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# Parabolic Systems with Nonlinear Competitive Interactions

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Engineering Physics and Mathematics

# Parabolic Systems With Nonlinear Competitive Interactions

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# ABSTRACT

A parabolic system with nonlinear local or nonlocal interactions is considered, with specific applications to the description of competition/combat situations. Global existence and comparison results are proven under suitable assumptions on the interaction terms. Comparisons with solutions of the "equivalent" ordinary differential (space independent) model, that has been previously used in modelling combat are of special interest. The stationary state of such a system – corresponding to a stalemate – is also considered. Under certain conditions on the interaction, domain size and boundary conditions, a uniqueness result is obtained for the stationary state. A simple example shows that these conditions are sharp and uniqueness may be lost when they are not fulfilled.

# 1. INTRODUCTION

Analytic models for combat (competition) originate in the seminal paper by Lanchester [1] who described the dynamics of fighting troops on the battlefield by simple ordinary differential equations. He proposed two sets of equations: the first is based on an aimed fire model under which the rate of attrition on one side is proportional to the number of troops on the other side:

$$\begin{aligned} \frac{du}{dt} &= -d_1 v \\ & \qquad\qquad d_i > 0 \\ \frac{dv}{dt} &= -d_2 u. \end{aligned} \tag{1.1}$$

The second is based on an area fire model of combat in which the rate of attrition is proportional to the product of the number of troops on each side:

$$\begin{aligned} \frac{du}{dt} &= -c_1 uv \\ & \qquad\qquad c_i > 0 \\ \frac{dv}{dt} &= -c_2 uv. \end{aligned} \tag{1.2}$$

The coefficients $d_i, c_i$ in (1.1), (1.2) are usually referred to as the aimed fire and area fire attrition coefficient. Many variants of these models have been developed since, some of them including the effects of smart weapons, guerilla fights, and stochastic sources. For a good account of these aspects the interested reader is referred to the monograph of Taylor[2]. One of the major drawbacks of the Lanchester equations is that they do not account for spatial dependence, in the absence of which, advance, retreat, maneuvers, deployment, etc., cannot be properly described.

To correct – at least in part – this deficiency, an analytic description of combat based on partial differential equations has been recently proposed.[3] The $n$-dimensional version of these equations is

$$\begin{aligned} u_t &= L_1 u \ - F_1(x,t,u,v) \\ v_t &= L_2 v \ - F_2(x,t,u,v), \end{aligned} \tag{1.3}$$

where $L_1$ and $L_2$ are the nondegenerate second-order operators

$$L_i w = \sum_{j,k=1}^{n} a^i_{jk}(x,t) w_{x_j x_k} + \sum_{j=1}^{n} b^i_j(x,t) w_{x_j} \qquad i = 1,2 \tag{1.4}$$

with

$$a^i_{jk} = a^i_{kj}, \ \ 0 < \nu_i \mid \xi \mid^2 \leq \sum_{j,k} a^i_{jk} \xi_j \xi_k \leq \mu_i \mid \xi \mid^2 \qquad i = 1,2 \tag{1.5}$$

In the evolution equations (1.3) the different terms have the following interpretation:

– the second-order terms describe (Fickian) diffusion that models the natural tendency of any force to lose its initial configuration;

– the first-order terms describe the large-scale, ordered flow through advection (convection) as opposed to the chaotic "flow" induced by the diffusion term and accounting for small-scale movement;

– the functions $F_1, F_2$ contain local and nonlocal nonlinear interactions between the opposing forces $u$ and $v$ as well as external sources. As an example, we can consider:

$$F_1(x,t,u,v) = u(x,t)(-g_1 + f_1 u(x,t) + \int_\Omega c_1(x-y)v(y,t)dy) + d_1 v(x,t) - e_1, \quad (1.6)$$

where $g_1 u$ represents resupply of the force $u$ at the rate $g_1 > 0$, $f_1 u^2$ ($f_1 > 0$) models self-repressing effects due to crowding, saturation, confusion, $u \int c_1 v$ and $d_1 v$ ($d_1 > 0$) describe possible interactions between the forces, and $e_1 > 0$ is an autonomous source. For $c_1(x-y) = \delta(x-y)$ we recapture Lanchester's form.[1,2]

The system (1.3) has to be supplemented with boundary conditions (BC)

$$\begin{aligned} \alpha_1(x,t)u + \beta(x,t)\frac{\partial u}{\partial n} &= h_1(x,t), \ (x,t) \in \partial\Omega \times \mathbf{R}_+ \\ \alpha_2(x,t)v + \beta_2(x,t)\frac{\partial v}{\partial n} &= h_2(x,t) \end{aligned} \quad (1.7)$$

and initial conditions

$$\begin{aligned} u(x,0) &= u_0(x) \qquad x \in \bar{\Omega} \\ v(x,0) &= v_0(x) \end{aligned} \quad (1.8)$$

Such systems, often called reaction-diffusion (RD) systems are of a very general nature and appear quite naturally in biology and ecology [4], as well as in chemistry, solid state physics, viscous fluid dynamics through porous media, etc. The study of R-D systems from both theoretical, experimental and computational viewpoints has already an impressive history and is steadily growing. Recent summaries of rigorous results and methods can be found in the monographs of Fife [5] and Smoller [6], where some of the applied aspects are also considered. For the computational aspects, the monographs of Oran and Boris [7] is probably the most complete.

Apart from their ubiquity, the interest for R-D systems lies in their extremely rich variety of responses, such as wave propagation, hysteresis, bifurcations, limit cycles, catastrophe-like behavior, pattern formation and propagation, mode locking, etc.

In the single equation case, when $a_{jk}$, $b_j$, $F$, $\alpha, \beta, h$ are time independent, the system (1.3) (which reduces actually to one equation) generates a strongly monotonic semigroup (flow) and its dynamics are "completely" determined by its stationary states.[8] Although no *a priori* indication is given about the number, location, and basins of attraction of the fixed points, the result is important because it reduces the study of the evolution equation to the study of its stationary solutions. This result does not apply to typical combat situations where more than one force is involved and one usually deals with time-dependent coefficients and data. As a result, the battle outcome contains potentially elements of both predictability and

uncertainty, order and chaos, stability and instability. The realization of any of these situations depends on the type of interactions, range of parameters, strength of coupling, boundary data, and size or even shape of the domain.

Assuming that by appropriate choice of the input parameters the system (1.3)-(1.5) may be adapted to provide an accurate representation of (some kind of) combat, we are interested here in studying its time-dependent and stationary features and in giving them a military interpretation.

We shall make the following additional assumptions:[9]

i) $\Omega$ is a smooth bounded domain;
ii) $a_{ij}^k$ are Hölder continuous functions of $(x,t)$ in $(\Omega \times (0,T))$ of orders $\alpha$ and $\alpha/2$, respectively;
iii) $b_i^k$, $c_i$ are continous functions of $(x,t)$ in $(\Omega \times (0,T))$ and Hölder continuous functions of $x$ in $\Omega$;
iv) $h_i, \alpha_i, \beta_i$ are continuous functions of $(x,t)$ in $(\partial\Omega \times (0,T))$;
v) $u_0, v_0$ are continuous functions of $x$ in $\Omega$;
vi) $F_i$ are continuous functions of $(x,t)$ in $(\Omega \times (0,T))$ and Hölder continuous functions of $x$ in $\Omega$.

Usually, we shall denote by $F_i(x,t,u,v)$ a general interaction, including local and nonlocal terms. Purely local interactions will be denoted by $F_i(x,t,u(x,t),v(x,t))$. Section 2 is devoted to the general existence and comparison theory for such systems. Solutions are constructed by a monotone iteration scheme using upper and lower solutions. In Section 3 we compare the solution of the PDE competitive system with the solutions of the corresponding Lanchester equations. In Section 4 we prove a general theorem on the uniqueness of stationary states for the competitive PDE system by using a generalized maximum principle. We show the uniqueness criteria are sharp by constructing an example showing nonuniqueness.

## 2. EXISTENCE AND COMPARISON RESULTS

We shall study the classical solutions of the system (1.3)-(1.5) with the boundary conditions (1.7)-(1.8). The system is assumed to satisfy the condition i)-vi) in the Introduction. Then, we can prove the following existence result:

**Proposition 2.1**

Suppose the following assumptions are true:

$F_1$ is increasing in the $v$ position. (A.1)
$F_2$ is increasing the $u$ position. (A.2)
There exists a constant $M$ such that
$F_1$ - $Mu$ is decreasing in the $u$ position. (A.3)
$F_2$ - $Mv$ is decreasing in the $v$ position. (A.4)
There exists a constant $C$ such that

$$\|F_1(x,t,u_1,0) - F_1(x,t,u_2,0)\|_\infty \leq C\|u_1 - u_2\|_\infty \tag{A.5}$$

$$\|F_2(x,t,0,v_1) - F_2(x,t,0,v_2)\|_\infty \leq C\|v_1 - v_2\|_\infty \tag{A.6}$$

$$F_1(x,t,0,v) \leq 0 \text{ on } \Omega \text{ for all } v \tag{A.7}$$

$$F_2(x,t,u,0) \leq 0 \text{ on } \Omega \text{ for all } u \tag{A.8}$$

$$\alpha_i, \beta_i, h_i \geq 0, \ \ \alpha_i + \beta_i > 0 \text{ on } \partial\Omega \times (0,T), \ \ i = 1,2 \tag{A.9}$$

$$u_0, v_0 \geq 0 \ \text{ on } \ \bar{\Omega} \tag{A.10}$$

Then there exists a classical solution to the system (1.3)-(1.5) with boundary/initial conditions (1.7), (1.8).

Proof: We shall construct the solutions as limits of sequences $\{u^n\}, \{v^n\}$ obtained by iteration. These sequences will have lower bounds $\underline{u} = 0, \underline{v} = 0$ and upper bounds $\bar{u}$, $\bar{v}$, respectively. The upper bounds are solutions of the equations

$$\bar{v}_t - L_2\bar{v} + F_2(x,t,0,\bar{v}) = 0 \ \text{ on } \ \Omega \times (0,T) \tag{2.1}$$

$$\bar{u}_t - L_1\bar{u} + F_1(x,t,\bar{u},0) = 0 \ \text{ on } \ \Omega \times (0,T) \tag{2.2}$$

satisfying the boundary conditions (1.7)-(1.8).[9,10 Ch. V, Th. 2.1]

Define $u^0 = 0$ and notice that by assumption (A7) and (2.1) $(u^0, v^0)$ is a subsolution of (1.3)-(1.5).

We define iteratively $u^n, v^n$ as solutions of

$$u_t^n - L_1u^n + Mu^n + F_1(x,t,u^{n-1},v^{n-1}) - Mu^{n-1} = 0$$

$$v_t^n - L_2v^n + Mv^n + F_2(x,t,u^{n-1},v^{n-1}) - Mv^{n-1} = 0$$

$$\text{in } \Omega \times (0,T)$$

satisfying boundary/initial conditions (1.7)-(1.8). Note $u^0 \leq u^1$ by assumptions (A.7), (A.9), and (A.10) and $v^0 = v^1$ by construction. Assuming $v^{n-2} \geq v^{n-1}$ and $u^{n-2} \leq u^{n-1}$, one obtains

$$v^{n-1} \geq v^n \text{ and } u^{n-1} \leq u^n$$

by using assumptions (A.1)-(A.4) and comparing RHS of the equations

$$\begin{aligned} u_t^n - L_1 u^n + M u^n &= -F_1(x,t,u^{n-1},v^{n-1}) + M u^{n-1} \\ u_t^{n-1} - L_1 u^{n-1} + M u^{n-1} &= -F_1(x,t,u^{n-2},v^{n-2}) + M u^{n-2} \\ v_t^n - L_2 v^n + M v^n &= -F_2(x,t,u^{n-1},v^{n-1}) + M v^{n-1} \\ v_t^{n-1} - L_1 v^{n-1} + M v^{n-1} &= -F_2(x,t,u^{n-2},v^{n-2}) + M v^{n-2}. \end{aligned} \tag{2.3}$$

The sequences $\{u^n\}, \{v^n\}$ are monotonically increasing and decreasing respectively. The $\{v^n\}$ sequence is bounded below by zero by assumptions (A.8)-(A.10). The $\{u^n\}$ sequence is bounded by the solution $\bar{u}$. Since the RHS in (2.3) converge, by standard regularity results the sequences $\{u^n\}, \{v^n\}$ and their derivatives converge to the solution $(u, v)$ of the system (1.3)-(1.5).

This monotone scheme is similar to Sattinger[11] and Pao[12], but our analysis includes both the construction of upper and lower solutions and *a priori* bounds on solutions, which will be needed later.

Local existence can be obtained without assumptions (A.7), (A.8).

Now we have a comparison result:

**Proposition 2.2.**

Suppose there exists functions $\tilde{F}_1, \tilde{F}_2, \bar{u}, \bar{v}, \underline{u}, \underline{v}$ satisfying

$$\begin{aligned} &\bar{u}_t - L_1\bar{u} + F_1(x,t,\bar{u},\underline{v}) \geq 0, \;\; \underline{u}_t - L_1\underline{u} + \tilde{F}_1(x,t,\underline{u},\bar{v}) < 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Omega \times (0,T) \\ &\underline{v}_t - L_2\underline{v} + F_2(x,t,\bar{u},\underline{v}) \leq 0, \;\; \bar{v}_t - L_2\bar{v} + \tilde{F}_2(x,t,\underline{u},\bar{v}) \geq 0 \\ &\alpha_1\bar{u} + \beta_1\frac{\partial\bar{u}}{\partial\eta} \geq h_1, \;\; \alpha_1\underline{u} + \beta_1\frac{\partial\underline{u}}{\partial\eta} \leq h_1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega \times (0,T) \\ &\alpha_2\underline{v} + \beta_2\frac{\partial\underline{v}}{\partial\eta} \leq h_2, \;\; \alpha_2\bar{v} + \beta_2\frac{\partial\bar{v}}{\partial\eta} \geq h_2 \\ &\underline{u}(x,0) \leq \bar{u}(x,0), \;\; \underline{v}(x,0) \leq \bar{v}(x,0) \;\; \text{on } \Omega. \end{aligned}$$

Suppose for all $u_i, v_i$, $i = 1, 2$, satisfying

$$\begin{aligned} \min(\underline{u}, \bar{u}) &\leq u_i \leq \max(\underline{u}, \bar{u}) \\ \min(\underline{v}, \bar{v}) &\leq v_i \leq \max(\underline{v}, \bar{v}), \end{aligned}$$

we have

$$\begin{aligned} F_1(x,t,u_i,v_i) &\leq \tilde{F}_1(x,t,u_i,v_i) \\ F_2(x,t,u_i,v_i) &\geq \tilde{F}_2(x,t,u_i,v_i), \end{aligned}$$

(A.1)-(A.4) hold for such $u_i, v_i$, and

$$\|\tilde{F}_i(x,t,u_1,v_1) - \tilde{F}_i(x,t,u_2,v_2)\|_\infty \leq c(\|u_1 - u_2\|_\infty + \|v_1 - v_2\|_\infty) \tag{2.4}$$

Then,

$$\underline{u} \leq \bar{u}, \quad \underline{v} \leq \bar{v} \quad \text{on} \quad \Omega \times (0,T).$$

Proof: The proof follows with arguments similar to the proof of the last proposition, using the method of Fife [5,Thm. 5.1] with suitable modifications to handle the nonlocal terms.

**Remarks**

1. Pao [12] obtained similar results with $\tilde{F} = F$. The advantage of using $\tilde{F}$'s is that one does not require $F$ to satisfy (2.4).
2. The strong maximum principle can be used to show

$$\underline{u} < \bar{u} \text{ on } \Omega \quad \text{or} \quad \underline{u} \equiv \bar{u} \quad \text{on} \quad \Omega$$
$$\underline{v} < \bar{v} \quad \text{on} \quad \Omega \quad \text{or} \quad \underline{v} \equiv \bar{v} \quad \text{on} \quad \Omega$$

   Also, if (A.5)-(A.9) hold, then the corresponding solutions are nonnegative. The original Lanchester model for aimed fire (system (1.1)) does not satisfy (A.7)-(A.8). In this case, any comparison results hold only while the solution remains nonnegative. A possibility to avoid this limitation is to replace the constant attritions $d_i$ with sigmoidic functions, e.g. $d_i(\cdot) = d_i \tanh(\cdot/\epsilon)$ such that $d_i \approx$ constant for almost all $u$ and $d(0) = 0$. This is in accordance with Lanchester's own observation that at low densities the aimed fire model is not accurate any longer. Another possibility is to use, instead of equations, variational inequalities in which additional constraints (e.g., $u \geq 0, v \geq 0$) can be naturally included as obstacles.[13]
3. Similar existence and comparison results hold for elliptic problems.[11]

# 3. COMPARISON WITH ODE SYSTEMS

From the proof of Proposition 2.1, we illustrated comparisons of the solution $u, v$ of (1.3)-(1.5), (1.7), (1.8) with the solutions of the system

$$\left.\begin{aligned} &\bar{u}_t - L_1\bar{u} + F_1(x,t,\bar{u},0) = 0 \\ &\bar{v}_t - L_2\bar{v} + F_2(x,t,0,\bar{v}) = 0 \end{aligned}\right. \qquad \Omega \times (0,T)$$

satisfying the boundary/initial conditions (1.7)-(1.8). The principal importance of this result is restricted by the fact that obtaining the bounds $\bar{u}$, $\bar{v}$ may be sometimes a very hard task. (Of course, if $\alpha_i > 0$, large positive constants can serve as upper bounds.) Now we turn our attention to obtaining more specific comparison functions in particular cases; namely, we compare our solution to a solution of a related ODE system.

**Proposition 3.1**

Suppose $u, v$ is a solution to (1.3)-(1.5) with $F_1$, $F_2$ given by

$$F_1(x,t,u,v) = u(x,t)\int_\Omega c_1(x-y)v(y,t)dt + f_1(x,t,u(x,t),v(x,t))$$

$$F_2(x,t,u,v) = v(x,t)\int_\Omega c_2(x-y)u(y,t)dt + f_2(x,t,u(x,t),v(x,t)),$$

and satisfying assumptions (A.1)-(A.4), (A.7)-(A.10), $c_1, c_2 \geq 0$. Suppose $\underset{\sim}{u}, \tilde{v}$ is a solution of the ODE system:

$$\left.\begin{aligned} &\underset{\sim}{u}_t + c_1^*\underset{\sim}{u}\tilde{v} + \tilde{f}_1(t,\underset{\sim}{u},\tilde{v}) = 0 \\ &\tilde{v}_t + c_2^*\underset{\sim}{u}\tilde{v} + \tilde{f}_2(t,\underset{\sim}{u},\tilde{v}) = 0 \end{aligned}\right. \qquad \text{on } (0,T)$$

with initial data satisfying

$$0 < \underset{\sim}{u}(0) \leq \inf_\Omega u_0(x) \qquad \text{and} \qquad \tilde{v}(0) \geq \sup_\Omega v_0(x)$$

and

$$c_1^* \geq \sup_\Omega \int_\Omega c_1(x_2 - y)dy \qquad \text{and} \qquad c_2^* \leq \inf_\Omega \int_\Omega c_2(x-y)dy.$$

Assume

$$\left.\begin{aligned} &\tilde{f}_1(t,p,q) \geq f_1(x,t,p,q) \\ &\tilde{f}_2(t,p,q) \leq f_2(x,t,p,q), \end{aligned}\right. \qquad \text{on } \Omega \times (0,T) \times \mathbf{R}^+ \times \mathbf{R}^+$$

$$f_1(t,0,q) \le 0 \qquad \text{on } (0,T), \text{ for all } q \in \mathbf{R}^+$$

and

$$\|\tilde{f}_i(t,u_1,v_1) - \tilde{f}_i(t,u_2,v_2)\|_\infty \le C(\|u_1-u_2\|_\infty + \|v_1-v_2\|_\infty)$$
$$\text{for all } \min(\underset{\sim}{u},u) \le u_i \le \max(\underset{\sim}{u},u)$$
$$i=1,2,$$
$$\min(\tilde{v},v) \le v_i \le \max(\tilde{v},v).$$

If

$$h_1 \ge \alpha_1 \underset{\sim}{u} \qquad \text{and} \qquad h_2 \le \alpha_2 \tilde{v} \qquad \text{on} \qquad \partial\Omega \times (0,T)$$

then

$$\underset{\sim}{u} \le u \qquad \text{and} \qquad v \le \tilde{v}.$$

<u>Proof:</u>
Notice that $\underset{\sim}{u}, \tilde{v}$ satisfy

$$\begin{aligned}\underset{\sim}{u}_t - L_1\underset{\sim}{u} &= -c_1^*\underset{\sim}{u}\tilde{v} - \tilde{f}_1(t,\underset{\sim}{u},\tilde{v}) \\ &\le -\underset{\sim}{u}(t)\tilde{v}(t)\int_\Omega c_1(x-y)dy - f_1(x,t,\underset{\sim}{u},\tilde{v})\end{aligned}$$

and

$$\begin{aligned}\tilde{v}_t - L_1\tilde{v} &= -c_2^*\underset{\sim}{u}\tilde{v} - \tilde{f}_2(t,\underset{\sim}{u},\tilde{v}) \\ &\ge -\underset{\sim}{u}(t)\tilde{v}(t)\int_\Omega c_1(x-y)dy - f_2(x,t,\underset{\sim}{u},\tilde{v})\end{aligned}$$

$$\alpha_1\underset{\sim}{u} + \beta_1\frac{\partial \underset{\sim}{u}}{\partial n} = \alpha_1\underset{\sim}{u} \le h_1 \quad \text{and} \quad \alpha_2\tilde{v} + \beta_2\frac{\partial \tilde{v}}{\partial n} = \alpha_2\tilde{v} \ge h_2$$
$$\text{on } \Omega \times (0,T),$$

and

$$\underset{\sim}{u} \ge 0, \ \tilde{v} \ge 0 \qquad \text{on } (0,T).$$

Using comparison result, Proposition 2.2, we conclude

$$\underset{\sim}{u} \le u \text{ and } v \le \tilde{v}.$$

**Remarks:**

If

$$f_1(x,t,u,v) = -g_1u + f_1u^2 + c_1uv$$

and

$$f_2(x,t,u,v) = -g_2v + f_2v^2 + c_2\,uv$$

with constant coefficients, this comparison result can be found in Pao [12, Th. 3.1]. For solutions of ODE systems with other forms of $f_i$'s occurring in Lanchester models, see Taylor [2].

This condition

$$u(x,0) = u_0(x) \geq \underset{\sim}{u}(0) > 0$$

is too restrictive because it is natural to expect $u_0(x)$ to be zero for certain $x$ values. Now we turn to a comparison result with the space integrated concentration of the forces involved.

To exemplify, we consider a different realization of the system (1.3)-(1.5), namely

$$L_1 u = \sum_{jk} (a^1_{jk}(x,t) u_{x_j})_{x_k} + \sum_j (b^1_j u)_{x_j} \tag{3.1}$$

$$F_1 = u(x,t) \int_\Omega c_1(x-y) v(y,t) dy + d_1(u) v \tag{3.2}$$

$$L_2 v = \sum_{jk} (a^2_{jk}(x,t) u_{x_j})_{x_k} + \sum_j (b^2_j v)_{x_j} \tag{3.3}$$

$$F_2 = v(x,t) \int_\Omega c_2(x-y) u(y,t) dy + d_2(v) u \tag{3.4}$$

with

$$d_1(0) = 0, \qquad \alpha_1, \beta_i, h_i \geq 0, \qquad \alpha_i + \beta_i > 0 \tag{3.5}$$

The conditions (3.5) allow us to use comparison methods, while the nonlocal convolution-like interaction in the integral term is well suited for integral estimates. Integrating over $\Omega$ and using the divergence theorem we get

$$\begin{aligned} \frac{d}{dt} \int_\Omega u(x,t) dx &= \int_{\partial\Omega} \left[ a_1(x,t) \nabla u + \vec{b}_1(x,t) u \right] \cdot \vec{n} ds \\ - \int_\Omega \int_\Omega c_1(x-y) u(x,t) v(y,t) dx dy &- \int_\Omega d_1(u(x)) v(x,t) dx \\ \frac{d}{dt} \int_\Omega v(x,t) dx &= \int_{\partial\Omega} \left[ a_2(x,t) \nabla u + \vec{b}_2(x,t) u \right] \cdot \vec{n} ds \\ - \int_\Omega \int_\Omega c_2(x-y) u(y,t) v(x,t) dx dy &- \int_\Omega d_2(v(x)) u(x,t) dx. \end{aligned} \tag{3.6}$$

Here we denote by $a_i$ the tensor $(a^i_{jk})$, $j,k = 1,...,n$ and by $\vec{b}_i$ the vector $(b^i_j)$, $j = 1,...n$, $i = 1,2$.

The boundary terms can be written as follows

$$\int_{\partial\Omega} (a_1 \nabla u + \vec{b}_1 u) \cdot \vec{n} ds = \int_{\partial\Omega} \left( a_1 \frac{\partial u}{\partial n} + \vec{b}_1 \cdot \vec{n} u \right) ds =$$

$$\int_{\partial\Omega} \left[ a_1 \frac{h_1}{\beta_1} + \left( \vec{b} \cdot \vec{n} - \frac{a_1 \alpha_1}{\beta_1} \right) u \right] ds = f_1(t) + \int_{\partial\Omega} \left( \vec{b} \cdot \vec{n} - \frac{a_1 \alpha_1}{\beta_1} \right) u ds \tag{3.7}$$

Similarly,

$$\int_{\partial\Omega}(a_2\nabla v+\vec{b}_2 v)\cdot\vec{n}ds = f_2(t)+\int_{\partial\Omega}\left(\vec{b}_2\cdot\vec{n}-\frac{a_2\alpha_2}{\beta_2}\right)vds. \tag{3.7'}$$

The terms $f_i$ in (3.7)-(3.7′) are (at least in principle) known nonnegative functions. The known factors $\vec{b}_i\cdot\vec{n}-\frac{a_i\alpha_i}{\beta_i}$ multiplying $u, v$, are nonnegative so the signs of the integrals left in (3.7)-(3.7′) can be controlled by hypotheses on $\vec{b}_i$, $a_i$, $\alpha_i$, and $\beta_i$.

In order to compare the solutions of the PDE system with the solutions of an associated ODE (Lanchester) system, we define

$$U(t)=\int_\Omega u(x,t)dx \tag{3.8}$$

$$V(t)=\int_\Omega v(x,t)dx$$

$$U(0)=\int_\Omega u_0(x)dx=U_0,\ V(0)=\int_\Omega v_0(x)dx=V_0 \tag{3.9}$$

and assume there exist constants $\underline{c}_i, \bar{c}_i, \bar{d}_i$ such that

$$\underline{c}_i\le c_i\le\bar{c}_i,\ \ 0\le d_i\le\bar{d}_i. \tag{3.10}$$

and that

$$\begin{aligned}\vec{b}_1\cdot\vec{n}-\frac{a_1\alpha_1}{\beta_1}&\ge 0\\ \vec{b}_2\cdot\vec{n}-\frac{a_2\alpha_2}{\beta_2}&\le 0\end{aligned}. \tag{3.11}$$

Then, taking into account (3.6)-(3.8) and (3.10) we get

$$\begin{aligned}\frac{dU}{dt}&\ge f_1-\bar{c}_1UV-\bar{d}_1V\\ \frac{dV}{dt}&\le f_2-\underline{c}_2UV\end{aligned} \tag{3.12}$$

The estimates (3.12) lead to the following result.

**Proposition 3.3**

Suppose $(\tilde{U},\ \tilde{V})$ is the solution of the system

$$\tilde{U}_t=f_1-\bar{c}_1\tilde{U}\tilde{V}-\bar{d}_1\tilde{V}$$

$$\tilde{V}_t=f_2-\underline{c}_2\tilde{U}\tilde{V}$$

with

$$\tilde{U}(0)=U_0,\ \ \tilde{V}(0)=V_0$$

Then, under conditions (3.10), (3.11), the functions U(t), V(t) defined by (3.8), (3.9) with, $u, v$ solutions of the system (1.3) - (1.5) with the specifications (3.1) - (3.5), satisfy the inequalities

$$U\ge\tilde{U},\ \ V\le\tilde{V}.$$

# 4. UNIQUENESS QUESTIONS FOR STATIONARY STATES

In general, positive steady states of the model need not be unique. However, under appropriate hypotheses, we can obtain a uniqueness result. This is significant because the results of Pao[12] and Hirsch[8] then imply stability of the unique steady state.

**Proposition 4.1**

Suppose that for $i = 1,2$ the eigenvalue problems

$$L_i\phi_i + \lambda^i\phi_i = 0 \quad \text{in } \Omega,$$

$$\alpha_i(x)\phi_i + \beta_i(x)\frac{\partial\phi_i}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where $L_i$ is defined in (1.3), admit first positive eigenvalues $\lambda_1^i$ with positive eigenfunctions $\phi_i$ on $\Omega$, that $\beta_i(x) > 0$ or $\beta_i(x) \equiv 0$ on $\partial\Omega$, and that the interaction terms have the form

$$F_1(x,u,v) = u\int_\Omega c_1(x,y)v(y)dy + d_1(u)v - e_1(x)$$

$$F_2(x,u,v) = v\int_\Omega c_2(x,y)u(y)dy + d_2(v)u - e_2(x)$$

with $c_1, c_2, d_1, d_2 \geq 0$, with $d_1, d_2$ differentiable and bounded below and $d_1', d_2' \geq 0$. If the model has a steady state $u, v$ with $0 \leq u \leq K_1$ and $0 \leq v \leq K_2$, the steady state will be unique among steady states satisfying those bounds provided that we can normalize $\phi_1$ and $\phi_2$ such that

$$\lambda_1^1 > \sup_{x\epsilon\Omega}(K_1\int_\Omega[c_1(x,y)\phi_2(y)/\phi_1(x)]dy) + \sup_{0\leq w\leq K_1} d_1(w) \tag{4.1}$$

and

$$\lambda_1^2 > \sup_{x\epsilon\Omega}(K_2\int_\Omega[c_2(x,y)\phi_1(\gamma)/\phi_2(x)]dy) + \sup_{0\leq w\leq K_1} d_2(w). \tag{4.2}$$

<u>Remarks:</u> The bounds $K_1$ and $K_2$ may be established for <u>all</u> positive steady state solutions, yielding global uniqueness provided we assume $d_1(0) = d_2(0) = 0$. Indeed, by comparing the steady state system with the linear system

$$L_1u - e_1(x) = 0$$

$$L_2v - e_2(x) = 0$$

with the same boundary conditions we obtain the result via the maximum principle since the interaction terms are nonnegative. The kernels $c_1$ and $c_2$ may be taken to be positive multiples of delta distributions. In the case $c_1 \equiv c_2 \equiv 0$, $d_1$, $d_2$ constant, the example below shows that conditions (4.1) and (4.2) are sharp.

Proof:

Suppose that $u_1 v_1$ and $u_2, v_2$ are steady state solution pairs. Let $p = u_1 - u_2$ and $q = v_1 - v_2$. Then $p$ and $q$ satisfy the system

$$\begin{aligned} 0 = L_1 p - p \int_\Omega c_1(x,y) v_1(y) dy - u_2 \int_\Omega c_1(x,y) q(y) dy \\ - d_1'(u^*) v_1 p - d_1(u_2) q \end{aligned} \tag{4.3}$$

$$\begin{aligned} 0 = L_2 q - q \int_\Omega c_2(x,y) v_2(y) dy - v_1 \int_\Omega c_2(x,y) p(y) dy \\ - d_2'(v^*) u_2 q - d_2(v_1) p \end{aligned} \tag{4.4}$$

for some $u^*$, $v^*$ with $u^*(x)$ between $u_1(x)$ and $u_2(x)$ and $v^*(x)$ between $v_1(x)$ and $v_2(x)$ for each $x$, and with $p, q$ satisfying the homogeneous boundary conditions $\alpha_1 p + \beta_1 \partial p/\partial n = 0$, $\alpha_2 q + \beta_2 \partial q/\partial n = 0$. We would like to conclude that $p \equiv q \equiv 0$ to establish uniqueness. To do so, we will transform the system (4.3), (4.4) into one where we may apply the maximum principle. The transformation is related to the generalized maximum principle discussed by Protter and Weinberger[14]; some similar ideas are used in [15]. First, we replace $q$ with $r = -q$, leading to the system

$$\begin{aligned} 0 = L_1 p - p \left[ \int_\Omega c_1(x,y) v_1(y) dy + d_1'(u^*) v_1 \right] \\ + u_2 \int_\Omega c_1(x,y) r(y) dy + d_1(u_2) r \end{aligned} \tag{4.5}$$

$$\begin{aligned} 0 = L_2 r - r \left[ \int_\Omega c_2(x,y) u_2(y) dy + d_2'(v^*) u_2 \right] \\ + v_1 \int_\Omega c_2(x,y) p(y) dy + d_2(v_1) p \end{aligned} \tag{4.6}$$

Next, we introduce new variables $\tilde{p}$ and $\tilde{r}$. In the case of mixed boundary conditions, we have $\phi_1$ and $\phi_2$ strictly positive on $\bar{\Omega}$, and set $\tilde{p} = p/\phi_1$, $\tilde{r} = r/\phi_2$. In the case of Dirichlet conditions, we must take a domain $\Omega'$ with $\bar{\Omega} \leq \Omega'$, but with $\Omega'$ sufficiently close to $\Omega$ that the first eigenvalues for the Dirichlet problems for $L_1$ and $L_2$ on $\Omega'$ still satisfy (4.1) and (4.2). A calculation shows that $\tilde{p}$ and $\tilde{r}$ satisfy

$$\mathcal{L}_1 \tilde{p} + \tilde{p} L_1(\phi_1)/\phi_1 = L_2(p)/\phi_1 \tag{4.7}$$

$$\mathcal{L}_2 \tilde{r} + \tilde{r} L_2(\phi_2)/\phi_2 = L_2(r)/\phi_2 \tag{4.8}$$

where

$$\mathcal{L}_1 w \equiv \sum_{k,\ell} a^i_{k\ell} w_{x_i x_j} + \sum_k (b^i_k + 2 \sum_\ell a^i_{k\ell} \phi_{x_\ell}/\phi) w_{x_k}$$

for $i = 1, 2$. Also, if $\beta_1 \equiv 0$ then $\tilde{p}$ still satisfies $\tilde{p} = 0$ on $\partial\Omega$; if $\beta_1 > 0$, then $\partial\tilde{p}/\partial\eta = -(p/\beta_1\phi_1^2)[\alpha_1\phi_1 + \beta_1 \partial\phi_1/\partial n] = 0$ on $\partial\Omega$, since $\phi_1$ is chosen to satisfy the homogeneous boundary conditions. Similarly, $\tilde{r}$ satisfies either homogeneous

Dirichlet or Neumann boundary conditions. Finally, we have $L_i\phi_i = -\lambda_1^i\phi_i$ for $i = 1,2$. (In the case of Dirichlet boundary conditions, $\lambda_1^i$ will be the first eigenvalue on $\Omega'$ rather than $\Omega$; but recall that $\Omega'$ was chosen so that (4.1) and (4.2) still hold.) Hence, (4.5), (4.6), (4.7), and (4.8) imply that $\tilde{p}$, $\tilde{r}$ satisfy

$$\begin{aligned} 0 = \mathcal{L}_1\tilde{p} - \lambda_1^1\tilde{p} - \tilde{p}\left[\int_\Omega c_1(x,y)v(y)dy + d_1'(u^*)v_1\right] \\ + (u_2(x)/\phi_1(x))\int_\Omega c_1(x,y)\phi_2(y)\tilde{r}(y)dy + d_1(u_2)\tilde{r} \end{aligned} \tag{4.9}$$

$$\begin{aligned} 0 = \mathcal{L}_2\tilde{r} - \lambda_1^2\tilde{r} - r\left[\int_\Omega c_2(x,y)u_2(y)dy + d_2'(v^*)u_2\right] \\ + (v_1(x)/\phi_2(x))\int_\Omega c_2(x,y)\phi_1(y)\tilde{p}(y)dy + d_2(v)\tilde{p}, \end{aligned} \tag{4.10}$$

along with homogeneous boundary conditions of Dirichlet or Neumann type. Suppose that $\tilde{p}$ has a positive maximum $\tilde{p}_M$ greater than or equal to that of $\tilde{r}$. (The cases where $\tilde{r}$ is assumed to have a positive maximum larger than that of $\tilde{p}$ or where $\tilde{p}$ or $\tilde{r}$ is assumed to have a negative minimum are treated analgously, with cases involving $\tilde{r}$ using (4.2) instead of (4.1).) If the maximum $\tilde{p}_M$ is attained at $x_0 \in \Omega$, then the maximum principle and the signs of the interaction terms in (4.9) imply

$$\begin{aligned} 0 \le -\mathcal{L}_1\tilde{p}\,|_{x_0} &\le -\lambda_1^1\tilde{p}_M + (u_2(x_0)/\phi_1(x_0))\int_\Omega c_1(x_0,y)\phi_2(y)\tilde{r}(y)dy + d_1(u_2)\tilde{r}(x_0) \\ &\le \tilde{p}_M\left[-\lambda_1^1 + \sup_{x\in\Omega}\left(K_1\int_\Omega [c_1(x,y)\phi_2(y)/\phi_1(x)]dy\right) + \sup_{0\le w\le K_1} d_1(w)\right], \end{aligned}$$

which contradicts (4.1). In the case of Dirichlet boundary conditions, no positive maximum can occur on the boundary. In the case of Neumann boundary conditions, if $\tilde{p}(x_0) = \tilde{p}_M$ for some $x_0 \in \partial\Omega$, we observe that from the last computation, continuity, and hypothesis (4.1), we must have $\mathcal{L}_1\tilde{p} \ge 0$ in a neighborhood of $x_0$ in $\bar{\Omega}$. Since $\partial\Omega$ is assumed to be smooth, there is an open ball $B \subseteq \Omega$ with $\partial B \cap \partial\Omega = \{x_0\}$ such that $\mathcal{L}_1\tilde{p} \ge 0$ on $B$. If $\tilde{p}(x) = \tilde{p}_M$ for some $x \in B$, then since $B \subseteq \Omega$, we again have $\tilde{p} \equiv \tilde{p}_M$, which yields a contradiction as before. If $\tilde{p} < \tilde{p}_M$ on $B$, then the Hopf maximum principle implies $\partial\tilde{p}/\partial n > 0$ at $x_0$, contradicting the Neumann boundary condition. (This argument is similar to that given in the discussion of maximum principles for weakly coupled parabolic systems in [14].) Hence, $\tilde{p}$ cannot have a positive maximum greater than or equal to that of $\tilde{r}$.

A similar analysis using (4.2) shows that $\tilde{r}$ cannot have a positive maximum greater than that of $\tilde{p}$; the case of negative minima can also be treated in this way. It follows that $\tilde{p} \equiv \tilde{r} \equiv 0$, which implies $u_1 \equiv u_2$, $v_1 \equiv v_2$, which implies uniqueness.

Note that (4.1) and (4.2) exhibit the interplay of the local and nonlocal interaction terms and of the diffusion coefficient, domain size and boundary conditions which are hidden in $\lambda_1^1$ and $\lambda_1^2$. We shall show concretely in the following example how uniqueness is lost when assumptions (4.1) and (4.2) are violated.

**Example**

In order to easily carry out the analytic calculations, we consider the rather special system

$$u_{xx} + gu - dv = 0 \qquad x \in (-L, L) \quad (4.11)$$

$$v_{xx} + gv - du = 0$$

$$u + \gamma \frac{\partial u}{\partial n} |_{\pm L} = v + \gamma \frac{\partial v}{\partial n} |_{\pm L} = h. \qquad (4.12)$$

We observe that the system admits the solution $u(x) = v(x)$. By inspection,

$$u(x) = A \cosh \sqrt{d-g}\ x$$

is a solution. The BC (4.12) require

$$A \cosh \sqrt{d-g} L + \gamma \sqrt{d-g} A \sinh \sqrt{d-g} L = h$$

which determines A

$$A = h(\cosh \sqrt{d-g} L + \gamma \sqrt{d-g} \sinh \sqrt{d-g} L)^{-1}.$$

In order to study the influence of $g, d, L, \gamma$ on the uniqueness of the stationary state, we try to find another solution. This can be done if there exist $p \equiv u_1 - u_2$, $q \equiv v_1 - v_2$ not identically zero such that

$$\begin{aligned} p_{xx} + gp - dq &= 0 \\ q_{xx} + gq - dp &= 0 \end{aligned} \qquad (4.13)$$

$$p + \gamma \frac{\partial p}{\partial n} |_{\pm L} = q + \gamma \frac{\partial q}{\partial n} |_{\pm L} = 0 \qquad (4.14)$$

We try a solution $p(x) = -q(x) = A' \cos \beta' x$. Equation (4.13) gives $\beta' = \sqrt{g+d}$. The BC (4.14) require

$$A'(\cos \sqrt{d+g} L - \gamma \sqrt{d+g} \sin \sqrt{d+g} L) = 0$$

Nonuniqueness ($A' \neq 0$) is obtained if

$$\gamma \sqrt{d+g} = \cot \sqrt{d+g} L. \qquad (4.15)$$

For fixed $\gamma$ and $L$, the equality (4.15) is satisfied only if $d + g$ equals the first eigenvalue of the eigenvalue problem

$$p_{xx} + \lambda p = 0$$

$$p + \gamma \frac{\partial p}{\partial n} |_{\pm L} = 0$$

Suppose we have

$$p_{xx} + (d+g)p = p_{xx} - (d+g)q = 0$$

and

$$d + g < \lambda_1.$$

Then (4.13), (4.14) are not satisfied unless $A' = 0$, i.e., $d+g < \lambda$, implies uniqueness.

We note that the example above fits into the more general case discussed in Proposition 4.1 if we take $c_1 = c_2 = e_1 = e_2 = 0$ and replace $\lambda_1$ by $\lambda_1 - g$. Then, the condition (4.1)-(4.2) for uniqueness reads

$$\lambda_1^1 - g = \lambda_1^2 - g > d.$$

The calculation above shows that uniqueness is indeed lost for $\lambda_1^1 = \lambda_1^2 = d + g$, implying that conditions (4.1)-(4.2) are sharp.

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