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A New Algorithm for Linear Programming That Is Easy To Implement: Application to the Transportation Problem

J.-C. Culioli
V. Protopopescu

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A NEW ALGORITHM FOR LINEAR PROGRAMMING THAT IS EASY TO IMPLEMENT: APPLICATION TO THE TRANSPORTATION PROBLEM

J.-C. Culioli and V. Protopopescu

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ABSTRACT

We propose a new algorithm for linear-programming and apply it to solve some transportation problems. We prove the convergence of the algorithm under reasonable sufficient conditions. The main motivation of this new linear-programming solver is its extreme simplicity and its parallel structure which allows for straightforward implementations on personal computers as well as on more sophisticated machines. We give in the Appendix a version of the algorithm programmed in C in less than 30 lines. We also comment in the Appendix the relationship of our algorithm with some models of neural networks.

1. INTRODUCTION

Since Khachiyan [1] in 1978 and Karmarkar [2] in 1984 have revived the search for new linear-programming algorithms, many attempts have been made to surpass or at least equal the robust Simplex method introduced by Dantzig [3] in the late fourties. There are now so many competitors to the Simplex (Ref. [4] to [7] to cite a few) that one could wonder if it is necessary to add one more stone to an already solid edifice. Another reticence could be linked to the following remark: if the number of algorithms increases linearly, the number of versions on different machines and the number of numerical tests to compare them, increase almost exponentially and this amounts to a non negligible investment in terms of coding and testing for the linear-programming community.

Despite these possible caveats, we do propose here a new linear-programming algorithm and show its virtues both by theoretical analysis and by applying it to the Transportation Problem. We do not claim that the sigmoidic algorithm (the reason for this name will be clear in the next Section) is faster than any of the recent excellent algorithms that have been discovered in the late few years or that it can beat the Simplex in some instances. But we do claim and show that is is extremely simple to understand, analyze, and implement - three properties which are only rarely shared simultaneously by the above-mentioned algorithms. In addition, the method is easy to parallelize. Some very large-scale integration circuits could also be designed to mimic its parallel structure without many difficulties. The second part of the Appendix stresses this point in showing the connection of our algorithm with a neural network model.

In the next Section, we briefly describe the linear-programming problem, the sigmoidic algorithm, and conduct the theoretical analysis. We apply it to the Transportation Problem in Section 3 and give some numerical results in Section 4. The Appendix contains the listing of a C program that executes the algorithm for a sample program.

2. THE SIGMOIDIC LINEAR-PROGRAMMING ALGORITHM

2.1 . The Linear-Programming Problem

The standard linear programming problem can be stated as follows:

Find

$$\text{Max} \quad \Sigma_i \, c_i \, . \, x_i \qquad (1)$$

subject to the constraints

$$x_i \geq 0, \quad i = 1,2,\ldots, N \qquad (2\text{-a})$$

and

$$\Sigma_i \, a_j^i \, . \, x_i = b_j, \quad j = 1,2,\ldots,m. \qquad (2\text{-b})$$

Usually the vectors $(x_i) = x$ and $(c_i) = c$ are referred to as the primal vector and the cost vector. We shall denote by A the matrix $[a_j^i]$ and by b the constraint vector. In this notation, (1)-(2) can be written as the primal problem

Find Max $\langle c,x \rangle$ subject to $Ax = b$ (PP).

One can introduce the dual problem associated with (PP):

Find Min $\langle b,p \rangle$ subject to $A'p \geq c$ (DP).

Here, A' denotes the transpose of A and $p = (p_j)$ is the dual vector associated with the constraints. The equivalence of (LP) and (DP) is summarized [3] by the following statements:

- (LP) has a bounded solution x^* if and only if (DP) has a bounded solution p^*.
- The optimal couple (x^*, p^*) is such that

(i) $Ax^*=b$,

(ii) $A'p^* \geq c$

(iii) $\langle c,x^* \rangle = \langle b,p^* \rangle$

In the following, we assume that (1)-(2) has a bounded solution, denoted by x^*, and that we know beforehand a bound for each x_i, denoted X_i. Typically, X_i should be greater than x_i^*.

2.2 The Algorithm.

In order to present the sigmoidic algorithm, a few preliminary definitions are necessary. A one dimensional parameter-dependent mapping $g_T : R \to R$ defined by

$g_T(y) = 1/(1+\exp(-y/T))$ is referred here to as a sigmoidic function.

The typical graph of g_T (we will only consider $T > 0$) is given in Figure 1.

The most important properties of sigmoidic functions are the following:

for every y in R and any T in R_+^*, we have:

(S1) $0 < g_T(y) < 1$.

(S2) $g_T'(y) = \frac{1}{T} g_T(y)\, g_T(-y)$.

(S3) $0 < g_T'(y) < \frac{1}{T}$.

The inverse mapping of g_T is easily computed as

$g_T^{-1}(z) = T \operatorname{Log}[z/(1-z)]$ for z in (0,1).

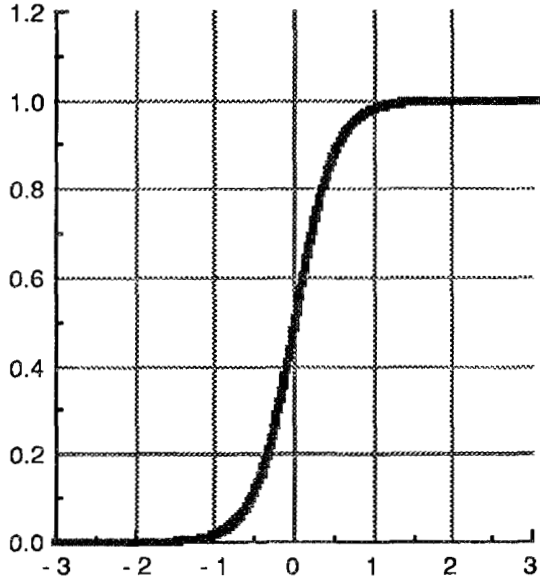


Figure 1. A sigmoidic function.

The sigmoidic function plays a crucial role in the design of the algorithm. The introduction of such "ad hoc" functions can be compared to the introduction of barrier functions like the "potential functions" of Karmarkar. As will be seen in Appendix A.1 (which describes Figure 2), the sigmoidic functions have the interesting geometric property of transforming a linear program into a parametrized non-linear optimization problem, enabling us to use non-linear search algorithms.

The sigmoidic algorithm represents a primal-dual approach similar to some non-linear saddle-point algorithms [8]. It consists in successive iterations on the primal and dual variables:

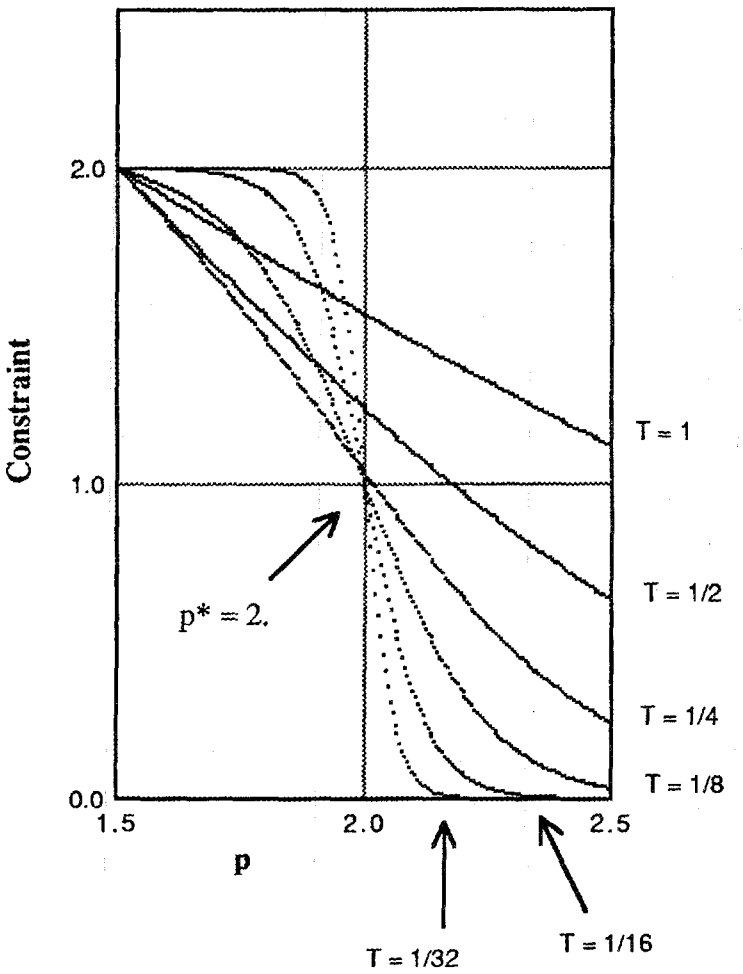


Figure 2. Non-linear interpretation (see Appendix A.1).

- initialize p_j^0 for $j = 1,2, ..., m$.
- for $k \geq 0$, update x_i and p_j according to:

for $i = 1, 2, ..., n$, $\quad x_i^{k+1} = X_i . g_T [c_i - \Sigma_j p_j^k . a_j^i]$ (3)

for $j = 1,2, ..., m$, $\quad p_j^{k+1} = p_j^k + \varepsilon .[\Sigma_i a_j^i . x_i^{k+1} - b_j]$ (4)

- stop when an optimality criterion (to be defined) is satisfied.

Here ε is a small positive parameter. Conditions on ε and T will be given in the next Section. Notice that the sigmoidic algorithm is already written in a parallel form.

2.3 Analysis of the Algorithm.

In order to prove convergence and optimality, we have to assume a set of conditions that are not always met by every linear-programming problem. These conditions are, however, satisfied by the transportation model (see Section 3). We also give in Section 4 a numerical example for which these conditions are not met but for which the algorithm works. Therefore, we think that our hypotheses are not the most general ones and weakening them would allow to include more general linear programming problems.

Now we are able to state the following

Theorem:

Assume that:

- all the components of b and the elements of A are non negative.
- for all $i \leq n$, there exists at least one j for which $a_j^i \neq 0$, so that we can define $X_i = 2 \, Max_j \{ b_j / a_j^i \mid a_j^i \neq 0 \}$. X_i is ≥ 0 and is the bound on x_i used in (3).
- the rank of the matrix A is m.

Then it follows that:

- for ε small enough, the sequence p^k converges to a finite vector $\hat{p}$ and consequently, the sequence x^k also converges to a limit denoted by $\hat{x}$.
- $\hat{x}$ is a feasible solution of (1) - (2) and $\hat{p}$ is a feasible solution of the dual problem corresponding to (1)-(2).
- The duality-gap $|\langle c,x\rangle - \langle b,p\rangle|$ is linearly bounded by T.

Proof:

Convergence.

By eliminating x_i^{k+1} between equations (3) and (4), we obtain a closed iteration formula for $p = (p_1, \ldots, p_m)$: $p^{k+1} = H(p^k)$, where H is applied componentwise to p, namely:

$$H(p_j) = p_j + \varepsilon \ \{ \Sigma_i \ a_j^i \ X_i \ g_T [c_i - \Sigma_h \ p_h \cdot a_h^i] - b_j \}$$

Let us compute the Jacobian of H:

We have $DH(j,j) = 1 - \varepsilon \ \{ \Sigma_i \ (a_j^i)^2 \ X_i \ g' [c_i - \Sigma_h \ p_h \cdot a_h^i] \}$

and for $k \neq j$, $DH(j,k) = - \varepsilon \ \{ \Sigma_i \ a_j^i a_k^i \ X_i \ g' [c_i - \Sigma_h \ p_h \cdot a_h^i] \}$.

According to property (S2), we have $g'[c_i - \Sigma_h \ p_h \ a_h^i] > 0$.

With the notations $g_i = \sqrt{g'[c_i - \Sigma_h \ p_h a_h^i] \ X_i}$

and $B_j^i = a_j^i \ g_i$, we have the simpler expressions:

$DH(j,j) = 1 - \varepsilon \ \Sigma_i \ (B_j^i)^2$ and $DH(j,k) = - \varepsilon \ \Sigma_i \ B_j^i B_k^i$.

Thus $DH = 1 - \varepsilon \ \ \mathrm{Gram}(u_1, \ldots, u_m)$ with $u_j = (B_j^1, \ldots, B_j^n)$.

Since rank(A) = m and every g_i is positive, then $B = [B_j^i] = [a_j^i \ g_i]$ is an mxn matrix of rank m. Gram(B) is an mxm symmetric and positive definite matrix of rank m (See [11], p. 155). Its spectrum contains m (possibly degenerate) real and positive eigenvalues

l_k, $1 \le k \le m$, which satisfy the following inequalities: $\|Gram(B)\| = l_1 \ge \ldots \ge l_m > 0$, where $\| . \|$ denotes the Euclidian norm.

Therefore, we can find $\varepsilon > 0$ such that $1 - \varepsilon\, l_1 < 1$ and $1 - \varepsilon\, l_m > -1$.

Thus, with $0 < \varepsilon < \frac{2}{l_1}$, $\|1 - \varepsilon\ Gram(B)\| < 1$ and the mapping H is a contraction, i.e. the sequence p^k converges to a fixed point.

Optimality.

Here, we refer to the properties of the optimal couple (x^*,p^*), given in Section 2. We prove that (i) and (ii) are satisfied by the limits of the x^k and p^k sequences. We then prove that (iii) can be satisfied as closely as desired, depending on the value of T. The derivation of the first optimality condition is conducted on a continuous version of the algorithm, i.e. we consider the updating equation for p as an ordinary differential equation ($0 < \varepsilon << 1$). With $dt = \frac{1}{\varepsilon}$, the evolution equations associated with (3)-(4) are

$$x(t) = X \,.\, g(c - A'p(t)) \qquad (5)$$

$$\frac{d}{dt} p(t) = Ax - b \qquad (6)$$

where $X \ge x^*$ and g_T is applied componentwise on its vector argument.

<u>Result (i) :</u>

We show that $\lim_{t \to \infty} Ax(t) = b$

To prove this, let us consider $V(t) = \frac{1}{2}\|Ax(t)-b\|^2 \ \ge\ 0$.

We have $\frac{d}{dt}V = \langle A\ \frac{d}{dt}x, Ax-b\rangle = \langle A\ \frac{d}{dt}x,\ \frac{d}{dt}p\rangle$ (with (6)).

If we take the derivative of (5), we get

$$\frac{d}{dt}x = - G A' \frac{d}{dt}p \qquad (7)$$

with $G = [\, X_i \,.\, g'(c_i - \Sigma_h\, p_h\, a_h^i)\, d_{ij}\,] = [\, g_i^2\, d_{ij}]$

Thus $\frac{d}{dt}V = - < A\, G\, A' \frac{d}{dt}p,\ \frac{d}{dt}p > = - < G\ A' \frac{d}{dt}p,\ A' \frac{d}{dt}p > \ \leq\ 0.$

The last inequality comes from the fact that rank(A') = rank(A) = m and G is a positive invertible matrix. The equality sign corresponds to the equilibrium situation, $\frac{d}{dt}p = 0$.

<u>Result (ii):</u>

Let us introduce new variables $y_i := c_i - \Sigma_j\, a_i^j\, p_j$.

We would like to prove that $\mathrm{Lim}_{k\to\infty}\ A'p^k \geq c$, which means that the variables y_i are negative whenever x^k and p^k have converged. We have $x_i = X_i \,.\, g_T\,(y_i)$.

Thus $y_i = T\,\mathrm{Log}[\frac{x_i}{X_i - x_i}]$. In the limit, if $x_i \leq X_i/2$, then $y_i \leq 0$.

On the other hand, if $x_i > X_i/2$, this implies that $y_i > 0$, but then we are able to prove that at least one constraint in Ax = b is not satisfied: suppose there exists an index i_0 such that $x_{i_0} > X_{i_0}/2$, then $x_{i_0} > \mathrm{Max}_j\ \{b_j / a_j^i\,,\ \text{for } a_j^i \neq 0\}$.

This maximum is attained at $j = j_0$. Thus $\Sigma_i\, a_{j_0}^i \,.\, x_i > b_{j_0}$ and p_{j_0} could not have converged.

<u>Result (iii):</u>

Consider the quantity $\quad W = \Sigma_i\, W_i = \Sigma_i\ x_i\, y_i = \Sigma_i\, x_i\, (\, c_i - \Sigma_j\, a_i^j\, p_j\,)$

$$= <c,x> - \Sigma_j\, (\, \Sigma_i\ x_i\, a_i^j\,)\, p_j \ = <c,x> - <b,p>$$

W is the duality-gap.

We have $W_i = X_i \; y_i \; g_T(y_i) \; = \; T \; X_i \; \frac{y_i}{T} \; \frac{2}{1+\exp(-y/T)}$

With $u_i = - \frac{y_i}{T} > 0$, we have $W_i = h(u_i) = - T X_i \frac{u_i}{1 + \exp(u_i)}$

Since $\mathrm{Lim}_{u_i \to +\infty} \; h(u_i) \; = 0$ and $h(0) = 0$, h has a minimum.

This minimum is attained at u^0, solution of $h'(u^0) = 0$, that is:

$1 + \exp(u^0) - u^0 \exp(u^0) \; = 0.$

One finds easily that $1 \leq u^0 \leq 2$. Thus, for all i, we have

$$0 \geq \quad W_i \quad \geq - T X_i \frac{u^0}{1 + \exp(u^0)} = - T X_i \exp(-u_0) \geq - T \frac{X_i}{e}$$

which proves that

$$0 \geq \quad W \quad \geq - T \frac{n}{e} \mathrm{Sup}\{X_i\}$$

The lower bound for W can be made as small as needed if T is chosen small enough. Although in some situations this bound can be actually attained, in general it gives a rather pessimistic estimate, as is shown in the numerical results Section.

3. APPLICATION TO THE TRANSPORTATION PROBLEM

The Transportation Problem can be formulated in the following way [3]: given N sources indexed by i with stocks S_i, i =1, 2,..., N, and M destinations indexed by j with demands D_j, j =1, 2,..., M, a transportation cost $c_{ij}\ x_{ij}$ is associated with the shipment of the (positive) quantity x_{ij} from the source S_i to the destination D_j. One assumes there is no loss during the process, i.e. for every source i, $\Sigma_j x_{ij} = S_i$ (a "house-keeping" constraint) and also that the demand is met for every destination, that is $\Sigma_i x_{ij} = D_j$ (a "demand-satisfaction" constraint). Notice that these two assumptions lead to

$$\Sigma_i \Sigma_j\ x_{ij} = \Sigma_j\ D_j = \Sigma_i\ S_i.$$

The problem of minimizing the cost of the transportation leads to the linear program:

$$\text{Min}\ \Sigma\, c_{ij}\ x_{ij} \quad \text{with} \quad x_{ij} \geq 0, \quad \Sigma_j\ x_{ij} = S_i \quad \text{and} \quad \Sigma_i\ x_{ij} = D_j$$

We show here how to apply the algorithm to the above model which enters perfectly the theoretical scheme discussed in Section 2. The first step is to find natural boundaries for the primal variables. In our case, since the stocks S_i and the demands D_j are known data, they both constitute natural boundaries on x_{ij}. Thus $0 \leq x_{ij} \leq X_{ij} = \text{Min}\{S_i\ ,\ D_j\}$. The existence of well-scaled boundaries is important for a good behavior of the algorithm.

Now, we have to define a dual variable for each constraint.

We denote by p_i and by q_j the dual variable associated to $\Sigma_j\ x_{ij} = S_i$ and to $\Sigma_i\ x_{ij} = D_j$, respectively. The initialization of the algorithm consists in assigning preliminary values to the dual variables p_i and q_j. A good guess of these dual variables - which have the economic interpretation of prices associated to the constraints - speeds up the algorithm. However, $p_i^0 = q_j^0 = 0$, works well too.

Once the initialization is done, the variables x_{ij}, p_i and q_j are updated, for $k \geq 1$, according to:

$$x_{ij}^{k} = X_{ij}\ g_{\Gamma}\ (-\ c_{ij} - p_i^{k} - q_j^{k}), \quad \text{for every i and j.}$$

$$p_i^{k+1} = p_i^{k} + \varepsilon\ (\ \Sigma_j\ x_{ij}^{k} -\ S_i), \quad \text{for every i.}$$

$$q_j^{k+1} = q_j^{k} + \varepsilon\ (\ \Sigma_i\ x_{ij}^{k} -\ D_j), \quad \text{for every j.}$$

Intuitively, the updating of x_{ij} consists in allocating a portion of the total possible X_{ij} that depends on the associated costs. The updating of p or q is intended to meet the "house-keeping" and the "demand-satisfaction" constraints with correct prices. The economics analog is a "one goods buyer (destination) or seller (source) market": prices are adjusted until the constraint is satisfied.

4. NUMERICAL RESULTS

4.1 Example 1

We first consider a very simple problem for which the assumptions of the theorem are fullfilled. This example is the sample problem that appears in the listing given in the Appendix.

$$\text{Max } z = 2x_1 + 4x_2 + 4x_3 - 3x_4, \quad x_i \geq 0$$

$$\text{subject to the constraints} \quad x_1 + x_2 + x_3 = 4$$

$$\text{and} \quad x_1 + 4x_2 + x_4 = 8$$

The optimal solution is given by $x^* = (0,2,2,0)$, $p^* = (4,0)$ and $z^* = 16$.

We have chosen $T = 0.2$, $\varepsilon = 0.02$. The boundaries we have used are equal to 4 for every variable. In Figures 3 and 4 we give the trajectories of the primal, dual variables, primal cost $\langle c,x \rangle$ and dual cost $\langle b,p \rangle$. The convergence takes about a hundred iterations.

4.2 Example 2.

Now, we consider another problem which does not satisfy the conditions of the theorem. However, the algorithm finds the right solution in approximately the same number of iterations.

The problem is

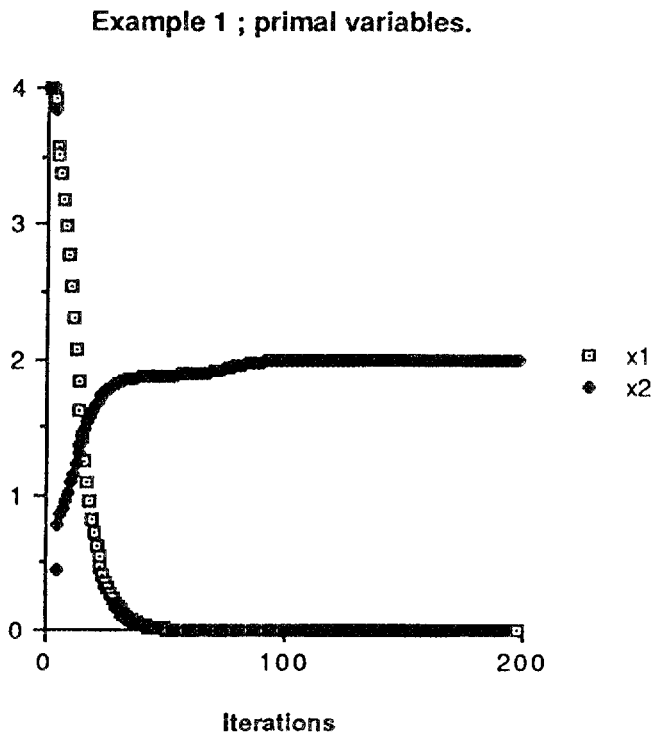
$$\text{Max } z = x_1 + 5x_2 + 3x_3, \quad x_i \geq 0$$

$$\text{subject to the constraints:} \quad x_1 + 2x_2 + x_3 = 3$$

$$\text{and} \quad 2x_1 - x_2 = 4$$

The optimal solution is given by $x^* = (2,0,1)$, $p^* = (3,-1)$ and $z^* = 5$.

We have chosen $T = 1$, $\varepsilon = 0.01$. The boundaries are 3., 3. and 1.5 respectively (computed from the first equation). In Figure 5, we present the trajectories of the primal and



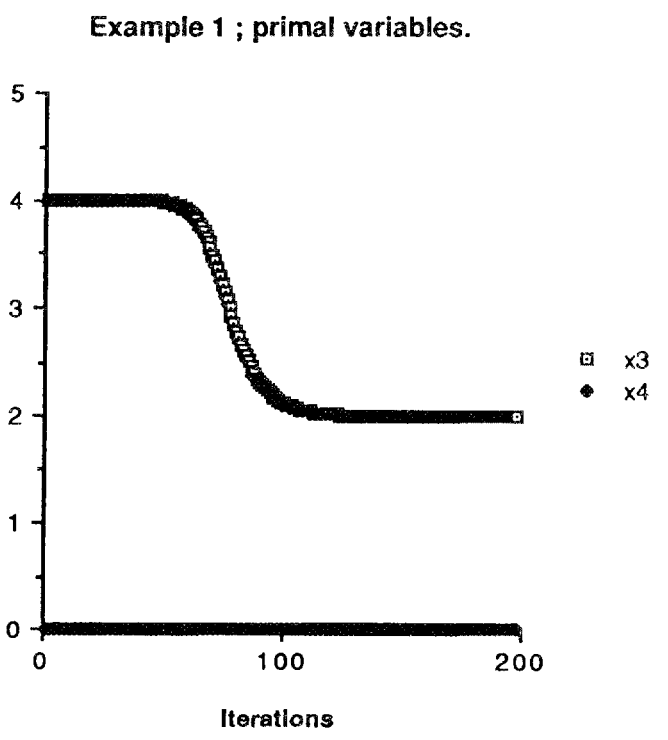
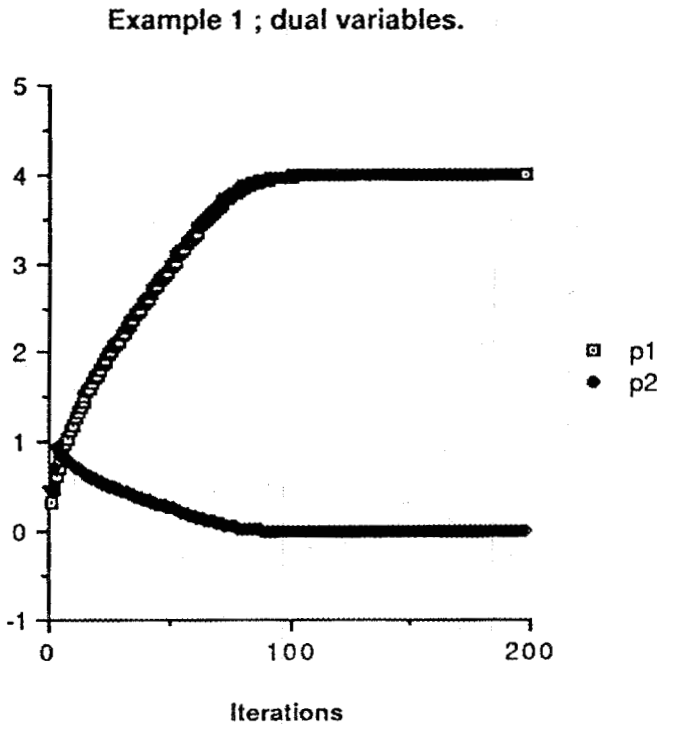


Figure 3. Trajectories of the primal variables for example 1.



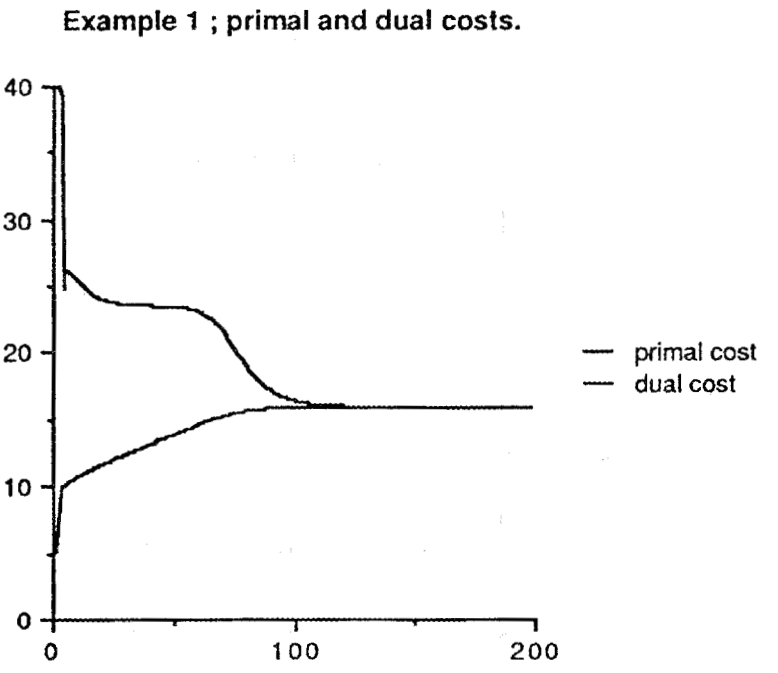
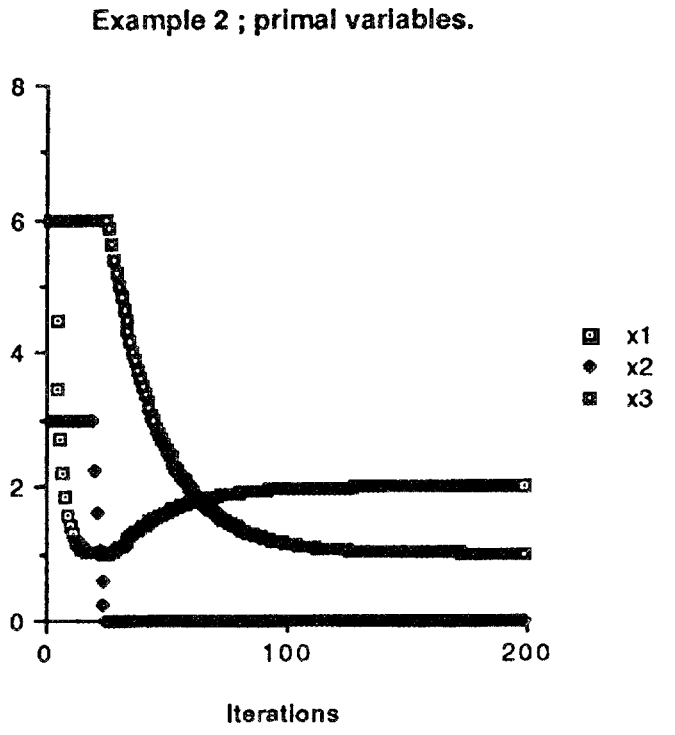


Figure 4. Trajectories of the dual variables and of the costs for example 1.



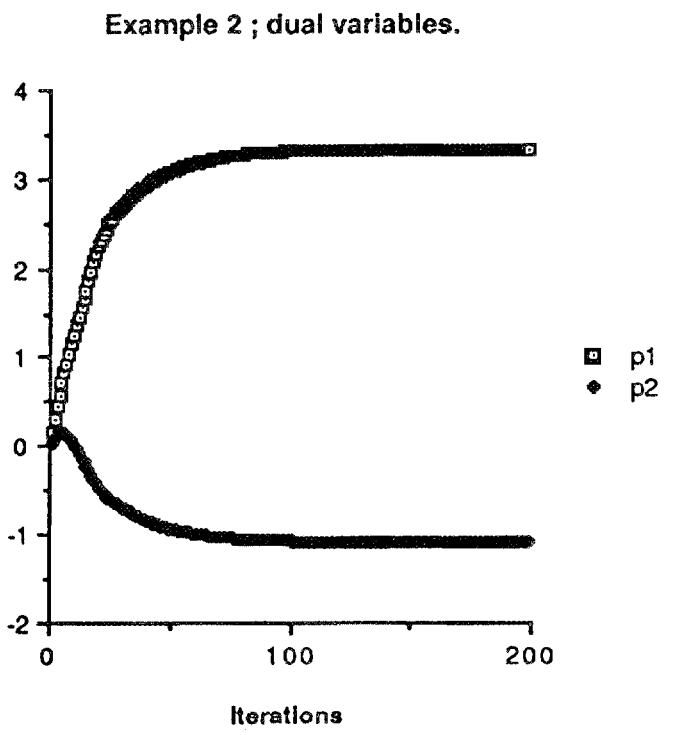


Figure 5. Trajectories of the primal dual variables for example 2.

dual variables. In order to illustrate that the duality gap is still linearly bounded by T, we have also run the algorithm with T = 0.2. Figure 6 shows clearly the consequent reduction of the duality-gap.

4.3 Example 3: a Transportation Problem.

We consider the following transportation problem (Figures 7 to 9):

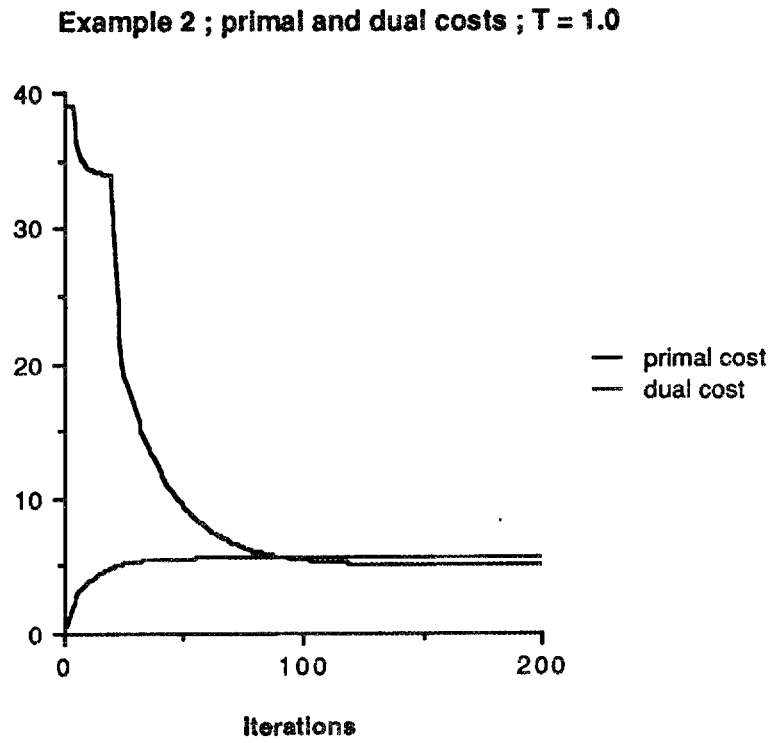
$$\text{Min} \quad z = 464\, x_{11} + 513\, x_{12} + 654\, x_{13} + 867\, x_{14} + 352\, x_{21} + 416\, x_{22} + 690\, x_{23} + 791\, x_{24} + 995\, x_{31} + 682\, x_{32} + 388\, x_{33} + 685\, x_{34}$$

subject to the standard constraints $\Sigma_j\ x_{ij} = S_i$ and $\Sigma_i\ x_{ij} = D_j$

with $S_i = 75, 125, 100,$ respectively and $D_j = 80, 65, 70, 85.$

We have taken $\varepsilon = 0.1$ and T = 12. The optimum value is 152, 535.

After one hundred and fifty iterations (Figure 9), z stabilizes at 153, 075, which is at 0.3 % of the optimum. Reducing T would - according to the above theorem - decrease this lack of optimality.



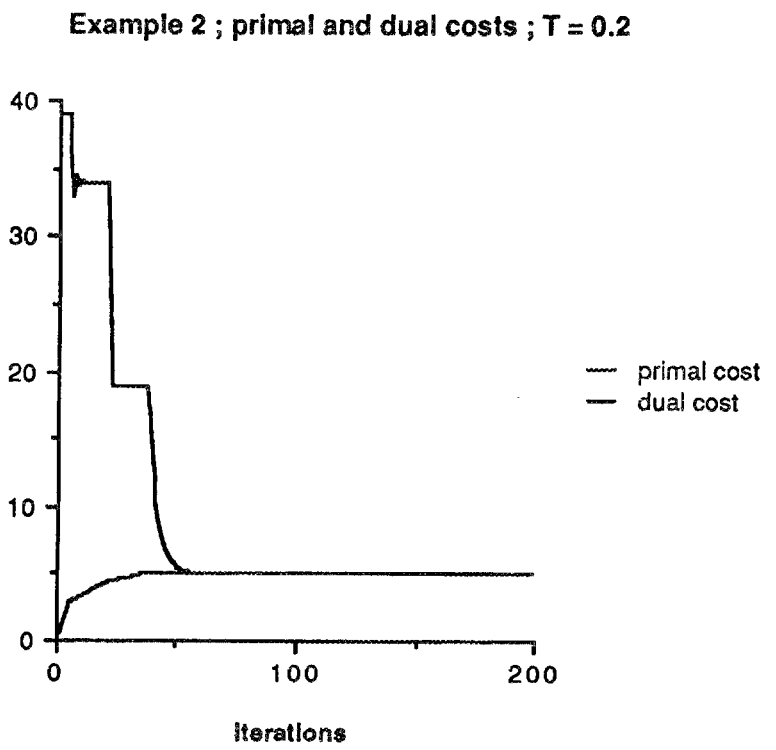
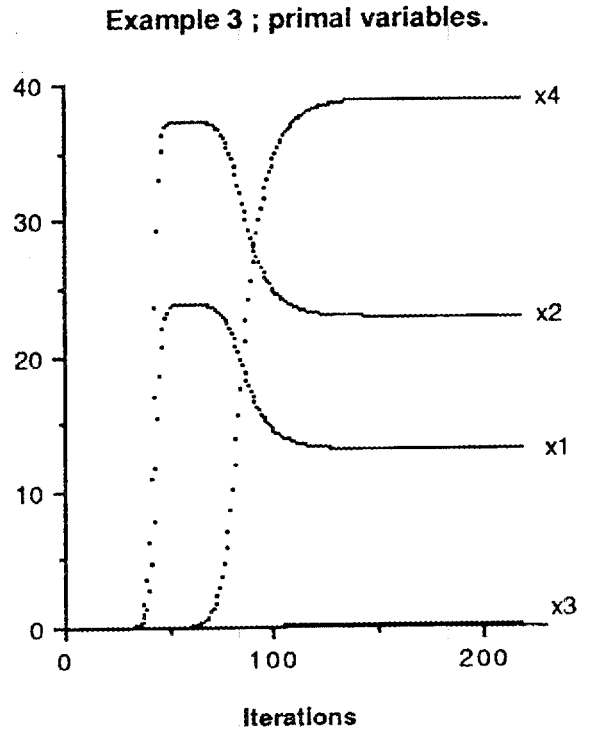


Figure 6. Primal and dual costs for example 2.



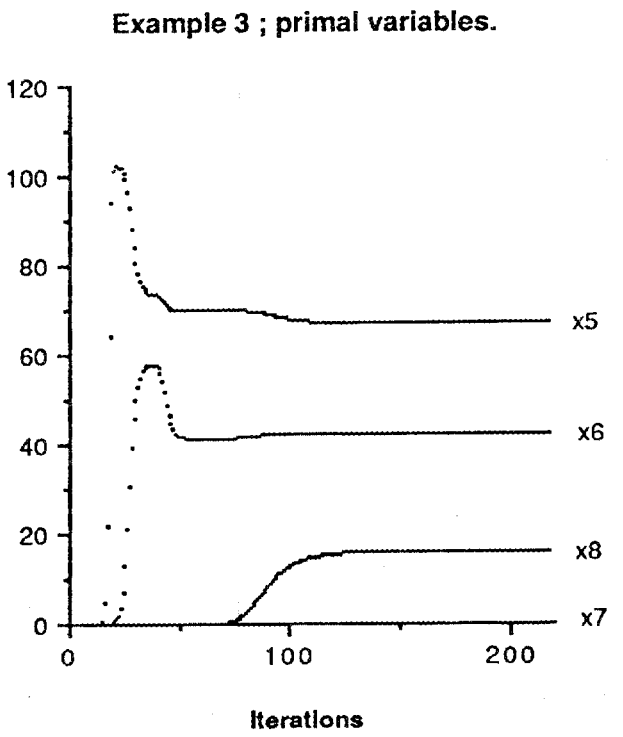
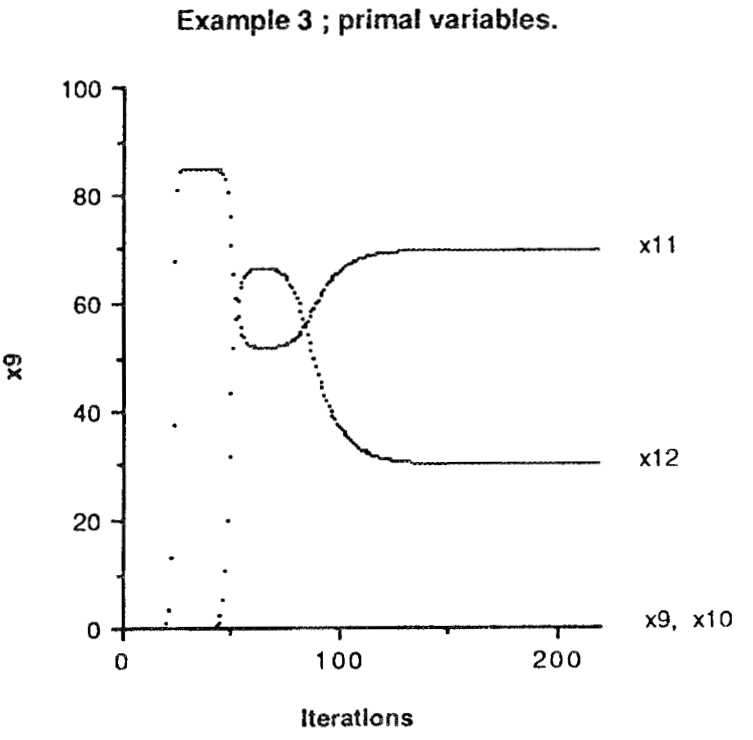


Figure 7. Trajectories of the primal variables for example 3.



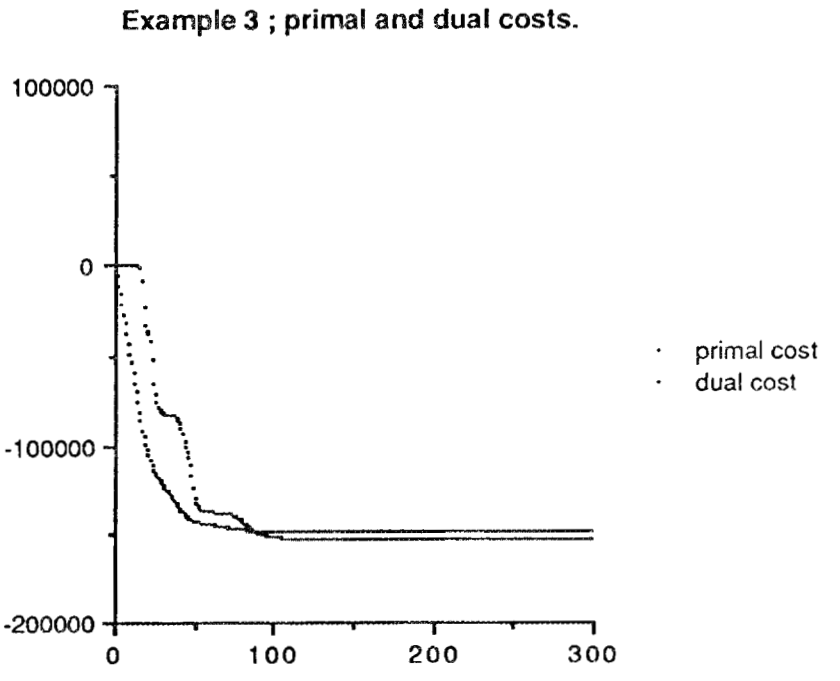
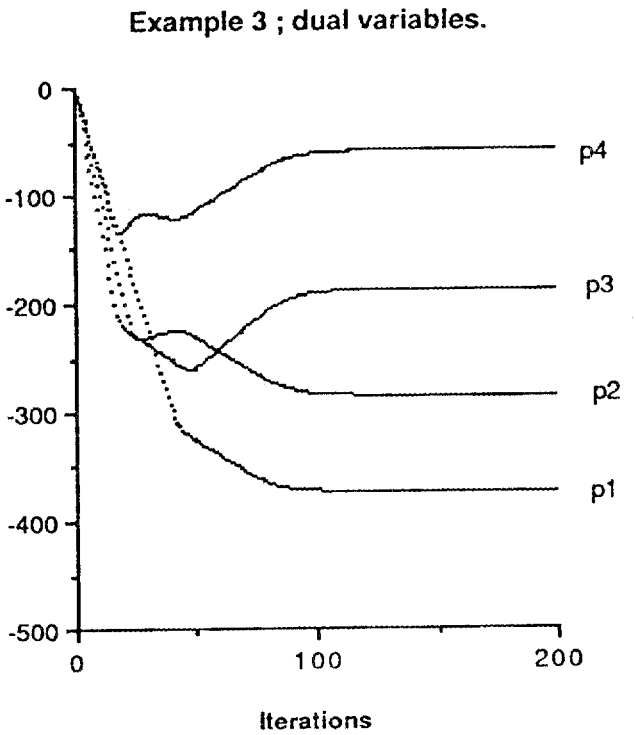


Figure 8. Trajectories of the primal variables and of the costs (example 3).



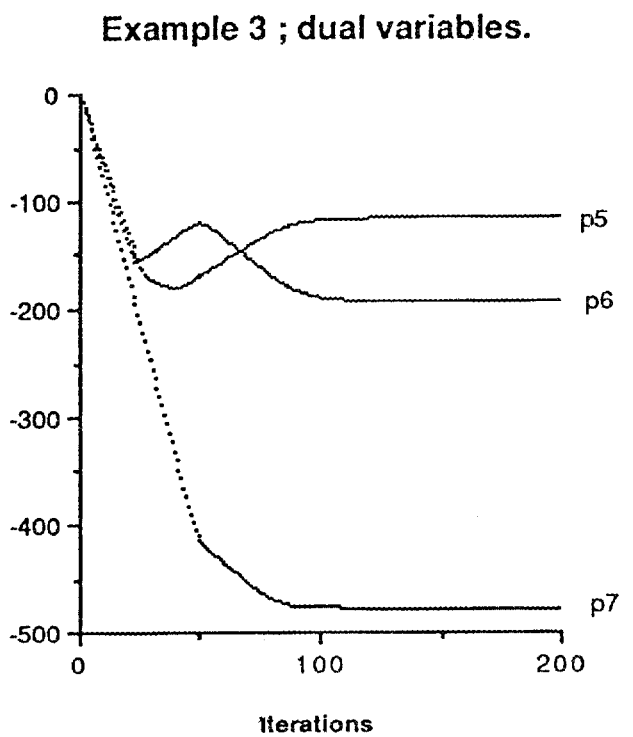


Figure 9. Trajectories of the dual variables (example 3).

5. CONCLUSIONS

In this preliminary theoretical and numerical study of the new sigmoidic linear-programming algorithm, we have shown that it is extremely simple to understand and to implement. It does not rely on any large numerical package nor does it use any complex algebraic transformation. It is fully parallel and can be implemented on very high speed parallel computers. We have shown its connection with some neural networks models.

We are currently in the process of testing the sigmoidic algorithm on some large-scale transportation models, of weakening the conditions in the theorem, and of extending the class of sigmoidic algorithms to non-linear optimization [13].
These algorithms have very interesting non-linear properties that, according to us, could be used in order to speed-up many optimization algorithms [14].

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APPENDIX

A.1 Non-Linear Interpretation.

In order to make the notations easier, we consider the simple problem

$$\text{Max } z = c_1 x_1 + c_2 x_2 \text{ with } c_1 > c_2, \qquad \text{(I)}$$

and x_1 and x_2 subject to the constraints $x_1 + x_2 = 1$, and $x_i \geq 0$,

the solution of which is $x^* = (1,0)$, $p^* = c_1$.

We assume a parametrization of x_i as a function of p and transform this problem into

$$\text{Max } z(p) = c_1 x_1(p) + c_2 x_2(p) \qquad \text{(II)}$$

subject to the constraint $x_1(p) + x_2(p) = 1$, and $x_i(p) \geq 0$.

The associated Lagrangian is (with q the dual variable associated with the constraint in p)

$$L(p,q) = c_1 x_1(p) + c_2 x_2(p) - q\,[\,x_1(p) + x_2(p) - 1\,]$$

and the optimality conditions are, if we assume that $x_i(p)$ is differentiable:

$$\frac{\partial L}{\partial p} = c_1 x_1'(p) + c_2 x_2'(p) - q\,[\,x_1'(p) + x_2'(p)\,] = 0 \qquad \text{(CO1)}$$

$$\frac{\partial L}{\partial q} = x_1(p) + x_2(p) - 1 = 0. \qquad \text{(CO2)}$$

These two conditions give p as a solution of (CO2) and, with a parametrization $x_i(p)$ such that $x_i'(p) \neq 0$ for every p, we get $q = \dfrac{c_1 x_1'(p) + c_2 x_2'(p)}{x_1'(p) + x_2'(p)}$.

Now, if $x_i(p) = 2\ g_T(c_i - p)$, which the choice of the sigmoidic algorithm, after a few algebraic manipulations, we obtain

$$q = c_1 + (c_2 - c_1)\frac{1}{1+\{r(T)\}^2} \quad \text{with } r(T) = \frac{\cosh[\frac{p - c2}{T}]}{\cosh[\frac{p - c1}{T}]}.$$

In Figure 2, we have represented (with $c_1 = 2$ and $c_2 = 1$) the left handside of (CO2) for several values of T. As T decreases, the values of p(T) tend rapidly to the optimal value $p^* = c_1 = 2$.

Thus we infer that $\mathrm{Lim}_{T->0}\ p(T) = c_1$, as could be easily derived from (CO2). This also implies that $\mathrm{Lim}_{T->0}\ q(T) = c_1 = p^*$. In conclusion, we have shown that the primal and dual solutions of the linear problem (I) can be approximated as closely as needed by the solutions of the parametrized problem (II). This is exactly what the sigmoidic algorithm does.

A.2 Listing of the Sigmoidic Algorithm.

```
#include <stdio.h>
#include<math.h>
 main()
{ double x[4],xbar[4],p[2],constraint[2],y[4],criprim,cridual;
 double c[4],a[4][2],b[2],eps,temp;
 int nmax,nvar,ncont,i,j,k;
 FILE *outdata ;

 outdata = fopen("sigmo_LP","w+");

      /*   Data for the linear program     */
c[0] = 2.00 ; c[1] = 4.00; c[2] = 4.00 ; c[3] = -3.00;
a[0][0] = 1.00  ; a[1][0] =1.00;  a[2][0] = 1.00 ;   a[3][0] =  0.00   ;
```

```
a[0][1] = 1.00 ; a[1][1] = 4.00 ;  a[2][1] = 0.00 ;  a[3][1] =  1.00   ;
b[0] = 4.00 ; b[1] = 8.00;


        /*   Data for the algorithm   */
nmax =199;   nvar = 3; ncont = 1;
p[0] = 0.00 ;  p[1] = 0.00 ; eps = .02 ; temp =.2 ;
xbar[0] = 4.00 ; xbar[1] = 4.00 ; xbar[2] = 4.00 ; xbar[3] = 4.00;
        /* The algorithm */
for  (i = 1;i<=nmax;i = i+1){
        /* iteration on primal variables */
      for  (j=0;j<=nvar;j=j+1){
       x[j]=xbar[j]/(1.0+exp(-(c[j]-p[0]*a[j][0] - p[1]*a[j][1]) /temp));}
        /* computation of the constraints  */
      constraint[0] = - b[0];   constraint[1] = - b[1];
      for  (j=0;j<=nvar;j=j+1){
              constraint[0] =  constraint[0] +  a[j][0]*x[j] ;
              constraint[1] =  constraint[1] +  a[j][1]*x[j] ; }
        /* iteration on dual variables */
      p[0] = p[0] +  eps*constraint[0] ;
      p[1] = p[1] +  eps*constraint[1] ;
        /* primal and dual criterions */
      criprim = c[0]*x[0]+c[1]*x[1]+c[2]*x[2]+c[3]*x[3];
      cridual = b[0]*p[0]+b[1]*p[1];
fprintf(outdata,"%f\t%f\t%f\t%f\t%f\t%f\t%f\t%f\n",
        x[0],x[1],x[2],x[3],p[0],p[1],criprim,cridual) ; }
fclose(outdata);}
```

A.3 Connection with Neural Networks.

The sigmoidic algorithm described in this paper has some common features with neural network models [10,11]. However, the ``computation energy" of our algorithm is not quadratic, but linear, which greatly simplifies the model.

To model a linear program with a neural network, we introduce two sets of neurons: ``primal" and ``dual". The primal neurons correspond to the x_i. Their states take values in the interval $[0,X_i]$, where X_i has been defined above. By analogy, we simply denote the state of the primal neuron i by x_i. We also define dual neurons. The dual neuron j models the cost associated to the satisfaction of the constraint j. Its state, denoted p_j, can take any value between -P and P, with P a given positive constant.

Our computation energy is, by analogy with non-linear optimization, the Lagrangian function L:

$$L(x,p) = \Sigma_i c_i \cdot x_i + \Sigma_j p_j \cdot [\Sigma_i a_j^i \cdot x_i - b_j].$$

The updating rules for the primal and the dual neurons are given by the sigmoidic algorithm. They conduct the network to local extrema of the Lagrangian function.

One notices that the sigmoidic algorithm is expressed in a parallel distributed processing way [12]. The separation of the neurons in two classes allows the dual neurons to provide information about the constraints to the primal neurons. Each dual neuron acts as a coordinator that announces to all the primal neurons what is the price associated to the constraint. A representation of the model is given in Figure 10. The updating of the primal neurons is similar to the discrete updating of many neural network models. The sigmoid function g on which the algorithm is based is reminiscent of the neuron firing function with its ``best efficiency" and ``saturation" regions. The variables c_i and b_j are respectively the ``threshold" associated to neuron x_i and p_j, and the matrix coefficients a_j^i are the

``connection strengths" between primal and dual neurons. Since the connection matrix is given by the optimization problem, it is not - in general - symmetric. Yet, when m = N, one can establish an analogy with Hopfield's quadratic energy. In that case, the connection matrix must be symmetric, all the neurons must be primal, and there is no collective information conveyed by any dual variable. In that case, one can only minimize quadratic functions without constraints.

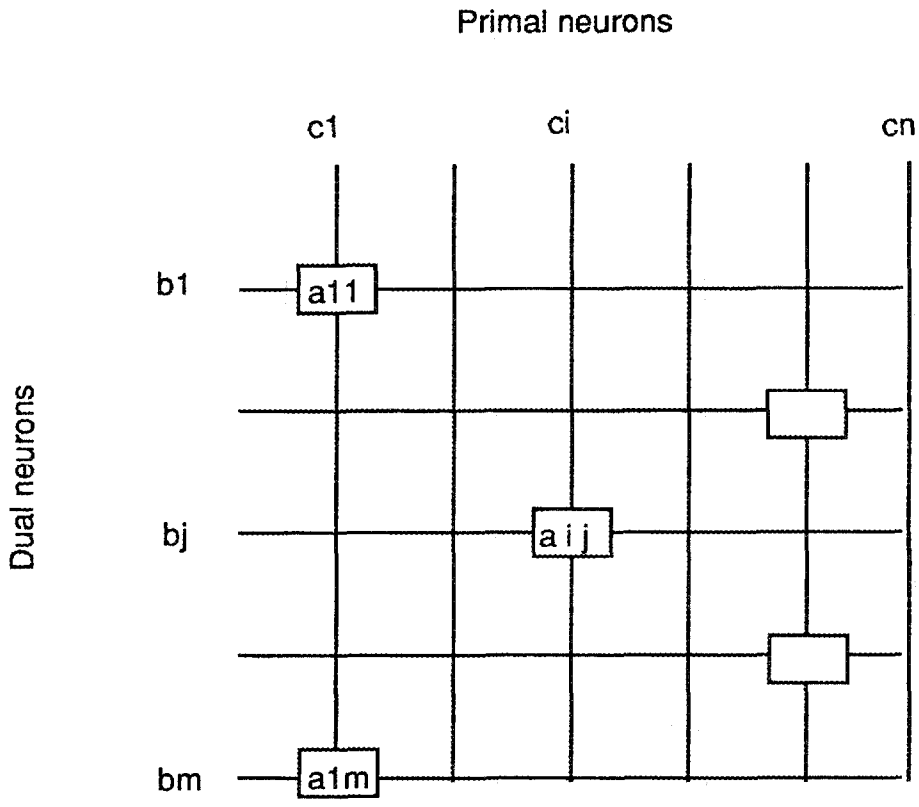

Figure 10. Architecture of a neural network based on the sigmoidic algorithm.

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