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The Bang-Bang Production of Depletable Natural Resources

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THE BANG-BANG PRODUCTION OF DEPLETABLE NATURAL RESOURCES

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ABSTRACT

This paper reconsiders the problem of determining the optimal production path for a depletable natural resource. The classical result of Hotelling is that the resource owner is indifferent between producing and not producing when the net price of the resource is increasing with the interest rate. However, the sharp increases and decreases in oil price in the last decade illustrate that the net price may not always increase with the interest rate. When the net price is not increasing with the interest rate, the Pontryagin Maximum Principle can be used to extend the classical result to a problem with a bang-bang production schedule and to problems with increasing extraction costs.

I. INTRODUCTION

Since the seminal paper by Hotelling (1931), economists have known that when the market is in equilibrium, the net price to an owner of a depletable natural resource must increase with the interest rate. To quote Solow (1974), "It is hard to overemphasize the importance of this tilt in the time profile for net price. If the net price were to rise too slowly, production would be pushed nearer in time and the resource would be exhausted quickly, precisely because no one would wish to hold resources in the ground and earn less than the going rate of return. If the net price were to rise too fast, resource deposits would be an excellent way to hold wealth, and owners would delay production while they enjoyed supernormal capital gains."

However, the behavior of the oil market since 1974 illustrates that the net price of a depletable natural resource can increase faster or slower than the interest rate. Since 1974, a small group of producers have caused two sharp increases in oil price and one sharp decrease. The oil market has not been in equilibrium and most producers have faced exogenous prices.

In this paper, we calculate the optimal production path for an owner of a depletable natural resource for the case where the market is not in equilibrium and the net price is not increasing with the interest rate. We shall begin with the case where the extraction costs are constant and subsequently consider the case where the extraction costs increase with cumulative consumption. We shall find that the Maximum Principle of Pontryagin (1962) is a fruitful method for solving the problems. When the price of the resource is exogenous, the optimal production path is bang-bang; that is, the resource owner is either at full production or at zero production. The key decision for the resource owner is the switch time, time, when to start or stop production. For our problem, the Hotelling rule is the switching rule, rather than a forecast of the net price. Our results are an extension of the work of Clark (1976).

II. THE BASIC PROBLEM

Consider an owner of a finite stock of a depletable natural resource who knows the future price (P(t)) for the resource and wishes to maximize his profits. If his extraction costs are C, then his profit is P(t)-C. If the owner uses a discount rate (r) to compare future profits to present profits, then the objective of the resource owner is to maximize the discounted value of his profits (J):

$$J = \int_0^T [P(t) - C]\, q(t)\, e^{-rt}\, dt \quad , \tag{1}$$

where q(t) is the production rate for the resource.

We will assume that the production rate is bounded:

$$D(t) \geq q(t) \geq 0 \quad , \tag{2}$$

where D(t) is given and finite. Since the resource is exhaustible, we assume that the owner's stock of the resource is finite:

$$Q(T) = \int_0^T q(t)\, dt \;\leq\; Q^* \quad . \tag{3}$$

The optimization problem for the resource owner is to find a production rate [q(t)] that satisfies the conditions of Eqs. (2) and (3) and maximizes Eq. (1). We shall call this optimization problem the basic problem.

We can obtain the solution of the basic problem from first principles. Let θ be the discounted present value of the net price:

$$\theta(t) = [P(t) - C]\ e^{-rt} \ . \tag{4}$$

Consider the case where θ has the values displayed in Fig. 1; that is, θ has two maxima and the first is larger than the second.

When should an owner of a finite stock of a depletable natural resource sell the resource? He should sell the first unit when θ is at a maximum and he should sell additional units near the maximum until he depletes his stock. If his stock of resource is large enough, he can produce during both maxima. For the case displayed in Fig. 1, the resource owner has the following bang-bang decision rule: produce at full capacity whenever θ is greater than K and stop production whenever θ is less than K.

For the values of $\theta(t)$ plotted in Fig. 1, each value of K between the maximum value of θ and zero is the solution of the basic problem for a mix of demand [D(t)] and total stock of the resource [Q^*]. If the resource owner has a small stock, then K should be near the maximum value of θ. If the owner has a large stock, then K can be near zero.

The quote from Solow (1974) suggests the following bang-bang production rule: If the net price is increasing too slowly, produce now. If the net price is rising faster than the interest rate, stop production. The Solow rule concerns the rate of change of the net price, while our rule concerns the level of the net price. If the net price is always increasing more slowly than the interest rate, then θ is monotonically decreasing and both rules recommend production at full capacity. For the first maximum in Fig. 1, Solow would not begin production until after the peak in the curve,

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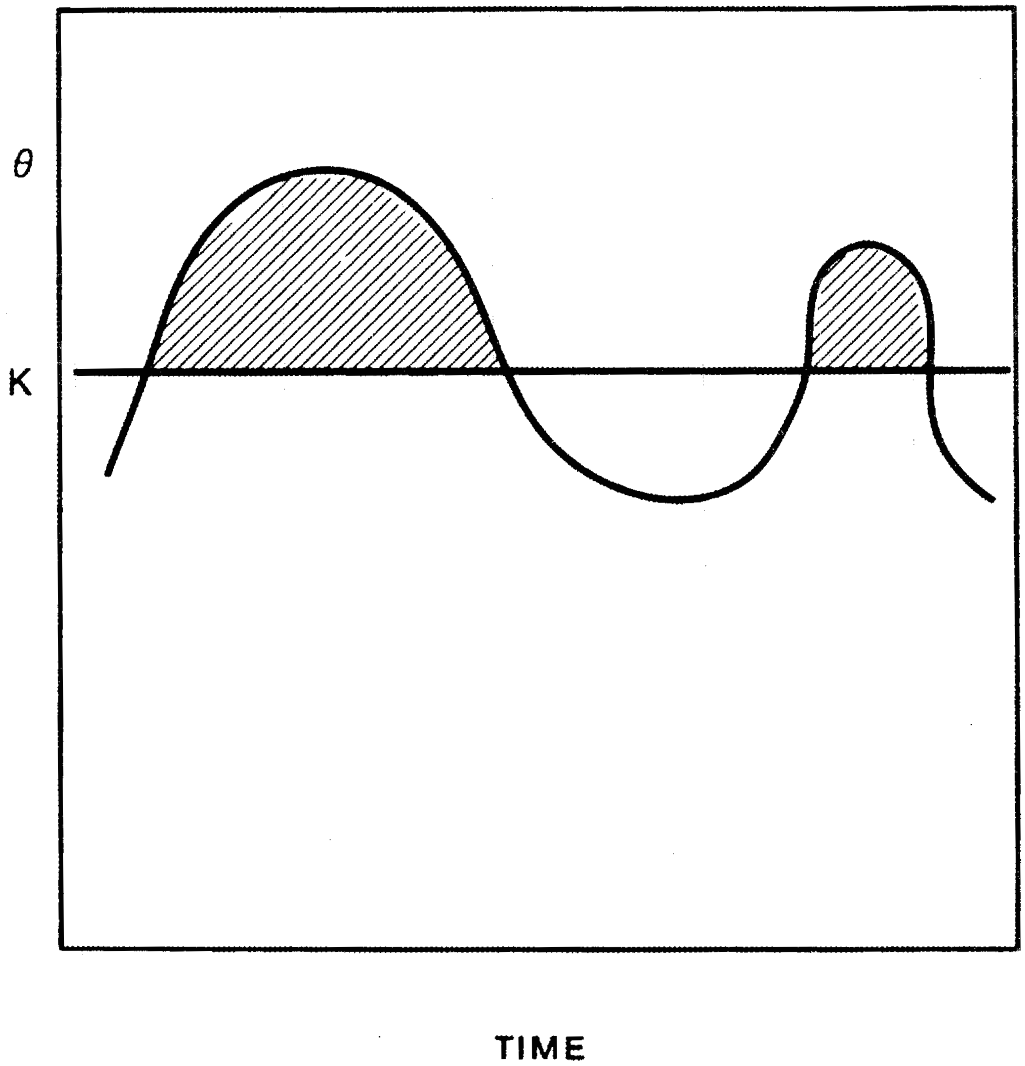

Fig. 1. The Discounted Present Value of the Net Price for a Depletable Natural Resource.

while we produce before and after the peak. If there were no limit on the production rate, we would both produce all of our resource at the peak of the curve.

Hotelling (1931) derived the equilibrium condition for Eq. (1) without formally posing an optimization problem. The general solution of Eq. (1) is bang-bang because the price is exogenous (see Clark [1976]). Most economists have followed Hotelling (1931) in assuming that the price depends on the production rate and time. The economic justification for Eq. (1) is that the resource owner has a small share of the market and cannot influence the market price. If all producers had the same values for θ, then everyone would produce when θ was at a maximum and the price would be driven down. Similarly, the production would be low when θ was at a minimum and the price would be driven up.

Although market equilibrium may require that the net price increase with the interest rate, the time delays inherent in discovering and developing a depletable natural resource may prevent the market from ever reaching equilibrium. For the oil market, the price in 1981 was probably too high, while the price in 1986 was probably too low.

Because the price is exogenous, the optimum solution to the basic problem is bang-bang; when the price is right, the resource owner produces as much as possible. The limit on production [Eq. (2)] is required to guarantee a finite solution. How realistic is the limit on production? Most mines or wells have an upper limit on production capacity. In most cases, mines or wells are designed to operate for several years rather than for days or weeks. In a later section, we will assume that the price depends on the production rate and time.

We can attempt to solve the basic problem using the Calculus of Variations [see Sagan (1969)]. Applying the Euler-Lagrange equation to the problem, the partial derivative of the integrand of Eq. (1) with respect to $q(t)$ is a constant:

$$[P(t) - C]\ e^{-rt} = K\ , \tag{5}$$

where K is a constant. Equation (5) may be rewritten:

$$P(t) - C = K\ e^{rt}\ . \tag{6}$$

Equation (6) is the fundamental result of Hotelling that the net price increases with the interest rate.

Equation (6) is a satisfactory solution of Hotelling's problem, in which the price is a function of the production rate. However, Eq. (6) is not a satisfactory solution to our optimization problem. Equation (6) places a condition on $P(t)$ [an exogenous input] and it does not help us determine $q(t)$. To find a bang-bang rule that will determine the production rate, we will use the Pontryagin Maximum Principle.

III. THE PONTRYAGIN MAXIMUM PRINCIPLE

In this section, we will briefly introduce the maximum principle. A full and rigorous presentation may be found in Pontryagin (1962). Consider the optimal control problem of finding a control vector [u(t)] that will move an object from one point in state space [x(0)] to another point [x(T)] and minimize a functional (J):

$$J = \int_0^T f^0[x(t),u(t)]\, dt\ . \tag{7}$$

The laws of motion for the object can be written in the form of a system of differential equations:

$$\frac{dx^i}{dt} = f^i[x,u]\ ,\ \text{for } i=1,..,n. \tag{8}$$

Note that the laws of motion and the integrand of the objective function are autonomous; that is, they do not depend explicitly on time.

To solve the problem, we introduce a system of auxiliary variables [ψ] that satisfy the following equations:

$$\frac{d\psi_i}{dt} = -\sum_{j=0}^{n} \frac{\partial f^j}{\partial x^i}\, \psi_j\ ,\ \text{for } i=0,1,..,n. \tag{9}$$

Using the auxiliary variables, we define a Hamiltonian function [H] by:

$$H[\psi,x,u] = \sum_{j=0}^{n} \psi_j \, f^j[x,u] \; . \qquad (10)$$

We shall say that u(t) is an admissible control if it is piecewise continuous for $0 \le t \le T$ and its range is in a set U. Let M be the least upper bound of the Hamiltonian with respect to u:

$$M[\psi,x] = \sup_{u \in U} H[\psi,x,u] \; . \qquad (11)$$

<u>The Pontryagin Maximum Principle</u>

Let u be an admissible control. Then u is an optimal control if:

1. u maximizes H; that is, $H[\psi,x,u] = M[\psi,x]$, and
2. at the terminal time (T), $\psi_0(T) \le 0$ and $M[\psi(T),x(T)] = 0$.

Furthermore, ψ_0 and $M[\psi,x]$ are constant.

To solve an optimal control problem using the Pontryagin Maximum Principle, we define the auxiliary variables and find the control that maximizes the Hamiltonian.

IV. SOLUTION OF THE BASIC PROBLEM USING THE MAXIMUM PRINCIPLE

For the basic problem, the control variable is the production rate, $q(t)$. The first state variable $[x_1]$ is the cumulative production, $Q(t)$. The Maximum Principle requires that the integrand of the objective function be autonomous. To make the integrand autonomous, we introduce time as a second state variable, x_2.

To summarize the basic problem, the components of the function f are given by:

$$f^0 = -\{P[x_2] - C\} \; u \; \exp[-rx_2] \; , \tag{12}$$

$$f^1 = u \; , \text{ and} \tag{13}$$

$$f^2 = 1 \; . \tag{14}$$

Since ψ_0 is a negative constant and the system of auxiliary equations is linear and homogeneous, we can make a arbitrary choice for ψ_0; let $\psi_0 = -1$. For the basic problem, the Hamiltonian function may be written:

$$H[\psi,x,u] = -f^0 + u \, \psi_1 + \psi_2 \; . \tag{15}$$

The optimal control maximizes the Hamiltonian. Since f^2 does not depend on u, the second auxiliary variable [ψ_2] does not influence the solution and we will ignore it.

The first auxiliary variable satisfies the following equation:

$$\frac{d\psi_1}{dt} = \frac{\partial f^0}{\partial x_1} . \tag{16}$$

For the basic problem, the extraction costs do not depend on cumulative production. Thus, the right side of Eq. (16) is zero and the first auxiliary variable is a constant. Later, we shall allow the extraction costs to depend on cumulative production.

If we rewrite the Hamiltonian in the original variables and let the first auxiliary variable equal -K, the Hamiltonian function may be written:

$$H = \phi\ q + \psi_2 , \tag{17}$$

where ϕ is defined by:

$$\phi(t) = [P(t) - C]\ e^{-rt} - K . \tag{18}$$

The optimal production rate [q] maximizes the Hamiltonian function. When ϕ is positive then q is at its upper bound, q=D(t); and when ϕ is negative then q is at its lower bound, q=0. In the jargon of optimal control theory, the optimal control is a bang-bang solution.

When we applied the Euler-Lagrange equation to the basic problem, we derived the condition that $\phi=0$. When we apply the Pontryagin Maximum Principle to the basic problem, we find that $\phi=0$ is not the solution; it is the condition for starting or stopping production. The Maximum Principle solution is much more satisfactory; it does not put a condition on P(t) [an exogenous input] and it provides a rule for determining the production rate.

A more comprehensive discussion of the bang-bang solution to the basic problem and of the application of the Pontryagin Maximum Principle to the optimal management of renewable and nonrenewable resources may be found in Clark (1976).

V. THE GENERAL PROBLEM

In this section, we will apply the Maximum Principle to a more general problem, for which the extraction cost increases with cumulative production and the demand depends on price. We will assume that the extraction cost depends on both cumulative production [Q] and time:

$$C = C[Q,t] \ . \tag{19}$$

The extraction cost depends on time because changes in technology can reduce production costs.

A basic economic principle is that sales depend on the interplay between supply and demand. The resource owner offers to sell his resource at a price and the market determines the quantity of resource that he will sell:

$$q = F[P,t] \ . \tag{20}$$

Both Hotelling (1931) and Stiglitz (1976) have considered the optimum production strategy for a monopolist. A monopolist controls the price by setting the level of production:

$$P = P[q,t] \ . \tag{21}$$

If the functions are single valued, an inverse function exists and there is no mathematical difference between Eqs. (20) and (21). Since we started with q as the control variable, we will continue with q as the control variable. However, we will consider both cases: competition and monopoly.

We have modified the integrand of the basic problem and must redefine the function f^0. The functions f^1, f^2, and the Hamiltonian function are unchanged. For the general problem the function f^0 is given by:

$$f^0 = -\{P[u,x_2] - C[x_1,x_2]\} \, u \exp[-rx_2] \quad . \tag{22}$$

Using the original variables and Eq. (16), the first auxiliary variable satisfies the following equation:

$$\frac{d\psi_1}{dt} = \frac{\partial C}{\partial Q} q \, e^{-rt} \quad . \tag{23}$$

If we assume that the partial derivative of the extraction cost with respect to cumulative production is positive, the first auxiliary variable increases whenever the production rate is positive. If we assume that the initial value of ψ_1 is ψ_1=-K, then the magnitude of ψ_1 decreases whenever q is positive. At the terminal time, either $\psi_1(T)$ is zero and $Q < Q^*$ or $\psi_1(T)$ is negative and $Q = Q^*$.

The optimal production rate maximizes the Hamiltonian. To find the optimal production rate, we differentiate the Hamiltonian with respect to q; the result is:

$$\{P[q,t] - C[Q,t] + \frac{\partial P}{\partial q} q\} \, e^{-rt} = -\psi_1 \quad . \tag{24}$$

Equation (24) is the solution to the general problem. Given a demand function $\{P[q,t]\}$ and a production cost function $\{C[Q,t]\}$, Eq. (23) can be solved to determine ψ_1 and Eq. (24) can be solved to determine the production rate [q].

Hotelling (1931) used the Euler-Lagrange equation to derive Eq. (24) for the case of constant extraction costs. When the extraction costs are constant, Eq. (24) states that the marginal revenue minus the production cost increases at the interest rate [see Gordon (1967)]. We are not aware of any previous derivation of Eq. (23). For the general problem, the price depends on the production rate and the Euler-Lagrange equation can be used to solve the problem. The Pontryagin Maximum Principle illuminates the solution by introducing the auxiliary variables.

Stiglitz (1976) has derived Eq. (24) for the special case of a constant elasticity of demand and extraction costs that depend on time but not on cumulative production:

$$P[q,t] = h(t)\ q^{\alpha-1}\ , \text{ and} \tag{25}$$

$$C[Q,t] = g(t)\ . \tag{26}$$

Weinstein and Zeckhauser (1975) have derived a result similar to Eq. (23) for a discrete time problem with increasing production costs. However, they do not define the auxiliary variable.

If the Euler-Lagrange equation is applied to the general problem, the condition for optimality is the equation that results when Eq. (24) is differentiated with respect to time and Eq. (23) is used to eliminate the auxiliary variable. The optimality condition without the auxiliary variable has been derived for the renewable resource problem by Clark (1976), Clark and Munro (1975), Berck (1981), and Pindyck (1984).

We can make Eq. (24) more understandable by defining an exhaustible resource owner's scarcity rent [R] by:

$$R = -\psi_1 \; e^{+rt} \; . \tag{27}$$

We define an objective function [L] by:

$$L = \{P[q,t] - C[Q,t] - R\} \; q \; . \tag{28}$$

If the resource owner chooses a production rate that maximizes L for each time period, then he will satisfy Eq. (24) and solve the general problem. In the objective function [L], a rent has been added to the extraction costs. The rent term summarizes the dynamics of the problem and incorporates the increases in extraction costs. The rent converts a multi-period optimization problem into a series of single period optimization problems.

VI. A LOGIT DEMAND FUNCTION

To proceed further, we must define a demand function. We assume that the owner sells the resource in a market where a competing resource is offered at price W(t). If the owner's price [P(t)] is greater than W, he will lose market share and vice versa. We consider a simple logit demand model:

$$q = D(t)\ s(t) \ , \tag{29}$$

where D(t) is the total demand, and the market share for the resource owner [s] is given by:

$$s(t) = \frac{P^\gamma}{P^\gamma + W^\gamma} \ , \tag{30}$$

where γ is a parameter. The logit share function [Eq. (30)] has been widely used in models of energy supply and demand; see Boyd, Phillips, and Regulinski (1982) and Reister (1983). Our logit demand model could be used to simulate whether a country uses domestic or imported oil.

Let σ be the price elasticity of demand:

$$\sigma = \frac{P}{q}\frac{\partial q}{\partial P} \ . \tag{31}$$

Using the definitions of σ and R, Eq. (24) may be written:

$$P \ \{ \ 1 + 1/\sigma \ \} = C + R \ . \tag{32}$$

For the logit demand function,

$$\sigma = \gamma (1 - s) . \tag{33}$$

Define p and c by: $p = P/W$ and $c = [C + R]/W$. Using the dimensionless parameters p and c, Eq. (32) may be written:

$$c = p (1 + 1/\sigma) = G(p). \tag{34}$$

Given c, we would like to find p. Since σ is function of p, G(p) is the inverse function. If we construct a table of G(p) as a function of p, then we can use the table to determine p as a function of c.

The logit exponent $[\gamma]$ controls the price elasticity of the demand model [see Eq. (33)]. In many applications in economics, a price elasticity of -2 is a large value. However, if γ=-2 and the owner's price was 10% higher than the competing price, the resource owner would capture 45% of the market. If the customers are choosing the least cost option, the market share for the more expensive resource would be zero. To reduce the market share for the expensive resource, we will raise the logit exponent to γ=-40. The functions G(p), s(p), and σ(p) are displayed in Table 1 for γ=-40.

If p is less than 0.91, then G(p) is negative. If p is greater than 0.92, then G(p) is positive. As p increases from 0.92, G(p) increases, s(p) decreases, and σ(p) becomes more negative. For large values of p, G(p) approaches p, s(p) approaches zero, and σ(p) approaches -40.

Table 1. The Price-Cost Function and the Market Shares for the Logit Demand Function.
Gamma= -40.0

p	G(p)	s(p)	σ(p)
0.90	-0.64	0.99	-0.58
0.91	-0.10	0.98	-0.90
0.92	0.25	0.97	-1.38
0.93	0.48	0.95	-2.08
0.94	0.64	0.92	-3.11
0.95	0.74	0.89	-4.56
0.96	0.81	0.84	-6.54
0.97	0.86	0.77	-9.13
0.98	0.90	0.69	-12.33
0.99	0.93	0.60	-16.03
1.00	0.95	0.50	-20.00
1.01	0.97	0.40	-23.93
1.02	0.98	0.31	-27.53
1.03	1.00	0.23	-30.61
1.04	1.01	0.17	-33.10
1.05	1.02	0.12	-35.02
1.06	1.03	0.09	-36.46
1.07	1.04	0.06	-37.50
1.08	1.05	0.04	-38.24
1.09	1.06	0.03	-38.77
1.10	1.07	0.02	-39.14

Given the extraction cost, rent, and competing price, we can calculate c and determine p and s from Table 1. The price ratio [p] is plotted in Fig. 2, while the market share [s] is plotted in Fig. 3. For all values of c, p is in the neighborhood of 1.0. Thus, the owner's price for the resource is always close to the competing price. If the competing price declines, the price offered by the owner will decline until c=1. If c is small, p is below 1.0 and the market share is near 100%. Thus, if the sum of the extraction cost and the rent is less than the competing price, the optimum strategy is to have a large market share. If c is greater than 1.0, then p is greater than c and the market share is small. If the competing price falls below the sum of the extraction cost and the rent, the optimum strategy is to have a small market share.

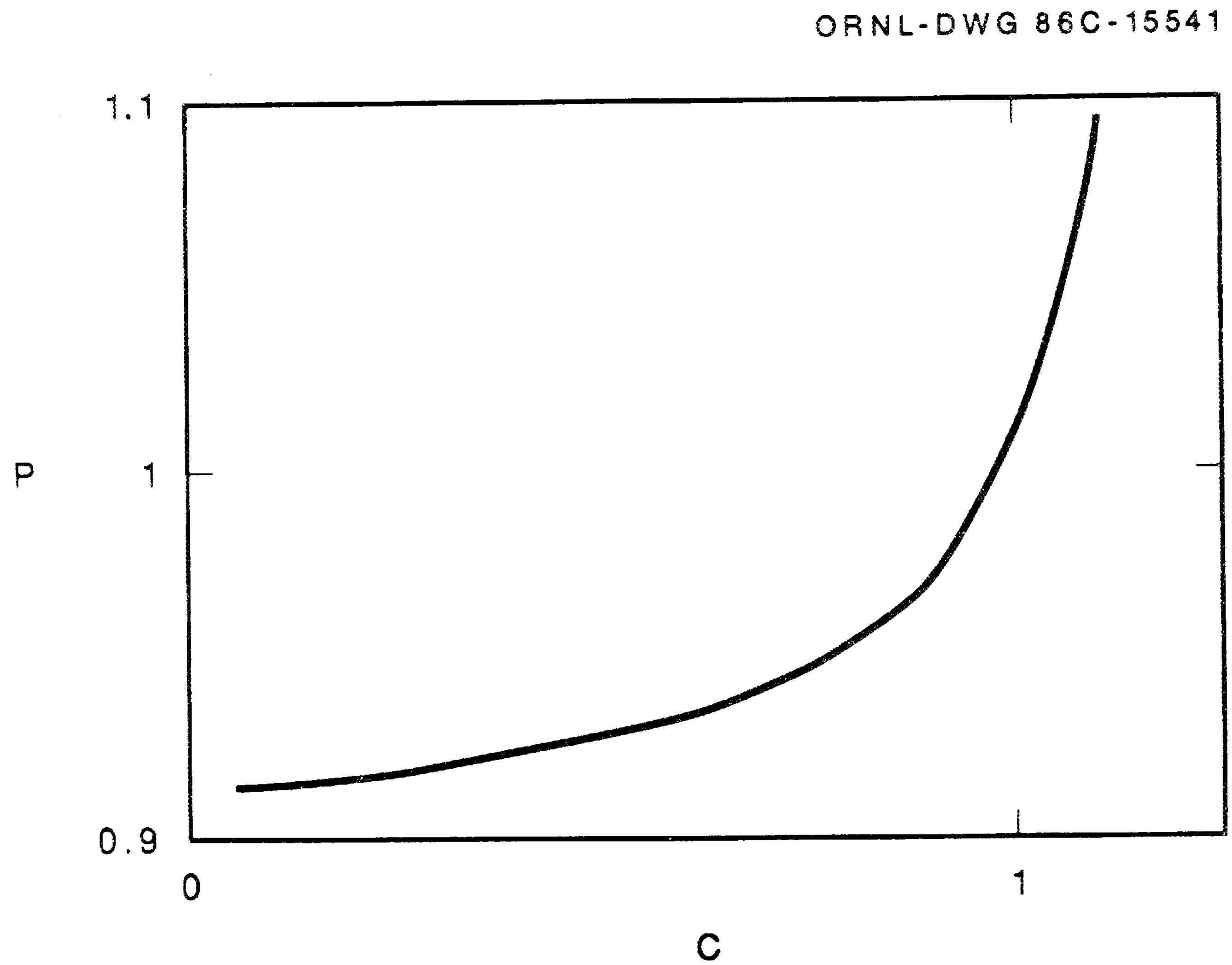

Fig. 2. The Price Ratio (p) as a Function of the Parameter c.

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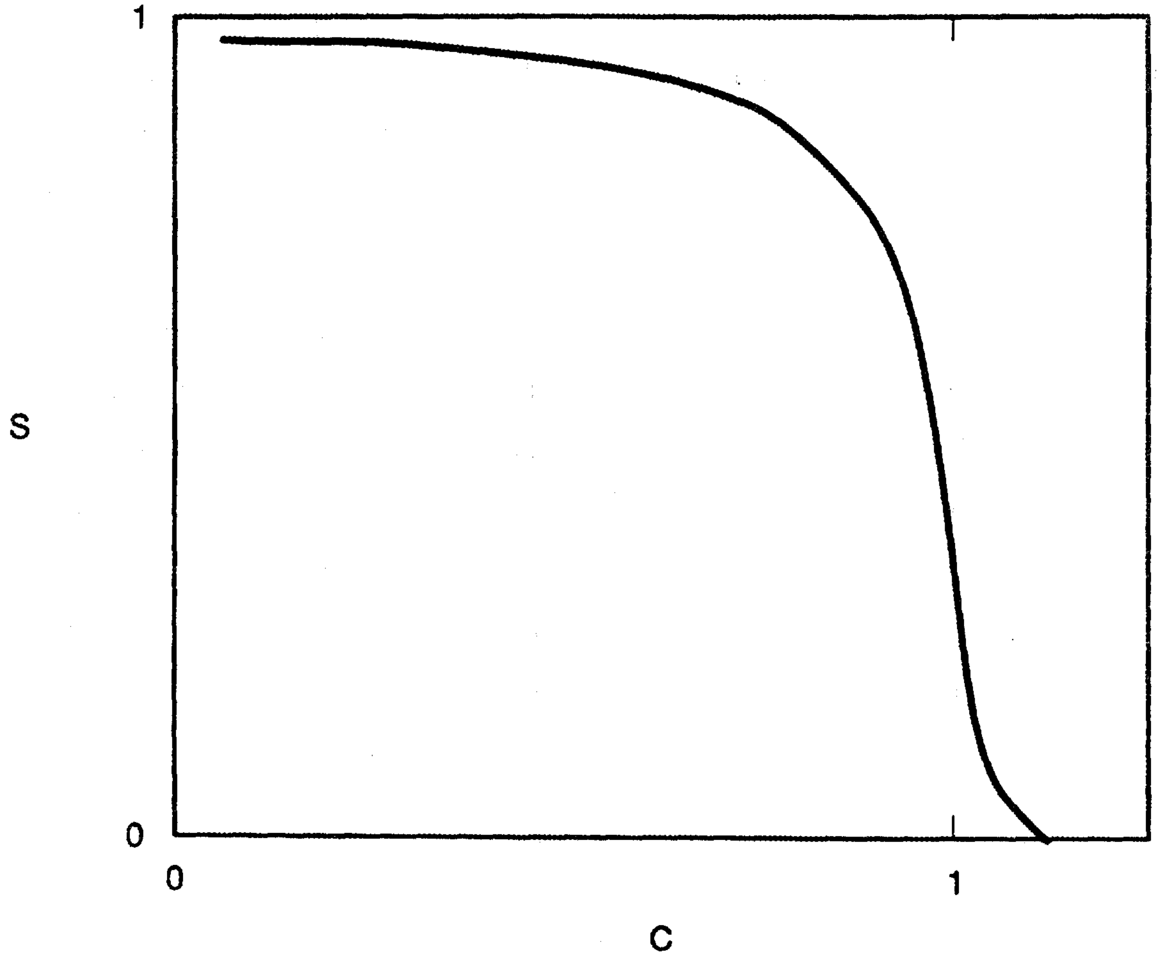

Fig. 3. The Market Share (s) as a Function of the Parameter c.

The relationship between the total resource [Q^*] and the initial value for ψ_1 [$\psi_1(0)$=-K] must be determined by numerical integration of Eq. (23). Given the total demand [D], the competing price [W], and the extraction cost function [Q], there will be a range of values for K that determine optimum solutions for various values of Q^*. In general, both the basic problem and the general problem with the logit demand function have bang-bang solutions and the auxiliary variable [ψ_1] controls the starting and stopping of production.

The solution of the general problem is illustrated in Figs. 4 through 7. For the example, the price of the competing resource [W] starts at $40, declines to $20 in year 10, before increasing to $60 in year 30. Because the discount rate is 1%, the maximum value for the discounted value of W occurs at the end of the period; that is, the discounted value of W is $45 in year 30. For the example, the total demand is D = 4 per year and the maximum production in 30 years is Q = 120. The extraction cost has a linear dependence on Q: C = 10 + 0.2*Q. As Q increases from 0 to 120, the cost increases from 10 to 34. For each value of K (the initial value for the rent), the equations can be solved to determine the rent, production rate, and discounted profits [J]. By varying K, the maximum value for the profits can be found.

The solution for K=0 is displayed in Fig. 4. When K=0, the rent is negative and the resource owner is near full production for the entire period. For the case displayed in Fig. 4, the cumulative production is Q=119 and the profits are J=1094.

The solution when K=10 is displayed in Fig. 5. For this case, the resource owner starts near full production; stops production for a few

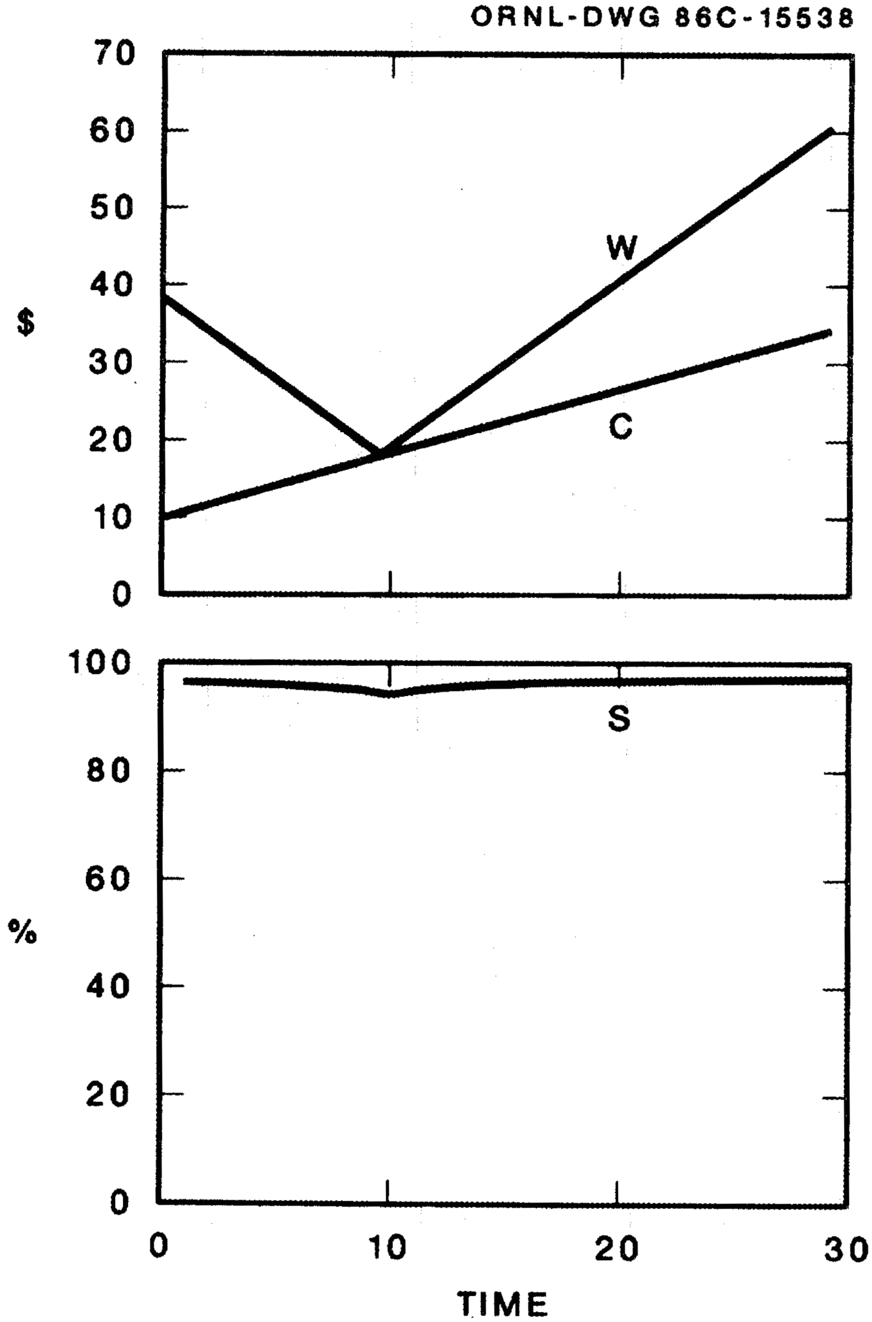

Fig. 4. The Solution of the General Problem When K = 0.0.

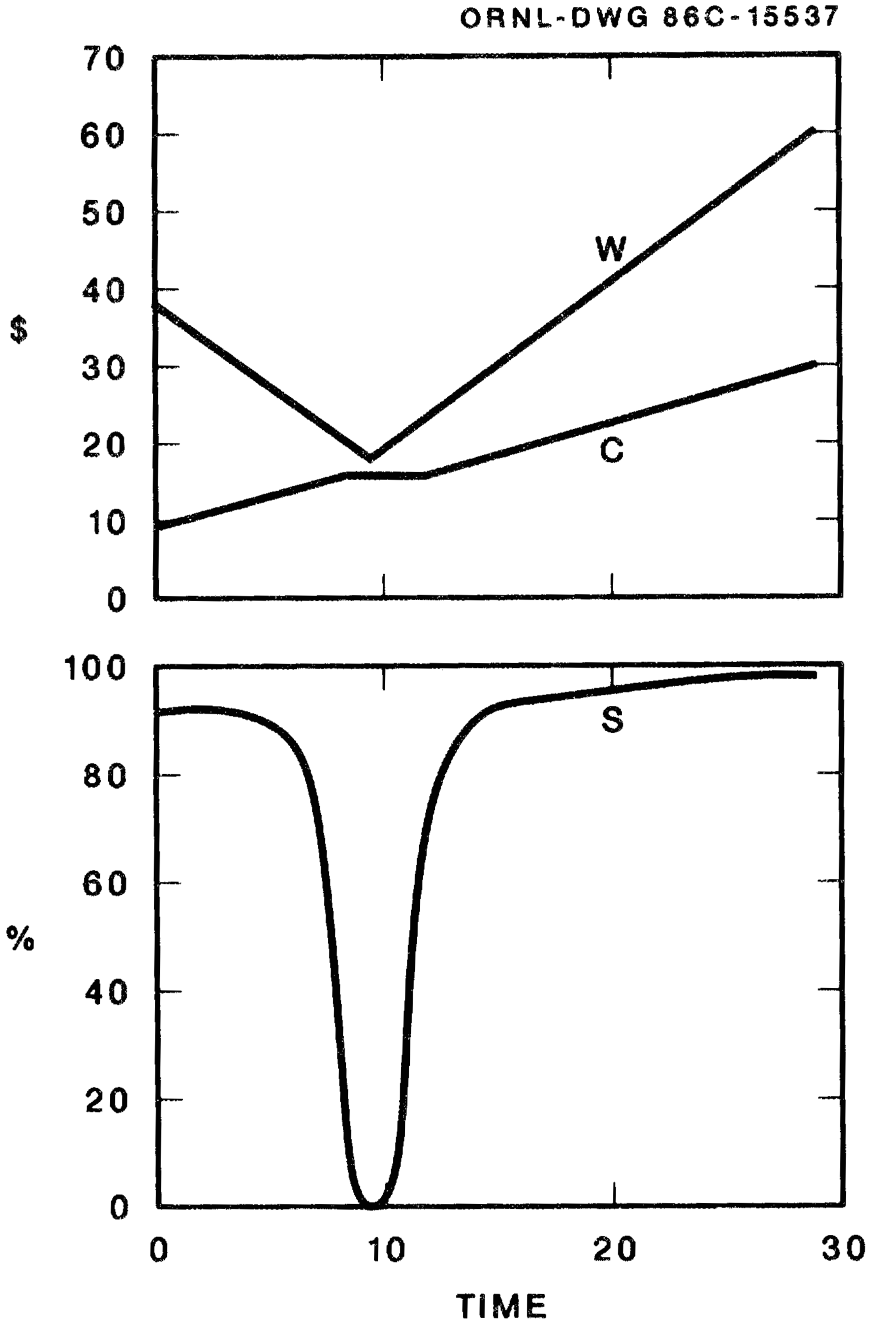

Fig. 5. The Solution of the General Problem When K = 10.0.

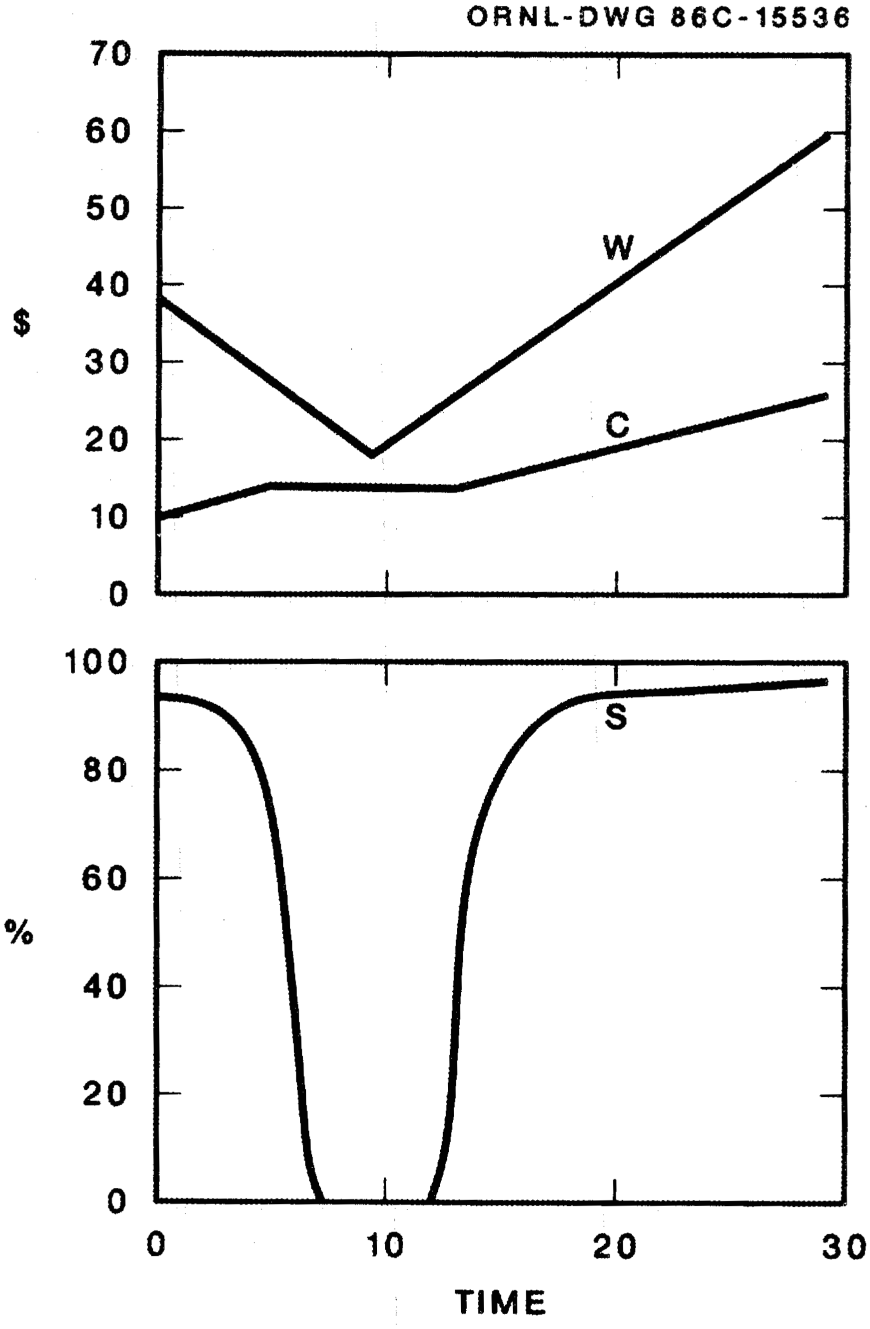

Fig. 6. The Optimal Solution of the General Problem When the Resource Constraint is Greater than 82 (k = 13.8).

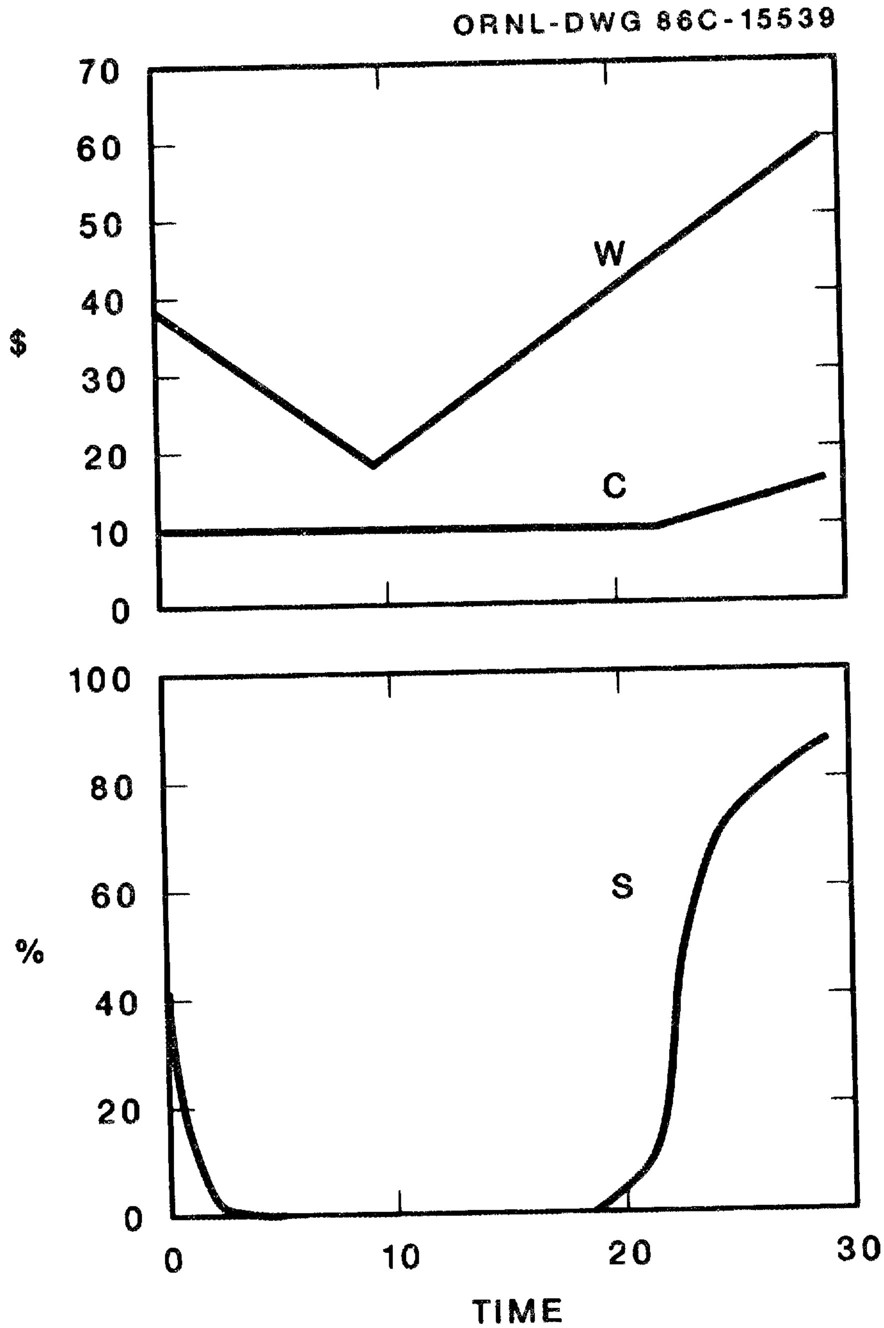

Fig. 7. The Optimal Solution of the General Problem When the Resource Constraint is 25 (k = 26.7).

years when the competing price is low; and returns to full production when the price recovers. By stopping production, the resource owner reduces his costs and increases his profits in the later period. For this case, the cumulative production is Q=100 and the profits are J=1312.

The optimum solution (without a resource constraint) is displayed in Fig. 6. For the optimum solution, K=13.8 and the resource owner stops production for a longer period, when the price is low. For this case, the cumulative production is Q=82 and the profits are J=1376. For the optimum solution without a resource constraint, the rent starts at \$13.8 and decreases to zero when t=30. For the cases shown in Figs. 4 and 5, the rent is negative when t=30.

The optimum solution when $Q^{*}=25$ is displayed in Fig. 7. For this case, K=26.7 and the profits are J=729. Most of the production occurs at the end of the period, when the discounted value of the competing price is at a maximum. The rent is large and positive throughout the period.

Equation (24) relates the marginal profits to the first auxiliary variable. If we use a logit demand function, Eq. (24) yields the following bang-bang decision rule: if extraction cost plus rent is less than the price of the competing resource, produce at full capacity; otherwise, stop production.

VII. THE SOCIAL OPTIMUM

Both Hotelling (1931) and Weinstein and Zeckhauser (1975) consider the optimal production of a depletable resource when the objective is to maximize the discounted sum of consumer plus producer surplus. For this case, the objective function is:

$$J = \int_0^T \{ \int_0^q P[y,t]\ dy - C[Q,t]\ q(t) \}\ e^{-rt}\ dt\ . \qquad (35)$$

The objective function for the social optimum problem is identical to the objective function for the general problem, except that an integral has replaced the product of price and quantity. The solution to the social optimum problem is identical to the solution of the general problem, with the exception that the price elasticity term disappears from Eq. (24):

$$\{P[q,t] - C[Q,t]\}\ e^{-rt} = -\psi_1\ . \qquad (36)$$

When the extraction costs are constant, Eq. (36) is the classic result of Hotelling that the net price increases with the interest rate. When the extraction costs are not constant, Eq. (23) determines the evolution of ψ_1.

If we assume a logit demand function and introduce the parameters p and c, Eq. (36) may be written p = c. When c is small, the socially optimum value for p is much lower than the optimal value for p for a resource owner (see Fig. 2). The optimum strategy for a low cost producer (like Saudi Arabia) is to charge high prices while the social optimum is to buy Saudi oil at the cost of production.

VIII. CONCLUSIONS

In this paper, we have considered the problem of determining the optimal production path for a depletable natural resource. The classical result of Hotelling is that when the market is in equilibrium, the net price paid to the owner of the resource must increase with the interest rate. We have considered three problems: the basic problem, the general problem, and the social optimum problem. For the basic problem, the market is not in equilibrium and the classical solution does not determine the production rate. For the general problem and the social optimum problem, the classical solution does work and we have extended the classical solution to the case of increasing extraction costs.

For the basic problem, we have used the Pontryagin Maximum Principle to find a bang-bang solution for the production rate. The classical solution determines the switch points; the times to stop or start production.

For all three problems, we have found a differential equation that determines the rent that a resource owner should charge to maximize his profits. The magnitude of the rent depends on its initial value. The proper initial value depends on the total resource and the level of demand.

For the general problem with a logit demand function, the classical solution results in a bang-bang production schedule. Thus, we have found two cases where the optimal production path for depletable natural resources is bang-bang.

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