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# Curvilinear Coordinates for Magnetic Confinement Geometries

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UNION CARBIDE CORPORATION
FOR THE UNITED STATES
DEPARTMENT OF ENERGY

Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, Virginia 22161
NTIS price codes—Printed Copy: A03; Microfiche A01

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ORNL/TM-8393
Dist. Category UC-20

Contract No. W-7405-eng-26

FUSION ENERGY DIVISION

# CURVILINEAR COORDINATES FOR MAGNETIC CONFINEMENT GEOMETRIES

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Date Published - August 1982

Prepared by the
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UNION CARBIDE CORPORATION
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DEPARTMENT OF ENERGY

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## ABSTRACT

The basic properties of curvilinear coordinates are reviewed. Some applications to the description of three-dimensional magnetic confinement geometries are cited. The notation used here attempts to be consistent with the literature, and the relation to differential geometry is stressed.

## 1. CURVILINEAR COORDINATES

Let any point P be denoted by its Cartesian coordinates $(x,y,z) \equiv (x_1,x_2,x_3)$. Then, provided the transformation of the coordinates $x_i = x_i(u_1,u_2,u_3)$ has a unique and differentiable inverse $u_i = u_i(x_1,x_2,x_3)$, the curvilinear coordinates of P are defined as $(u_1,u_2,u_3)$.

The surfaces $u_i = c_i$, where $c_i$ = constant, are coordinate surfaces. The curve along which a pair of surfaces $u_i = c_i$ and $u_j = c_j$ intersect is a $u_k$ coordinate curve (with $i \neq j \neq k$). Thus, for example, a $u_1$ coordinate curve is a curve along which $u_2$ and $u_3$ are constant but along which $u_1$ varies (see Fig. 1).

## 2. COVARIANT AND CONTRAVARIANT BASIS VECTORS

Consider the position vector $\vec{r} = x_j\vec{i}_j$ to the point P, where $\vec{i}_j$ are the orthogonal unit Cartesian coordinate vectors. The <u>covariant</u> basis vectors $\vec{e}_i$ are defined as

$$\vec{e}_i \equiv \frac{\partial \vec{r}}{\partial u_i} \quad . \tag{1}$$

Since the derivative is taken along the $u_i$ coordinate curve (i.e., with $u_j$ and $u_k$ held fixed), $\vec{e}_i$ is the tangent vector to the $u_i$ curve through P (see Fig. 2). Note that $\vec{e}_i$ and $\vec{e}_j$ are generally <u>not</u> orthogonal vectors (i.e., for $i \neq j$, $\vec{e}_i \cdot \vec{e}_j \neq 0$) nor are they unit vectors (i.e., $\vec{e}_i \cdot \vec{e}_i \neq 1$). The <u>contravariant</u> basis vectors $\vec{e}^{\,i}$ are defined as the vectors normal to the $u_i$ coordinate surfaces:

$$\vec{e}^{\,i} \equiv \nabla u_i \quad . \tag{2}$$

Since $\vec{e}_i$ is tangential to the surfaces of constant $u_j$ and $u_k$, and $\vec{e}^{\,i}$ is normal to the constant $u_i$ surface, it follows that $\vec{e}_i \cdot \vec{e}^{\,j} = 0$ for $i \neq j$. In fact, since $\vec{e}_i = (\partial x_k/\partial u_i)\vec{i}_k$, where the summation convention on repeated indices is implied, we find upon applying the chain rule $du_j = (\partial u_j/\partial x_k)dx_k$ that

$$\vec{e}_i \cdot \vec{e}^{\,j} = \left(\frac{\partial x_k}{\partial u_i}\right)\left(\frac{\partial u_j}{\partial x_k}\right) = \frac{du_j}{du_i} = \delta_{ij} \quad .$$

Thus, $\vec{e}_i$ and $\vec{e}^{\,j}$ are reciprocal, or adjoint, vectors:

$$\vec{e}_i \cdot \vec{e}^{\,j} = \delta_{ij} \quad . \tag{3}$$

In particular, for any cyclic permutation (i,j,k), Eq. (3) implies

$$\begin{aligned}\vec{e}_i &= \sqrt{g}\,(\vec{e}^{\,j} \times \vec{e}^{\,k}) \\ &= \sqrt{g}(\nabla u_j \times \nabla u_k) \quad ,\end{aligned} \tag{4}$$

where $\sqrt{g} \equiv (\vec{e}^{\,1} \cdot \vec{e}^{\,2} \times \vec{e}^{\,3})^{-1} = (\nabla u_1 \cdot \nabla u_2 \times \nabla u_3)^{-1}$ will be shown to be the Jacobian of the transformation from Cartesian coordinates to curvilinear coordinates $(u_i)$. Taking the cross product of $\vec{e}_i$ and $\vec{e}_j$ and using Eq. (4) yields

$$\vec{e}^{\,i} = g^{-1/2}\vec{e}_j \times \vec{e}_k \quad . \tag{5}$$

Thus, Eqs. (3) and (5) imply

$$\sqrt{g} = \vec{e}_1 \times \vec{e}_2 \cdot \vec{e}_3$$

$$= \frac{\partial \vec{r}}{\partial u_1} \times \frac{\partial \vec{r}}{\partial u_2} \cdot \frac{\partial \vec{r}}{\partial u_3} , \tag{6}$$

which shows that $\sqrt{g}$ is, indeed, the Jacobian.

Using the adjoint relation given in Eq. (3), it is possible to decompose any vector $\vec{A}$ in terms of the basis vectors $\vec{e}_i$ or $\vec{e}^i$:

$$\vec{A} = A^i \vec{e}_i = A_i \vec{e}^i , \tag{7}$$

where

$$A^i = \vec{A} \cdot \vec{e}^i = \vec{A} \cdot \nabla u_i \tag{8a}$$

and

$$A_i = \vec{A} \cdot \vec{e}_i = \sqrt{g} \vec{A} \cdot \nabla u_j \times \nabla u_k . \tag{8b}$$

The coefficients $A^i$ or $A_i$ are the contravariant components or the covariant components of $\vec{A}$, respectively. In Eq. (7), and in what follows, summation over repeated indices is implied.

## 3. VECTOR OPERATORS IN CURVILINEAR COORDINATES

For any scalar function $\Phi$, the gradient operator $\nabla\Phi$ is defined as

$$d\Phi \equiv \frac{\partial\Phi}{\partial u_i} du_i = \nabla\Phi \cdot d\vec{r} \quad , \tag{9}$$

where

$$d\vec{r} = \frac{\partial\vec{r}}{\partial u_i} du_i = \vec{e}_i du_i \quad . \tag{10}$$

Equating coefficients of $du_i$ in Eq. (9) yields the covariant components of $\nabla\Phi$: $\vec{e}_i \cdot \nabla\Phi = \partial\Phi/\partial u_i$. Thus, using Eq. (7) yields

$$\nabla\Phi = \frac{\partial\Phi}{\partial u_i} \vec{e}^{\,i} = \frac{\partial\Phi}{\partial u_i} \nabla u_i \quad . \tag{11}$$

A mnemonic for Eq. (11) is obtained by setting $\Phi = u_i$, from which the identity $\nabla u_i = \nabla u_i$ results.

Now, consider the vector identities:

$$\nabla \cdot (g^{-1/2}\vec{e}_i) = \nabla \cdot (\nabla u_j \times \nabla u_k) = 0 \tag{12a}$$

and

$$\nabla \times \vec{e}^{\,i} = \nabla \times (\nabla u_i) = 0 \quad . \tag{12b}$$

Then, using Eq. (7) in the form $\vec{A} = (g^{1/2}A^i)(g^{-1/2}\vec{e}_i)$, together with Eq. (12a), yields the curvilinear formula for the divergence operator:

$$\nabla \cdot \vec{A} = g^{-1/2}\vec{e}_i \cdot \nabla(g^{1/2}A^i)$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial u_i}(\sqrt{g}A^i) \quad . \tag{13}$$

Using Eqs. (7) and (12b) yields the curl operator

$$\nabla \times \vec{A} = \nabla A_j \times \vec{e}^j = \frac{\partial A_j}{\partial u_i} \vec{e}^i \times \vec{e}^j \tag{14a}$$

or, using Eq. (4),

$$\nabla \times \vec{A} = \frac{1}{\sqrt{g}} \varepsilon_{ijk} \frac{\partial A_j}{\partial u_i} \vec{e}_k \quad . \tag{14b}$$

Finally, the Laplacian operator $\Delta\Phi \equiv \nabla \cdot (\nabla\Phi)$ can be obtained by using Eq. (13) with $A^i = (\partial\Phi/\partial u_j)\nabla u_i \cdot \nabla u_j$. Denoting $\vec{e}^i \cdot \vec{e}^j \equiv g^{ij}$ yields

$$\Delta\Phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u_i} \left( \sqrt{g}\, g^{ij} \frac{\partial\Phi}{\partial u_j} \right) \quad . \tag{15}$$

## 4. METRIC TENSOR AND DIFFERENTIAL GEOMETRY RELATIONS

The differential arclength $ds^2 \equiv d\vec{r} \cdot d\vec{r}$ can be written [using Eq. (10)]

$$ds^2 = g_{ij}du_i du_j \quad , \tag{16}$$

where the covariant metric tensor components are

$$g_{ij} \equiv \vec{e}_i \cdot \vec{e}_j = g_{ji} \quad . \tag{17}$$

Alternate and useful forms for $g_{ij}$ follow from the various forms for $\vec{e}_i$. For example, $g_{ij}$ may be computed from the following relation:

$$g_{ij} = \sum_k \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \quad . \tag{18}$$

Using $g_{ij}$, the covariant base vectors $\vec{e}_i$ may be linearly decomposed in terms of the contravariant base vectors $\vec{e}^i$:

$$\vec{e}_i = g_{ij}\vec{e}^j \quad . \tag{19a}$$

Similarly, using $g^{ij} = \vec{e}^i \cdot \vec{e}^j$, which was introduced in Eq. (15), yields

$$\vec{e}^i = g^{ij}\vec{e}_j \quad . \tag{19b}$$

The $g^{ij}$ matrix is the <u>inverse</u> of the metric tensor $g_{ij}$, since from

Eq. (3) $\vec{e}_i \cdot \vec{e}^j = \delta_{ij} = \sum_{k,\ell} g_{ik} g^{\ell j} \vec{e}^k \cdot \vec{e}_\ell = \sum_k g_{ik} g^{kj}$ . Note that when $\sqrt{g} \neq 0$, $g^{ij}$ can be computed in terms of the metric tensor elements by using Eq. (5) to express $\vec{e}^i$ and $\vec{e}^j$ in terms of the adjoint vectors.

Equation (19a) can be used to express the Jacobian $\sqrt{g}$ in terms of $g_{ij}$. From Eq. (6), note that

$$\sqrt{g} = g_{1i} g_{2j} g_{3k}\, \vec{e}^i \times \vec{e}^j \cdot \vec{e}^k$$

$$= \epsilon_{ijk} g_{1i} g_{2j} g_{3k} g^{-1/2} = \frac{1}{\sqrt{g}} \det g_{ij} \quad . \tag{20}$$

Therefore,

$$g = |\det g_{ij}| \quad . \tag{21}$$

Now, the element of the line segment $d\vec{s}_i$ along the coordinate curve $u_i$ is given by

$$d\vec{s}_i = \left(\frac{\partial \vec{r}}{\partial u_i}\right) du_i = \vec{e}_i du_i \quad . \tag{22}$$

(Here, there is no implied summation.) The element of surface area $d\vec{\sigma}_i$ directed normal to the coordinate surface $u_i$ = constant is

$$d\vec{\sigma}_i \equiv \frac{\partial \vec{r}}{\partial u_j} \times \frac{\partial \vec{r}}{\partial u_k} du_j du_k$$

$$= \vec{e}_j \times \vec{e}_k du_j du_k \quad . \tag{23}$$

Using Eq. (5), this becomes (with i,j,k comprising a cyclic triplet)

$$d\vec{\sigma}_i = \sqrt{g}\vec{e}^i du_j du_k \quad . \tag{24a}$$

Note from Eq. (23) that

$$\begin{aligned} |d\vec{\sigma}_i| &= |\vec{e}_j \times \vec{e}_k| du_j du_k \\ &= (g_{jj}g_{kk} - g_{jk}^2)^{1/2} du_j du_k \quad . \end{aligned} \tag{24b}$$

Finally, the element of volume is given by

$$\begin{aligned} dV &= \frac{\partial \vec{r}}{\partial u_1} du_1 \times \frac{\partial \vec{r}}{\partial u_2} du_2 \cdot \frac{\partial \vec{r}}{\partial u_3} du_3 \\ &= \sqrt{g} du_1 du_2 du_3 \quad . \end{aligned} \tag{25}$$

## 5. TOROIDAL DIFFERENTIAL GEOMETRY

Let $(R,\phi,Z)$ be a cylindrical coordinate system, with $\phi$ the toroidal angle, R the major radius measured from the center of the torus bore, and Z the height measured from the torus midplane. Then,

$$\vec{r} = R\cos\phi\hat{x} + R\sin\phi\hat{y} + Z\hat{z} = R\hat{r} + Z\hat{z} \quad . \tag{26}$$

Now, let $(u_1,u_2,u_3)$ represent an arbitrary flux coordinate system, and consider the map $(R,\phi,Z) \rightarrow (u_1,u_2,u_3)$. The metric elements $g_{ij} = (\partial\vec{r}/\partial u_i) \cdot (\partial\vec{r}/\partial u_j)$ are readily evaluated from Eq. (26), noting

$(\partial\vec{r}/\partial u_i) = (\partial R/\partial u_i)\hat{r} + (R\partial\phi/\partial u_i)\hat{\phi} + (\partial Z/\partial u_i)\hat{z}$, where $\hat{\phi} = \partial\hat{r}/\partial\phi$ is a unit vector orthogonal to $\hat{r}$ and $\hat{z}$. Thus,

$$g_{ij} = \frac{\partial R}{\partial u_i}\frac{\partial R}{\partial u_j} + R^2 \frac{\partial \phi}{\partial u_i}\frac{\partial \phi}{\partial u_j} + \frac{\partial Z}{\partial u_i}\frac{\partial Z}{\partial u_j} \quad . \tag{27}$$

Using Eq. (6) for the Jacobian yields

$$\sqrt{g} = R\,\frac{\partial(R,\phi,Z)}{\partial(u_1,u_2,u_3)} \quad . \tag{28}$$

## 6. APPLICATION TO MAGNETIC GEOMETRY

Now consider the MHD equilibrium equations, which determine the magnetic flux coordinate geometry.[1] Let $u_1 = \rho$ (flux-surface label), $u_2 = \theta$ (poloidal angle), and $u_3 = \zeta$ (toroidal angle) be cyclic flux coordinates. From the pressure balance equation

$$\nabla p = \vec{J} \times \vec{B} \quad , \tag{29}$$

note that $\vec{B} \cdot \nabla u_1 = \vec{J} \cdot \nabla u_1 = 0$. Thus, from $\nabla \cdot \vec{B} = 0$ in curvilinear coordinates yields

$$\frac{\partial}{\partial u_2}(\sqrt{g}B^2) + \frac{\partial}{\partial u_3}(\sqrt{g}B^3) = 0 \quad , \tag{30}$$

from which

$$B^2 = -\frac{1}{\sqrt{g}}\frac{\partial\eta}{\partial u_3} \tag{31a}$$

and

$$B^3 = \frac{1}{\sqrt{g}}\frac{\partial\eta}{\partial u_2} \quad . \tag{31b}$$

Since $B^2$ and $B^3$ must be periodic functions of the angles $u_2$ and $u_3$, $\eta = -\chi' u_3 + \Phi' u_2 + \tilde{\eta}(u_1,u_2,u_3)$, where $\tilde{\eta}$ is a periodic function of $(u_2,u_3)$; $\chi(u_1)$ and $\Phi(u_1)$ are (within numerical factors) the poloidal and toroidal magnetic fluxes, respectively, outside or enclosed by the surface $u_1$ = constant; and prime denotes $\partial/\partial u_1$. For example, the toroidal flux is $\Phi_T(u_1) \equiv \int^{u_1} \vec{B}\cdot d\vec{\sigma}_3 = \int^{u_1} du_1 \int_0^{2\pi} du_2\sqrt{g}\,\vec{B}\cdot\vec{e}^3$, where $d\vec{\sigma}_3$ is given in Eq. (23). Using Eq. (31b), it follows that $\Phi_T(u_1) = 2\pi\int^{u_1} du_1\Phi' = 2\pi\Phi(u_1)$. Note that only the secular part of $\eta$ contributes to the enclosed flux.

Using this form for $\eta$, the contravariant representation for $\vec{B}$ becomes

$$\vec{B} = \Phi'\nabla u_1 \times \nabla u_2 + \chi'\nabla u_3 \times \nabla u_1 + \nabla u_1 \times \nabla\tilde{\eta} \quad . \tag{32a}$$

It is always possible to eliminate the last term in Eq. (32a) by a change of variables (e.g., $u_2' = u_2 + \tilde{\eta}/\Phi'$, $u_3' = u_3$). Then, in the primed coordinates, the magnetic field lines are straight on each magnetic surface, since $du_2'/du_3' \equiv B^2/B^3 = \chi'/\Phi' = \iota(u_1)$ (i.e., $u_2' - \iota(u_1)u_3'$ = constant). Here, $\iota(u_1)$ is the rotational transform, which is the change in poloidal angle for each unit change in toroidal angle. In this straight line system, Eq. (32a) reduces to

$$\vec{B} = B^2\vec{e}_2 + B^3\vec{e}_3 \tag{32b}$$

with $B^2 = \chi'/\sqrt{g}$ and $B^3 = \Phi'/\sqrt{g}$. Note that the covariant components $B_i = \vec{B} \cdot \vec{e}_i$ can be readily expressed in terms of $B^i$ by taking the inner product of Eq. (32b) with $\vec{e}_i$, yielding

$$B_i = B^2 g_{i2} + B^3 g_{i3} \quad , \tag{32c}$$

where $g_{ij}$ is the metric tensor. This relation will be subsequently used to determine the metric elements.

Let us digress momentarily and compare these relations with the well-known expression for $\vec{B}$ in an axisymmetric torus:

$$\begin{aligned} \vec{B} &= \chi'\nabla\zeta \times \nabla u_1 + F(u_1)\nabla\zeta \\ &= \chi'\nabla\zeta \times \nabla u_1 + \sqrt{g}R^{-2}\, F(u_1)\nabla u_1 \times \nabla\theta \ , \end{aligned} \tag{33a}$$

where $\sqrt{g} = (\nabla u_1 \cdot \nabla\theta \times \nabla\zeta)^{-1}$, $|\nabla\zeta| = R^{-1}$ ($\zeta$ is the geometric toroidal angle), and $\nabla\zeta \cdot \nabla u_1 = \nabla\zeta \cdot \nabla\theta = 0$. Here, $F(u_1) = RB_T$ is a flux function by virtue of $\vec{J} \cdot \nabla u_1 = \nabla \times \vec{B} \cdot \nabla u_1 = 0$. (Sometimes, the radial coordinate $u_1 = \chi$ is used, so that $\chi' = 1.0$.) Apparently, Eq. (33a) does not represent straight field lines unless $\sqrt{g}\, R^{-2} = f(u_1)$, which is not generally the case. However, comparing with Eq. (32a), note that $\eta = -\chi'\zeta + \langle\sqrt{g}\, R^{-2}\rangle_\theta\, F(u_1)\theta + \tilde{\eta}$, where

$\tilde{\eta} = F(u_1) \int_0^\theta \left(\sqrt{g}\, R^{-2} - \langle\sqrt{g}\, R^{-2}\rangle_\theta\right) d\theta$, $\Phi' = \langle\sqrt{g}\, R^{-2}\rangle_\theta F(u_1)$, and $\langle A\rangle_\theta \equiv (2\pi)^{-1} \int_0^{2\pi} A\, d\theta$. Thus, transforming to the angle

$$\theta' = \theta + \int_0^\theta \left( \frac{\sqrt{g}\, R^{-2}}{\langle\sqrt{g}\, R^{-2}\rangle_\theta} - 1 \right) d\theta$$

in Eq. (33a) yields

$$\vec{B} = \chi' \nabla\zeta \times \nabla u_1 + \Phi' \nabla u_1 \times \nabla\theta' \quad . \tag{33b}$$

The Jacobian in this straight field line system is $\sqrt{g}' = R^2 \langle\sqrt{g}\, R^{-2}\rangle_\theta$. In this coordinate system, $du_1 d\theta_2 d\zeta = (\sqrt{g}')^{-1} d^3x \sim R^{-2} d^3x$; hence, $u_1 \sim \int R^{-2} d^3x \sim \int B_v \cdot \nabla\phi d^3x$. Thus, the surface coordinate $u_1$ is proportional to the vacuum magnetic flux, which is nearly an adiabatic invariant.[2] Also note that the safety factor $q(u_1) \equiv \Phi'/\chi' = \langle\sqrt{g}\, R^{-2}\rangle_\theta F(u_1)/\chi' = F(u_1)\langle R^{-2}(\vec{B} \cdot \nabla\theta)^{-1}\rangle_\theta$ reduces to its well-known value $rB_T/RB_p$ in the large aspect ratio, circular surface limit.

Now consider the calculation of the current. Since $\vec{J} \cdot \nabla u_1 = 0$ and $\nabla \cdot \vec{J} = 0$, the contravariant form for $\vec{J}$ follows in analogy with Eq. (32a):

$$\vec{J} = \nabla u_1 \times \nabla\nu + J' \nabla u_1 \times \nabla u_2 - I' \nabla u_1 \times \nabla u_3$$

$$= J^2 \vec{e}_2 + J^3 \vec{e}_3 \quad , \tag{34a}$$

where

$$J^2 = \frac{(I' - \partial\nu/\partial u_3)}{\sqrt{g}} \quad , \tag{34b}$$

$$J^3 = \frac{(J' + \partial\nu/\partial u_2)}{\sqrt{g}} \quad , \tag{34c}$$

$I'(u_1)$ and $J'(u_1)$ are the poloidal and toroidal current densities, respectively, and $\nu$ is an arbitrary periodic function of $u_2$ and $u_3$. (The Hamada choice of coordinates makes $\nu = 0$, but this unnecessarily restricts the Jacobian to be constant.)

Next, the $\vec{e}^k$ components of Ampere's law $\nabla \times \vec{B} = \vec{J}$ yield, using Eq. (14b),

$$\varepsilon_{ijk} \frac{\partial B_j}{\partial u_i} = \sqrt{g}\, J^k \quad . \tag{35}$$

Solving Eq. (35) using $J^1 = 0$ and Eqs. (34b,c) yields

$$\vec{B} = \nabla\phi - \nu\nabla u_1 + J\nabla u_2 - I\nabla u_3 = B_i\vec{e}^i \quad ,$$

where

$$B_1 = -\nu + \frac{\partial\phi}{\partial u_1} \quad , \tag{36a}$$

$$B_2 = J + \frac{\partial\phi}{\partial u_2} \quad , \tag{36b}$$

and

$$B_3 = -I + \frac{\partial\phi}{\partial u_3} \quad . \tag{36c}$$

Here, $\phi$ is the scalar magnetic potential ($\vec{B} = \nabla\phi$ in the absence of internal currents $I' = J' = 0$). Let

$$\chi = -Iu_3 + Ju_2 + \phi \quad , \tag{37a}$$

where $\phi$ can be chosen to be a periodic function of $u_2$ and $u_3$. (Any secular part of $\phi$ can be absorbed into the flux functions I and J.) Then, the covariant decomposition of $\vec{B}$ becomes

$$\vec{B} = \nabla\chi + \beta\,\nabla\rho \quad , \tag{37b}$$

where $\beta = -\nu - J'u_2 + I'u_3$. [See Ref. 3, where $g = -I$, I is the parameter J, and $\beta_* = -\nu$. In Ref. 3 $\phi = 0$, which gives a special value for the Jacobian, as shown in Eq. (43).]

Equating Eq. (32c) with Eqs. (36a-c) yields

$$-\nu + \frac{\partial\phi}{\partial u_1} = B^2 g_{12} + B^3 g_{13} \quad , \tag{38a}$$

$$J + \frac{\partial\phi}{\partial u_2} = B^2 g_{22} + B^3 g_{23} \quad , \tag{38b}$$

and

$$-I + \frac{\partial\phi}{\partial u_3} = B^2 g_{32} + B^3 g_{33} \quad . \tag{38c}$$

Recall that $B^2 = \chi'/\sqrt{g}$ and $B^3 = \Phi'/\sqrt{g}$. [These are Eqs. (16)-(18) of Ref. 1.] These equations determine the flux coordinates once $\nu$, J, I, and $\phi$ are prescribed. For example, the metric elements $g_{ij}$ may be evaluated in terms of $(R,\phi,Z)$ using Eq. (27). Then, Eqs. (38a-c) are three coupled partial differential equations for the inverse equilibrium coordinates (as a function of the flux coordinates). These equations form the basis of a variational moment method for obtaining inverse equilibria. [See Ref. 4 for a two-dimensional solution of Eq. (11); Ref. 5 generalizes this method to three dimensions.]

Finally, let us determine $\nu$ from the pressure balance Eq. (29). Note that $\nabla p = \vec{e}^1 p'$, where prime denotes $\partial/\partial\rho$, and $(\vec{J} \times \vec{B}) \cdot \vec{e}^i \times \vec{e}^j = J^i B^j - J^j B^i$. Taking the $\vec{e}_i$ components of Eq. (29) and using Eqs. (3) and (4) yields

$$p'\delta_{i1} = \sqrt{g}\ \epsilon_{ijk} J^j B^k \quad . \tag{39}$$

For i = 2 or 3, this yields a trivial identity, since $B^1$ and $J^1$ both vanish. For i = 1, Eq. (39) yields

$$\sqrt{g}\ p' = I'\Phi' - J'\chi' - \sqrt{g}\ \vec{B} \cdot \nabla\nu \quad , \tag{40}$$

where $\sqrt{g}\vec{B} \cdot \nabla = \chi'\partial/\partial u_2 + \Phi'\partial/\partial u_3$. Integrating this equation over a field line yields a solubility constraint:

$$p'V' = I'\Phi' - J'\chi' \quad , \tag{41}$$

where $V' = \iint \sqrt{g}du_2du_3 \equiv \langle\sqrt{g}\rangle$. Subtracting Eq. (41) from (40) yields a magnetic differential equation for $\nu$:

$$\vec{B} \cdot \nabla\nu = p'\left(\frac{\langle\sqrt{g}\rangle}{\sqrt{g}} - 1\right), \tag{42}$$

which is Eq. (12) of Ref. 1. [To see that this is also Eq. (23) in Ref. 3, it suffices to note that for Ref. 3, $B^2 = (\imath J - I)/\sqrt{g}$. In general, $|B|^2 = B_2B^2 + B_3B^3 = g^{-1/2}(\chi' J - \Phi' I) + \vec{B} \cdot \nabla\phi$ or

$$\sqrt{g} = \frac{\chi' J - \Phi' I + \sqrt{g}\,\vec{B} \cdot \nabla\phi}{|B|^2} \quad . \tag{43}$$

It should be noted that the equilibrium Eqs. (38a-c) and (40) can be obtained from variational principles. For example, the variation of the Lagrangian[6]

$$L = \int \left(\frac{B^2}{2} - p\right) dV \tag{44}$$

with respect to the flux coordinates $(\rho,\theta,\zeta)$, when $\vec{B}$ is written in the contravariant form given by Eq. (32b), yields

$$- \chi'\vec{J} \cdot \nabla\zeta\ \Phi'\vec{J} \cdot \nabla\theta - p' = 0 \tag{45a}$$

and

$$\vec{J} \cdot \nabla\rho = 0 \quad , \tag{45b}$$

with $\vec{J} = \nabla \times \vec{B}$. Equations (45a,b) are the only nontrivial components of the pressure balance equation, since $\vec{B} \cdot \nabla\rho = 0$ is built into Eq. (32b). Equation (44) can also be used to obtain the three-dimensional inverse equilibrium equations:[5]

$$B^2 \frac{\partial B_2}{\partial \rho} + B^3 \frac{\partial B_3}{\partial \rho} - \vec{B} \cdot \nabla B_1 + p' = 0 \qquad (46a)$$

and

$$\frac{\partial B_3}{\partial \theta} - \frac{\partial B_2}{\partial \zeta} = 0 \quad . \qquad (46b)$$

These equations, with the covariant components $B_i$ given by Eq. (32c) in terms of the metric coefficients, can be used to determine the inverse equilibrium (i.e., the real coordinates $\vec{x}$ in terms of flux coordinates).

Finally, another variational principle[7] minimizing

$$W = \int \left( \frac{B^2}{2} + p \right) dV \quad , \qquad (47)$$

with respect to $\Phi$, $\nu$, and p when $\vec{B}$ is expressed in covariant form [Eq. (37b)] yields $\nabla \cdot B = \vec{B} \cdot \nabla p = 0$ and the equilibrium equation

$$\vec{B} \cdot \nabla\nu + J'\vec{B} \cdot \nabla\theta - I'\vec{B} \cdot \nabla\zeta + p' = 0 \quad , \qquad (48)$$

which is a generalization (to systems where $\vec{B}$ is not straight) of Eq. (40).

## REFERENCES


1. V. D. Shafranov, Nucl. Fusion 8, 253 (1968).
2. S. P. Hirshman and S. C. Jardin, Phys. Fluids 22, 731 (1979).
3. A. H. Boozer, Phys. Fluids 24, 1999 (1981).
4. L. L. Lao, S. P. Hirshman, and R. M. Wieland, Phys. Fluids 24, 1431 (1981).
5. A. Bhattacharjee, J. C. Wiley, and R. L. Dewar, IFS #48, University of Texas, Austin, Texas (1981).
6. H. Grad, Phys. Fluids 7, 1283 (1964).
7. S. P. Hirshman and J. C. Whitson, ORNL Theory Memo (1982).

## FIGURE CAPTIONS

FIG. 1. Curvilinear coordinate surfaces ($u_i = c_i$) and coordinate curves in the neighborhood of the point $P(x,y,z)$.

FIG. 2. Covariant and contravariant basis vectors ($\vec{e}_i$ and $\vec{e}^i$, respectively) at the point $P(x,y,z)$.

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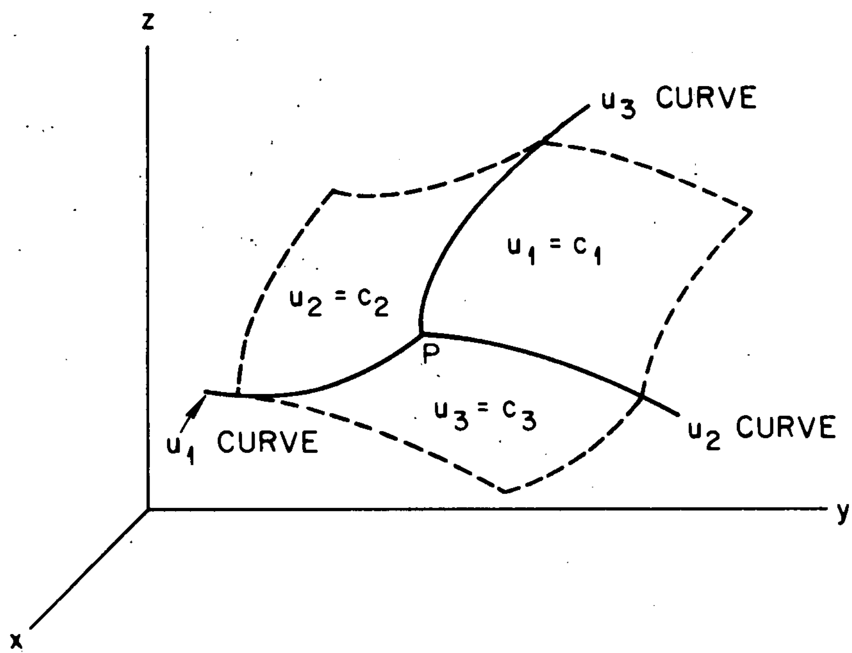


Fig. 1

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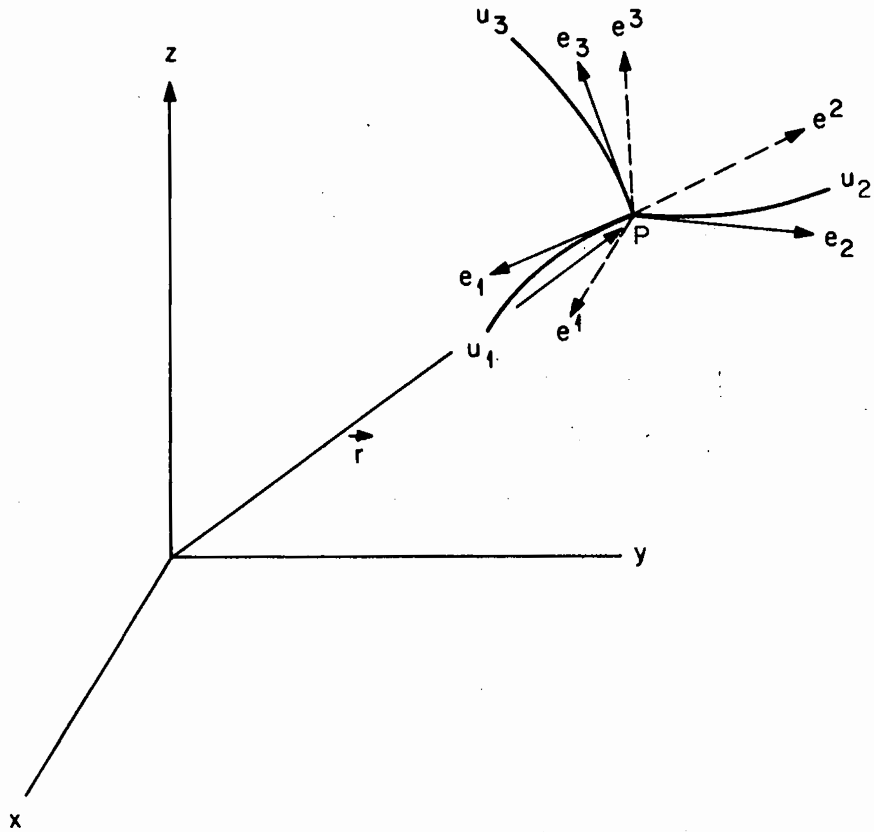


Fig. 2

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