stability of plasma waves which extend indefinitely in z, propagate in the y-direction, and are polarized so that the wave magnetic field is in the z-direction. We restrict our attention to some localized region in x, within which any x-dependence of the wave properties can be neglected.

The waves must satisfy Maxwell's equations with charge and current densities calculated from the linearized Vlasov equation for each species:[6]

$$\underset{\sim}{E} = -\nabla\varphi - \partial\underset{\sim}{A}/\partial t \; ; \; \underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

$$\epsilon_0 \nabla \cdot \underset{\sim}{E} = \rho = \sum_{\text{species}} q_S \int d^3v \, f_{1S}$$

$$\nabla \times \underset{\sim}{B} = \mu_0(\underset{\sim}{j} + \epsilon_0 \partial\underset{\sim}{E}/\partial t) \tag{1}$$

$$= \mu_0 \sum_{\text{species}} q_S \int d^3v \, \underset{\sim}{v} \, f_{1S} + \mu_0 \epsilon_0 \, \partial\underset{\sim}{E}/\partial t$$

$$M_S f_{1S} = - q_S \int dt \, (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}) \cdot \partial f_{0S}/\partial \underset{\sim}{v} \quad ,$$

where q_S, M_S, and f_S are the charge, mass, and distribution function of each species. We have described the waves by potentials, φ and $\underset{\sim}{A}$, which in the Coulomb gauge must satisfy

$$\epsilon_0 \nabla^2\varphi = - \rho \qquad \text{(Poisson's equation)} \quad ,$$

and (2)

$$- \nabla^2\underset{\sim}{A} + c^{-2} \, \partial(\partial\underset{\sim}{A}/\partial t + \nabla\varphi)/\partial t = \mu_0 \underset{\sim}{j} \quad \text{(Ampere's Law)} \; .$$

For the present case, we require only the x-component of Ampere's Law and must therefore solve

$$\epsilon_0 k^2 \varphi = \rho$$

and (3)

$$\epsilon_0 (c^2k^2 - \omega^2) A_x = j_x \quad ,$$

with ρ and j_x obtained from the linearized Vlasov equation. For gentle gradients, the two moments are given approximately by

$$\rho = - \sum_{\text{species}} \frac{q^2 N}{T} \left\{ \varphi \left[1 + (\omega - \omega_*) \sum_{m=-\infty}^{\infty} \left\langle \frac{J_m^2}{\omega_D - \omega - m\Omega} \right\rangle \right] \right.$$

$$\left. + \, i A_x (\omega - \omega_*) \sum_{m=-\infty}^{\infty} \left\langle \frac{v_\perp J_m J_m'}{\omega_D - \omega - m\Omega} \right\rangle \right\} \tag{4}$$

and

$$j_x = \sum_{\text{species}} \frac{q^2 N}{T} \left\{ - i\varphi(\omega-\omega_*) \sum_{m=-\infty}^{\infty} \left\langle \frac{v_\perp J_m J_m'}{\omega_D - \omega - m\Omega} \right\rangle \right.$$

$$\left. + A_x(\omega-\omega_*) \sum_{m=-\infty}^{\infty} \left\langle \frac{(v_\perp J_m')^2}{\omega_D - \omega - m\Omega} \right\rangle \right\} . \tag{5}$$

In these expressions N is the particle density at the point of interest; T is the temperature, here assumed independent of x; ω_*, ω_D and Ω are the diamagnetic frequency, the guiding-center drift frequency, and the gyrofrequency of each species:

$$\omega_* = \frac{kT}{qB} \frac{d \ln N}{dx} \qquad \begin{matrix} v_{*e,y} > 0 \\ v_{De,y} < 0 \end{matrix}$$

$$\omega_D = \frac{kMv_\perp^2}{2qB} \frac{d \ln B}{dx} \cong \frac{kT}{qB\, R_B} \tag{6}$$

$$\Omega = qB/M \quad .$$

All Bessel functions, J_m, have the same argument, $kv_\perp/\Omega$, where k is the magnitude of the wave vector; and angular brackets denote averages over the unperturbed Maxwellian distribution function (here normalized to unity). If we neglect the velocity dependence of ω_D, these averages become

$$\langle J_m^2 \rangle = e^{-\lambda} I_m(\lambda) \equiv Q_m(\lambda)$$

$$\langle v_\perp J_m J_m' \rangle = \frac{kT}{qB} Q_m' \tag{7}$$

$$\langle (v_\perp J_m')^2 \rangle = \left(\frac{kT}{qB}\right)^2 \left(\frac{m^2}{\lambda^2} Q_m - 2Q_m'\right)$$

where $\lambda \equiv k^2T/M\Omega^2 = k^2a^2/2$, and a is the average gyroradius.

Poisson's equation and Ampere's Law can now be put in the following convenient forms:

$$\varphi\left[1 + \sum_{\text{species}} (k\lambda_D)^{-2}\left(1 + \sum_{m=-\infty}^{\infty} \frac{\omega-\omega_*}{\omega_D-\omega-m\Omega}\, Q_m\right)\right]$$

$$+ \frac{iA_x}{k\epsilon_0 B} \sum_{\text{species}} qN \sum_{m=-\infty}^{\infty} \frac{\omega-\omega_*}{\omega_D-\omega-m\Omega}\, Q_m' = 0 \tag{8}$$

and

$$\varphi\left[\frac{\mu_0}{B^2} \sum_{\text{species}} qN \sum_{m=-\infty}^{\infty} \frac{\omega-\omega_*}{\omega_D-\omega-m\Omega}\, Q_m'\right]$$

$$- \frac{ikA_x}{B}\left[1 - \frac{\omega^2}{c^2k^2} - \beta \sum_{\text{species}} \sum_{m=-\infty} \frac{\omega-\omega_*}{\omega_D-\omega-m\Omega}\left(\frac{m^2 Q_m}{2\lambda^2} - Q_m'\right)\right] = 0 \quad . \tag{9}$$

with $\lambda_D^2 = \epsilon_0 T/q^2 N$ and $\beta = 2\mu_0 NT/B^2$. The condition for existence of non-trivial solutions provides the general dispersion relation:

$$\left[1 - \frac{\omega^2}{c^2k^2} - \beta \sum_{\text{species}} \sum_{m=-\infty} \frac{\omega-\omega_*}{\omega_D-\omega-m\Omega} - Q_m'\right)\right] \times$$

$$\left[1 + \sum_{\text{species}} (k\lambda_D)^{-2}\left(1 + \sum_{m=-\infty}^{\infty} \frac{\omega-\omega_*}{\omega_D-\omega-m\Omega}\, Q_m\right)\right] \tag{10}$$

$$+ \left[\frac{1}{ck\epsilon_0 B} \sum_{\text{species}} qN \sum_{m=-\infty}^{\infty} \frac{\omega-\omega_*}{\omega_D-\omega-m\Omega}\, Q_m'\right]^2 = 0 \quad ,$$

which we shall abbreviate as $D_{EM}D_{ES} + (CT)^2 = 0$.

To exhibit the instability of immediate concern, we apply this general dispersion relation to a hot-electron, cold-ion plasma and first investigate long-wavelength ($\lambda_i, \lambda_e \to 0$), low-frequency ($\omega^2 \ll \Omega_i^2$) modes. We then obtain familiar expressions for D_{EM} and D_{ES}:

$$D_{EM} \cong 1 - \frac{\omega^2}{v_A^2 k^2} + \beta - \beta \frac{\omega_* - \omega_D}{\omega - \omega_D}$$

$$D_{ES} \cong 1 + \frac{\omega_{pi}^2}{\Omega_i^2} + (k\lambda_D)^{-2} \frac{\omega_* - \omega_D}{\omega - \omega_D} \frac{\omega_D}{\omega} \tag{11}$$

and

$$CT \cong (k\lambda_D)^{-1} \sqrt{\beta}/2 \frac{\omega_* - \omega_D}{\omega - \omega_D}$$

The Alfven speed is v_A, and β, λ_D, ω_*, and ω_D all refer to the electrons. The dispersion relation can then be written in a very compact form if we define $x \equiv \omega/\omega_D$ and

$$x_0 \equiv (1 - \omega^2/v_A^2 k^2 + \beta\omega_*/\omega_D)/(1 - \omega^2/v_A^2 k^2 + \beta) \quad : \tag{12}$$

$$(k\lambda_D)^2 \left(1 + \frac{\omega_{pi}^2}{\Omega_i^2}\right)\left(1 - \frac{\omega_*}{\omega_D}\right) = \frac{1}{x-1}\left(\frac{1}{x} + \frac{1}{2}\frac{x_0 - 1}{x - x_0}\right) \quad . \tag{13}$$

Notice the following features of this dispersion relation:

1. If $x_0 \to 1$, we recover the dispersion relation describing ordinary flute modes.[7]
2. If $x_0 > 1$ and $\omega_*/\omega_D < 0$ the dispersion relation has three real roots corresponding to stable propagating waves.
3. If $x_0 < 0$, the dispersion relation may have a complex conjugate pair of roots if the left-hand side, $(k\lambda_D)^2(1 + \omega_{pi}^2/\Omega_i^2)(1 - \omega_*/\omega_D)^{-1}$, lies within the unstable region depending on x_0 as shown in Fig. 1.

For the slab model with straight magnetic lines of force, $\beta\omega_*/\omega_D = -2$, so that

$$-x_0 = (1 + \omega^2/v_A^2 k^2)/(1 - \omega^2/v_A^2 k^2 + \beta) \quad . \tag{14}$$

If $\omega^2/v_A^2k^2 \leqslant 1$, $x_0 \leqslant -0.5$, and a rough approximation to the stability boundary of Fig. 1 is simply $-2/x_0$; that is, <u>for instability</u>

$$(k\lambda_D)^2 \; (1 + \omega_{pi}^2/\Omega_i^2) \; (1 - \omega_*/\omega_D)^{-1} \leqslant -2/x_0 \tag{15}$$

Now in a cylindrical plasma, k must satisfy a periodicity condition such that $k = \ell/r$, where $\ell \doteq 1,2,3,\ldots$ and r is the location of the point of interest. The instability condition then becomes

$$\ell \frac{a_e}{r} \leqslant 2\sqrt{\frac{m}{M}} \left[\left(1 + \frac{2}{\beta}\right)\left(\frac{1 + \beta - \omega^2/v_A^2k^2}{1 + \omega^2/v_A^2k^2}\right)\right]^{1/2} , \tag{16}$$

where a_e and m are the electron gyroradius and mass. The factor in square brackets varies from $[(2 + \beta)(1 + \beta)/\beta]^{1/2}$ for $\omega^2 \ll v_A^2k^2$, to $[1 + \beta/2]^{1/2}$ for $\omega^2 = v_A^2k^2$. Thus, this instability is only possible if the plasma radius exceeds the electron gyroradius by a factor of order $(M/m)^{1/2}$; smaller plasmas would lie in the stable region of Fig. 1. The unstable growth can be understood in elementary terms by noting that in the magnetic field of the wave, electrons will drift in x so as to reinforce the wave if B_1, the wave magnetic field, is in phase with φ, the electrostatic potential of the wave. The relative phase of φ and B_1 is determined by Ampere's Law:

$$\frac{B_1}{B} = \frac{e\varphi}{2T_e} \frac{1-x_0}{x-x_0} > 0 \quad \text{if} \quad x_0 < x < o \quad .$$

Since waves which satisfy the criterion for instability have $\lambda_e \leqslant m/M$, our assumption of long wavelengths is clearly justified. The low-frequency approximation is not, however, generally valid. Unstable waves have frequencies $\omega \sim \omega_D$ and

$$\frac{\omega_D}{\Omega_i} \simeq \frac{\ell}{2}\frac{a_e}{r}\frac{a_e}{R_B}\frac{M}{m} \leqslant \frac{2}{\ell}\frac{r}{R_B}\left(1+\frac{\beta}{2}\right) , \tag{17}$$

where we have imposed the condition for instability on a_e/r. Thus, if $r \gg R_B$, as is true of the annular plasma in ELMO,[5] waves satisfying the condition for instability could violate the assumption $\omega^2 \ll \Omega_i^2$. This assumption can be relaxed by including the cyclotron terms in the dispersion relation. Evaluating (10) for the general case $\omega \sim \Omega_i$ leads to

$$D_{EM} \simeq 1 - \frac{\omega^2}{c^2k^2}\left[1 + \frac{\omega_{pi}^2}{\Omega_i^2-\omega^2} + \frac{\omega_{pe}^2}{\Omega_e^2\omega^2}(\omega-\omega_*)(\omega-\omega_D)\right] + \beta\frac{\omega-\omega_*}{\omega-\omega_D}$$

$$D_{ES} \simeq 1 + \frac{\omega_{pi}^2}{\Omega_i^2-\omega^2} + \frac{\omega_{pe}^2}{\Omega_e^2} + (k\lambda_D)^{-2}\frac{\omega_*-\omega_D}{\omega-\omega_D}\frac{\omega_D}{\omega} \tag{18}$$

$$CT \simeq (k\lambda_D)^{-1}\sqrt{\beta/2}\left[\frac{\omega_*-\omega_D}{\omega-\omega_D} - \frac{\omega^2}{\omega^2-\Omega_i^2}\right]$$

The electron cyclotron term in D_{ES} has been simplified in the anticipation that $\omega \sim \omega_D$. The resulting dispersion relation is

$$k^2\lambda_D^2\frac{1+\omega_{pe}^2/\Omega_e^2}{1-\omega_*/\omega_D} = \frac{x^2-\Omega_i^2/\omega_D^2}{x^2-\omega_{LH}^2/\omega_D^2}\frac{1}{x-1}\left\{\frac{1}{x} + \frac{1}{2}\frac{x_0-1}{x-x_0}\left[1 + \frac{x^2(x-1)}{(1-\omega_*/\omega_D)(x^2-\Omega_i^2/\omega_D^2)}\right]^2\right\} \tag{19}$$

where $\omega_{LH}^2 \simeq \omega_{pi}^2/(1+\omega_{pe}^2/\Omega_e^2)$ and now

$$x_0 \equiv (1 - \Delta + \beta\omega_*/\omega_D)/(1 - \Delta + \beta) \tag{20}$$

with

$$\Delta \equiv \frac{\omega^2}{c^2k^2}\left[1 + \frac{\omega_{pi}^2}{\Omega_i^2-\omega^2} + \frac{\omega_{pe}^2}{\Omega_e^2\omega^2}(\omega-\omega_*)(\omega-\omega_D)\right] \tag{21}$$

In the limit $\omega \ll \Omega_i$ equations (18-20) revert to the previous expressions (11-13). Before studying this dispersion relation, it is instructive to consider the limit $\Omega_i \ll \omega \ll \omega_{LH}$, for which (19) becomes

$$\frac{\omega_{pi}^2}{\omega_D^2} \frac{k^2\lambda_D^2}{1-\omega_*/\omega_D} = \frac{x^2}{1-x}\left[\frac{1}{x} + \frac{1}{2}\frac{x_0-1}{x-x_0}\left(\frac{\omega_*-\omega}{\omega_*-\omega_D}\right)^2\right] \tag{22}$$

The equilibrium property $\beta\omega_*/\omega_D = -2$ implies that

$$\left(\frac{\omega_*-\omega}{\omega_*-\omega_D}\right)^2 = \left(\frac{2+\beta x}{2+\beta}\right)^2 \approx 1 \tag{23}$$

The righthand side of the dispersion relation is shown in Fig. 2. Now we find only stable roots for $x_0 < 0$, a consequence of the fact that the dielectric constant has changed sign for frequencies between Ω_i and ω_{LH}.

These simple limits, $\omega \ll \Omega_i$ and $\omega \gg \Omega_i$, incorporate most of the features of the full dispersion relation (19). For $|\omega_D/\Omega_i| \simeq 1$, the dispersion relation must be treated numerically. We find that the growth rate goes to zero when $|\omega_D/\Omega_i|$ exceeds unity by a complicated factor which involves ω_H^2/c^2k^2. (ω_H^2/c^2k^2 is a coupling factor which appears in $D_{EM} = 0$ or $x = x_0$. We observe that the growth rate goes to zero when the roots of $D_{EM} = 0$ become real.) In Fig. 3 we show the growth rate versus β for several choices of the parameter $D \equiv \frac{1}{2}\frac{c^2}{r^2\Omega_e^2}$. Here guided by the ELMO series of experiments, we have set

$$\omega_{pe}^2/\Omega_e^2 = 1 \quad ; \quad |d\,\ln p/dx|/r = 0.2 \quad .$$

The mode number, ℓ, was taken to be 2. Under these circumstances

$$\lambda = 4D\beta$$

$$|\omega_D/\Omega_i| = 9.2 \times 10^3 \, D\beta^2$$

It must be noted that when these modes are stabilized, $|\omega/\omega_D| \simeq 1$ so that our neglect of the dispersion in the drift speed is questionable. However, inclusion of this effect and a more careful inclusion of ion gyroresonance effects does not seem likely to give rise to any rapidly growing waves, although some microscopic instabilities may be possible. This question is under continuing study.

In conclusion, we have used a Vlasov description of a simple plasma equilibrium to exhibit a low-frequency mode which, like the ballooning modes of variational analyses, is not stabilized by the "magnetic well" produced by the plasma diamagnetic currents. However, the characteristic frequency of the unstable wave is around the guiding-center drift frequency; and in a hot-electron plasma of moderate size, this frequency may easily exceed the ion gyrofrequency, thus invalidating the MHD description. From the Vlasov analysis it appears likely that the present modes may in fact be stabilized in such a high-frequency regime.

Finally, one should notice that a "ballooning" character is suggested for the present instabilities by Ampere's Law, in that although φ may be maintained nearly constant in z by the rapid flow of electrons along field lines, the fluctuating magnetic field, B_1, is allowed a significant variation in z-dependent equilibria. This aspect becomes apparent when the present slab model is generalized to axisymmetric equilibria with curved magnetic lines of force. Alternatively, the relation of the present instability to the traditional ballooning modes can be

established through detailed comparison of the normal mode calculation and a parallel variational analysis. This is treated in a separate paper.

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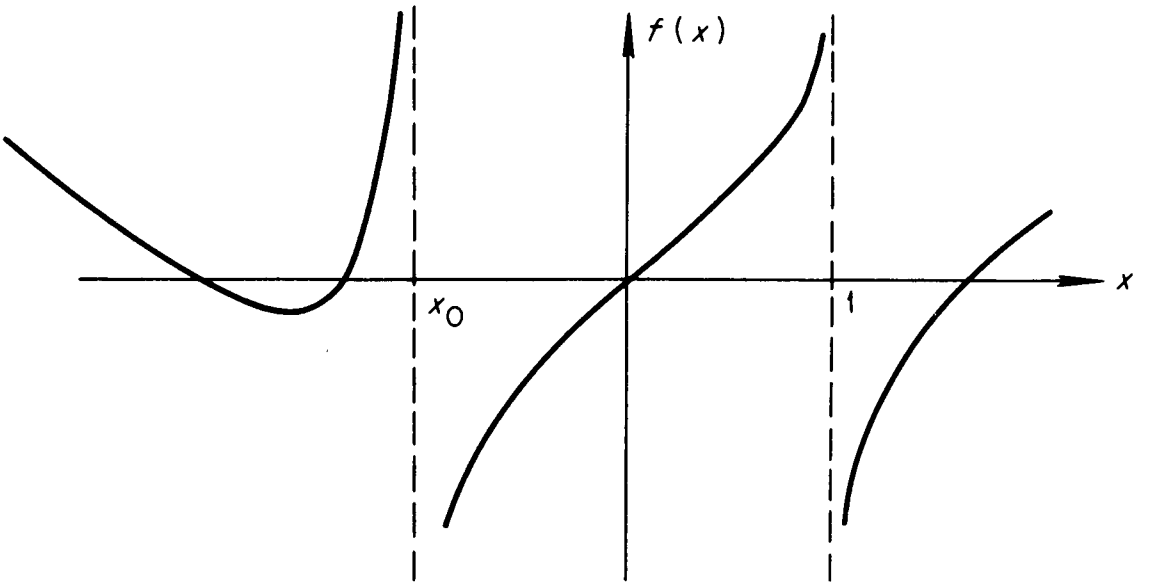

$$f(x) = \frac{x^2}{1-x}\left[\frac{1}{x} + \frac{1}{2}\,\frac{x_0-1}{x-x_0}\left(\frac{2+\beta x}{2+\beta}\right)^2\right]$$

Fig. 1 Stability boundaries in the low frequency regime.

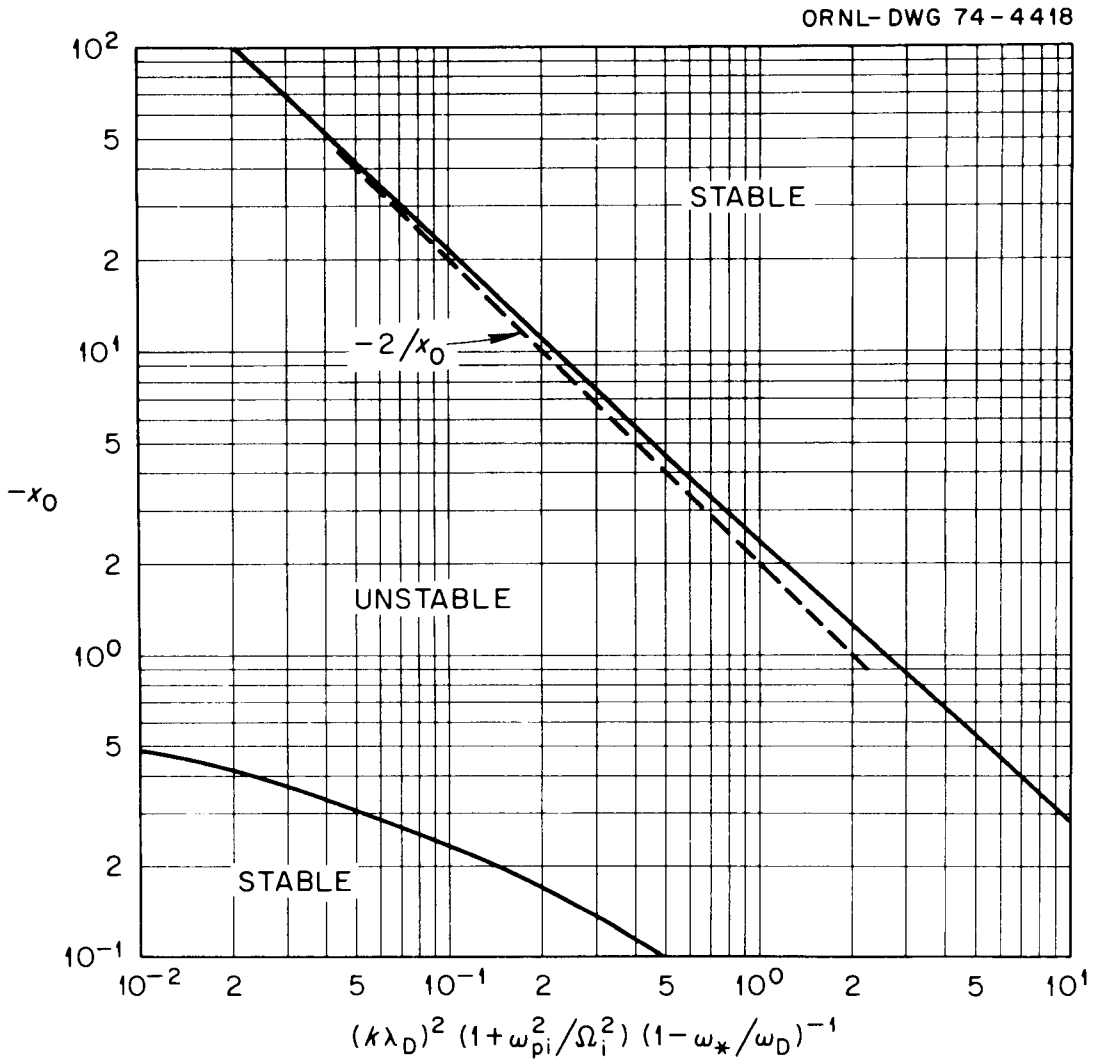

Fig. 2 Right Hand Side of Equation (21).

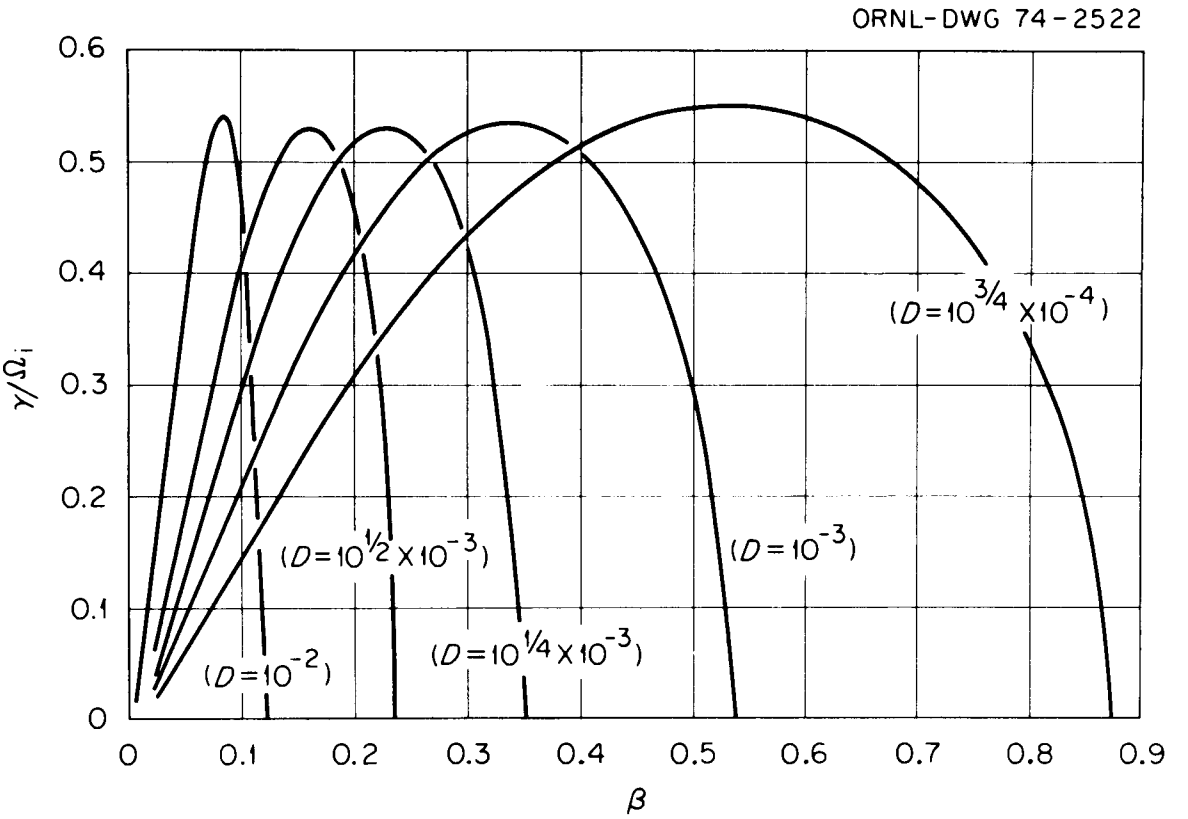

Fig. 3 Growth rate as a function of plasma pressure, β, for several values of the parameter $D \equiv c^2/(2r^2\,\Omega_e^2)$.