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# Propagators for General Nonlinear Equations

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# PROPAGATORS FOR GENERAL NONLINEAR EQUATIONS

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*ABSTRACT*: We introduce propagators – advanced and retarded – for nonlinear problems and show that they generalize the Green's functions used to solve linear problems. These propagators are generated by using the *dual* operator which plays, for the nonlinear problem, the same role as the customary adjoint operator for linear problems. The propagators obey a reciprocity relationship and satisfy a closed nonlinear integral equation or, alternatively, a linear integral equation that depends parametrically on the problem's solution. These equations are formulated in a canonical way, independent of dimensionality, boundary conditions and type of the underlying nonlinear problem.

We consider a general nonlinear problem represented as

$$N(u(x);\alpha(x)) = 0, \quad \text{in } \Omega \ , \tag{1}$$

where $N$ contains integral, multiplicative, and differential operators and $x = (x_1, \ldots, x_n) \in \Omega \subset R^n$ is the phase-space position vector. When the problem is time dependent, the set $(x_1,\ldots,x_n)$ contains $t$ as one of the independent variables, and Eq. (1) may reduce to an evolution equation. The state vector $u \in H_1$ and its image $N(u) \in H_2$ are considered to be scalar-valued functions; $H_1$ and $H_2$ are real Hilbert spaces endowed with inner products $<,>$ and $[,]$, respectively. (Distinct Hilbert spaces are chosen to highlight the duality interplay.) Furthermore, $\alpha(x) = (\alpha_1, \ldots, \alpha_I)$ is the vector of parameters describing the physical system. In general, a set of boundary/initial conditions (B.C.)

$$B(u,\alpha) = 0, \quad \text{on } \partial\Omega \ , \tag{2}$$

is needed to specify the problem. Inhomogeneous terms (sources) are represented by the nonzero components of $N(0,\alpha)$ and $B(0,\alpha)$.

A general analytical formalism for solving Eqs. (1) and (2), such as the Green's function method in linear theory, is presently unavailable. Also, the known methods, e.g., inverse scattering transform,[1] Lax-pair representation,[2] group invariance,[3] and similarity, impose (sometimes severe) restrictions on $N$, $\Omega$, and $B$.

This Letter presents a new, canonical, and practical method for solving general nonlinear equations by recasting them into a form amenable to treatment by powerful techniques from linear theory.

*The dual operator.* — The crux of our method is the construction of an operator $N^+(u)$, henceforth referred to as the operator *dual* to $N(u)$, which will play the role of the customary adjoint operator in linear theory. This dual is defined via the relationship

$$[N(u),\ v] = \langle u,\ N^{+}(u)v\rangle \quad , \tag{3}$$

for appropriate $v \epsilon H_2$, and is constructed by using the fundamental theorem of calculus in non-linear functional analysis,[4]

$$N(u_o + h) - N(u_o) = \int_0^1 \delta N(u_o + \epsilon h,\ h) d\epsilon \ , \tag{4}$$

where $\delta N(u_o;\ h) := dN(u_o + \epsilon h)/d\epsilon|_{\epsilon=0}$ is the first Gâteaux variation of $N(u)$ at $u_o$ along $h$. When $\delta N(u_o;\ h)$ is linear in $h$, then $\delta N(u_o;\ h) = N'(u_o)h$, where $N'(u_o)$ operates linearly on $h$ and is called the first Gâteaux derivative of $N(u)$ at $u_o$. In this case, we define the formal adjoint of $N'(u)$, $[N'(u)]^*$, via the usual (linear) relation

$$[N'(u)h,\ v] = \langle h,\ [N'(u)]^* v\rangle + \{h, \sigma^*(u)v\} \ , \tag{5}$$

where $\sigma^*(u)v$ represents the non-vanishing boundary terms and the braces $\{,\}$ denote the corresponding bilinear form on the boundary space. The equations $N'(u)h = 0$, $[N'(u)]^* v = 0$ are usually referred to as the direct and adjoint variational (gradient, tangential) equations associated with Eq. (1). They have been used previously to some extent (mostly the direct equation), for investigating the (possibly Hamiltonian) structure of the original nonlinear equation.[1,2,3,5] We mention also the interesting use of the adjoint variational equation in Ref. 6, enabling one to write closed equations for suitable generating functionals of the solution.

Using (5) and (4) taken at $u_o = 0$ and $h = u$, together with (3), gives the expression of the dual operator $N^{+}(u)$ as

$$N^{+}(u)v = \overline{u^{-1}N(0)v} + L^{*}(u)v + \Sigma^{*}(u)v \ , \tag{6}$$

where the overbar denotes complex conjugation,

$$L^{*}(u)v := \int_0^1 [N'(\epsilon u)]^{*} v d\epsilon \quad , \tag{7}$$

and

$$\Sigma^{*}(u)v := \int_0^1 \sigma^{*}(\epsilon u) v d\epsilon \quad . \tag{8}$$

Just like the operators $[N'(u)]^{*}$ and $\sigma^{*}(u)$, the operators $L^{*}(u)$ and $\Sigma^{*}(u)$ act linearly on the vector $v$ (while depending parametrically on $(u)$ in a (generally) nonlinear way). The operator $L^{*}(u)$, which we shall call the *predual* of *N(u)*, will be used to construct the propagator for the Eq. (1).

*Propagators and solutions.* — Consider $G_u^{*}(x,x')$ to be the unique solution of the *linear* system

$$L^{*}[u(x)]G_u^{*}(x,x') = \delta(x-x'), \quad \text{in } \Omega \quad , \tag{9}$$

$$\gamma^{*}[G_u^{*}(x,x')] = 0, \text{ on } \partial\Omega \quad , \tag{10}$$

where the operator $\gamma^{*}$, acting *linearly* on $G_u^{*}$, represents B.C. for $G_u^{*}$ on $\partial\Omega$ as specified below.

Using (6) together with (1) and (9) shows that calculating the difference between the inner products of *N(u)* with $G_u^{*}$ and of $u$ with $L^{*}G_u^{*}$, leads to the equation

$$u(x') = -\{u(x),\Sigma^*(u(x))G_u^*(x,x')\} - \langle u(x),\ \overline{u^{-1}N(0)}G_u^*(x,x')\rangle \quad . \tag{11}$$

The B.C. represented by $\gamma^*(G_u^*)$ in Eq. (10) are now chosen so as to maximize the simplification of (a) the boundary term appearing on the right-side of (11) and (b) the procedure needed to solve Eqs. (9) and (10) to obtain $G_u^*(x,x')$.

The selection of $\gamma^*(G_u^*)$, which in general is not unique, does not affect the uniqueness of the solution $u(x)$ given by (11). Equation (11) highlights the role of $G_u^*(x,x')$ as an "advanced propagator" for the solution $u(x)$, since $G_u^*$ propagates the source term $N(0)$ and the B.C. for $u$, from $u(x)$ to $u(x')$, just like the customary propagator does for the linear problems in field theory. Note, though, that in contrast to these linear problems, the advanced propagator $G_u^*(x,x')$ for the nonlinear system represented by Eqs. (1) and (2) depends implicitly and, in general, nonlinearly on the solution $u(x)$.

Since $L^*(u)$ acts linearly on $G_u^*$, we can construct its formal adjoint

$$L(u) := \int_0^1 N'(\epsilon u)d\epsilon \quad , \tag{12}$$

which we shall call the *antidual* of $N(u)$, and consider the system

$$L(u)G_u(x,x') = \delta(x-x') \ , \qquad \text{in } \Omega \ , \tag{13}$$

$$\gamma(G_u) = 0, \qquad \text{on } \partial\Omega \ . \tag{14}$$

In (14), the operator $\gamma$ acts linearly on $G_u$ and represents B.C. for $G_u$ chosen such that $\{G_u,\Sigma^*(u)G_u^*\}=0$ when $G_u^*$ and $G_u$ satisfy (10) and (14), respectively.

Forming the inner products of (13) and (9) with $G_u^*$ and $G_u$, respectively, and using (10), (12) and (14) leads to

$$G_u^*(x,x') = G_u(x',x) \quad . \tag{15}$$

Equation (15) is the reciprocity relationship between the propagators $G_u^*$ and $G_u$. Formally, this reciprocity relationship is identical to that encountered between the advanced and retarded propagators for *linear* problems. Furthermore, it follows from Eqs. (11) and (15) that the solution $u(x)$ of the nonlinear problem represented by (1) and (2) can also be expressed in terms of the retarded propagator $G_u$.

*Integral equations for propagators.* – Equation (11) gives the solution $u(x)$ of the nonlinear system represented by (1) and (2) as an integral equation involving the advanced propagator. Both this equation and its equivalent representation in terms of $G_u$ are formally similar to those encountered in linear theory, e.g., field theory. But, in contradistinction to the linear theory, both Eq. (10) and its equivalent representation in terms of $G_u$ are *nonlinear integral equations* for $u$ since both the retarded and the advanced propagators depend parametrically on $u$. The closed nonlinear integral equation for $u$ is obtained by solving the linear system (9) and (10) and replacing the solution $G^*(u)$ in (11). Clearly, this procedure requires inverting the operator $L^*(u)$, which is difficult but in principle is possible, since one can apply all the methods of linear theory.

Alternatively, a useful integral equation for the advanced propagator $G_u^*$ (or, respectively, the retarded propagator $G_u$) can be obtained by using a known, but otherwise arbitrary, vector $u_o$ in Eqs. (13) and (14) (or, respectively, in Eqs. (9) and (10)) to obtain a particular solution $G_o$ (or, respectively, $G_o^*$). Then, substracting the inner product $[L(u_o)G_o,G_u^*]$ from the inner product $<G_o,L^*(u)G_u^*>$, using (15), and recalling that $L(u)$ is the formal adjoint of $L^*(u)$ gives

$$G_u^*(x,x') = G_o^*(x,x') + \langle G_o(x'',x),$$
$$\left[L^*(u_o(x'')) - L^*(u(x''))\right]G_u^*(x'',x')\rangle + \left\{G_o(x'',x),\Sigma^*(u_o(x''))G_u^*(x'',x')\right\} . \tag{16}$$

Replacing $u$ everywhere in (16) by its expression (11) gives a closed integral equation for the forward propagator. A similar equation can be obtained for the backward propagator $G_u$.

In many cases, e.g. KdV-type equations, (16) becomes a purely integral equation. For general nonlinear problems, though, both $G_u^*$ and $G_u$ will satisfy nonlinear *integro-differential* equations. Nevertheless, the order of the derivative(s) in these equations is *lower* than the order of the derivative(s) appearing in the highest nonlinear term of the original equation; this fact is advantageous for computations.

*Series expansions.* — Consider, as before, that $u_o(x)$ represents an arbitrary but known state vector and that $h(x) \equiv u(x)-u_o(x)$ represents the difference between $u_o(x)$ and the solution $u(x)$ of (1) and (2). The equation for $h(x)$ can be obtained by applying (4) to (11). Furthermore, the relationship between the advanced propagator $G_u^*(x,x')$ and $h(x)$ is obtained by applying (4) to (16) and expanding $L^*(u)$ in a functional Taylor series in $h$ and $u_o$. This yields

$$G_u^*(x,x') = G_o^*(x,x') - \langle G_o(x'',x),$$
$$\left[\sum_{k=1}^{\infty}(1/k!)L^{*(k)}(u_o)h^k\right]G_u^*(x'',x')\rangle + \left\{G_o(x'',x),\Sigma^*(u_o(x''))G_u^*(x'',x')\right\} , \tag{17}$$

where the $k$-linear operator $L^{*(k)}(u_o)$ is the $k$-th Gâteaux derivative of $L^*$ at $u_o$ along $h$.

Equation (17) represents a nonlinear generalization of the well-known Lippman-Schwinger equation and therefore can be used just like the Lippman-Schwinger expansion is used in linear problems. Equation (17) is particularly useful when $L(u)$ has a polynomial dependence on $u$, so

that the series involves only a finite number of terms. In addition, Eq. (17) can be used to obtain closed-form expressions for perturbation diagrams, which is difficult to accomplish for nonlinear problems by present methods. Moreover, when $u_o$ is a nominal solution of the nonlinear system (1) and (2), Eq. (17) can be used to perform an efficient sensitivity analysis[7] to assess the variations around $u_o$ (or any function thereof) induced by variations in the system's parameters.

*Conclusions.* — This Letter has presented a new and canonical formalism for solving general nonlinear equations without restrictions on dimensionality, boundary conditions, or special structure (e.g., evolution) of the underlying nonlinear operators.

The predual and antidual operators $L^*(u)$ and $L(u)$ surpass the information contained in the variational operators $[N'(u)]^*$ and $N'(u)$ in that they incorporate the full nonlinear structure of $N$ (observe that $L(u)u = N(u) - N(0)$). Therefore, $G_u^*$ and $G_u$ also carry the information needed to solve the original nonlinear problem as expressed by (11).

The propagators $G_u^*$ and $G_u$ generalize the customary Green's functions used for solving linear problems; in particular, $G_u^*$ and $G_u$ reduce to these Green's functions if the original problem, described by (1) and (2), is linear. In such a case, $N^+(u)$ becomes the actual adjoint of $N(u)$, $L^*(u)$ becomes the formal adjoint of $N(u)$, and (9) and (10) describe the problem adjoint to (1) and (2). Furthermore, both $L$ and $L^*$ become independent of $u$, so $G$ becomes the Green's function, while $G^*$ becomes the adjoint Green's function for the linear problem described by (1) and (2); of course, $G$ and $G^*$ will also be independent of $u$.

The propagators $G_u^*$ and $G_u$ are defined as solutions of the linear equations (9)-(10) and (13)-(14), respectively, which still contain a (nonlinear) parametric dependence on the solution $u$. This parametric dependence can be eliminated so that $G_u^*$ and $G_u$ satisfy *closed* nonlinear integral equations. These integral forms open the way to using contraction principle and fixed point

arguments for proving existence and uniqueness of the solution $u$. Furthermore, the integral forms facilitate the computation and numerical analysis of the solution.

The formalism presented in this Letter has been successfully tested on two problems that admit analytical solutions: (a) a Riccati equation with linear and nonlinear B.C. and (b) a heat equation with nonlinear B.C. The detailed derivation of the formalism and the analysis of various examples will be published elsewhere.[8]

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