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Combat Modeling with Partial Differential Equations The Bidimensional Case

V. Protopopescu
R. T. Santoro
R. L. Cox
P. Rusu

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ORNL/TM-11343

Engineering Physics and Mathematics

Combat Modeling with Partial Differential Equations

The Bidimensional Case

V. Protopopescu, R. T. Santoro,
Engineering Physics and Mathematics Division

R. L. Cox
Computing and Telecommunications Division

P. Rusu
University of Colorado
Boulder, Colorado

DATE PUBLISHED — JANUARY 1990

Prepared by the
OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee 37831
operated by
MARTIN MARIETTA ENERGY SYSTEMS, INC.
for the
U.S. DEPARTMENT OF ENERGY
under contract DE-AC05-84OR21400

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ACKNOWLEDGMENTS

This research has been partially sponsored by DARPA under contract #1868-A037-A1 with Martin Marietta Energy Systems, Inc., which is under contract #DE-AC05-84OR21400 with U.S. Department of Energy.

ABSTRACT

The system of partial differential equations recently introduced to model combat in one spatial dimension has been extended to include two spatial dimensions and has been numerically integrated to demonstrate its capability to describe maneuver. Engagement scenarios wherein the attacking force(s) employs the frontal attack, turning movement, envelopment, or infiltration against a fixed defensive force are presented for various combinations of troop and firepower ratios. The time and spatial distributions of the forces are displayed in graphical form along with approximate attrition rates as a function of battle duration. The results establish that the PDE formalism replicates these maneuver forms within the constraints and present development of the model.

1. INTRODUCTION

In 1914, F. W. Lanchester[1] introduced the first successful mathematical model to describe military combat. His model is a system of nonlinear ordinary differential equations (ODEs)

$$\begin{aligned} u_t &= c_1 uv + d_1 v + e_1 \\ v_t &= c_2 uv + d_2 u + e_2 \end{aligned} \tag{1.1}$$

giving the time evolution of the total number of troops, $u(t)$ and $v(t)$, during a classical, usually small size, engagement. In this model the mutual attrition is controlled by the negative coefficients c_1, c_2, d_1, d_2 while autonomous sources or sinks are represented by the nonhomogeneous terms e_1 and e_2.

Lanchester's equations and their direct generalizations have been used for more than 70 years to study combat to guide military researchers in the assessment of concentration of troops in combat, and to predict the outcome of battle and resolve issues including but not limited to force concentration, duration of the battle, and tactical assessment of the engagement.

Since WWI, when the Eqs. (1.1) were proposed, the modern battlefield environment has changed dramatically. The lethality of firepower, mobility, and logistics capabilities have increased with improved technology. In addition, highly interconnected command, control, and communications (C^3) networks, hierarchical fire control, and joint forces operations have led to changes in tactical deployment strategies and mission planning. Although the Lanchester equations have undergone numerous improvements and refinements,[2] the ODE format cannot account for the movement of opposing forces and thus relevant factors such as maneuver (advance and retrograde), terrain effects, obstacles, replacements, target priority/fire allocation, etc., could not be systematically included in the analyses of battle.

A model for combat should take into account its two basic features, namely *attrition and maneuver*, and should be able to include, in a more refined stage, the nonhomogeneous character of the modern army, the importance of the principle of command and control, and the impact of logistics and intelligence.

A new, more comprehensive, analytic model[3] was introduced to describe both the spatial and temporal evolution of forces in combat. This new model is based on partial differential equations (PDEs) and contains the Lanchester model as one of its limiting cases but goes beyond any generalization of Eqs. (1.1) that has been tried over the years. It is a new and original tool for military research. Indeed, this model provides, for the first time, an analytic alternative to war games and combat simulation methods for obtaining time–dependent solutions of the attrition rates of opposing forces during movement to contact and engagement. However, in the form given in Ref. 3, the model constrains the force movement to one dimension, i.e., along

the axis of attack, and, accordingly, cannot represent realistic maneuvers on the battlefield (e.g., circumvention of obstacles by moving forces) and in general, it cannot account for the full dynamics of modern tactical situations. To include these aspects, and to have the capability for representing realistic engagement scenarios, the model proposed in Ref. 3 was extended to two spatial dimensions.

2. THE MODEL

Two situations have been considered: two opposing forces consisting of an attacker (offensive troops), A, and a defender, D, and two attacking forces $A1$, and $A2$ against a single defending force. For the former case, the PDEs describing the engagement are given by

$$\begin{aligned}
\partial_t u_A &= \partial_{\vec{r}}(\hat{D}_A \partial_{\vec{r}} u_A) \\
&\quad + \partial_{\vec{r}}(\vec{C}_A u_A) + u_A(a_A + b_A u_A + c_A * u_D) + d_A u_D + e_A \\
\partial_t u_D &= \partial_{\vec{r}}(\hat{D}_D \partial_{\vec{r}} u_D) \\
&\quad + \partial_{\vec{r}}(\vec{C}_D u_D) + u_D(a_D + b_D u_D + c_D * u_A) + d_D u_A + e_D
\end{aligned} \tag{2.1}$$

and for the latter, the first equation in (2.1) is replaced by two equations to account for the added offensive force and are given by

$$\begin{aligned}
\partial_t U_{A1} &= \partial_{\vec{r}}(\hat{D}_{A1} \partial_{\vec{r}} u_{A1}) \\
&\quad + \partial_{\vec{r}}(\vec{C}_{A1} u_{A1}) + u_{A1}(a_{A1} + b_{A1} u_{A1} + c_{A1} * u_D) + d_{A1} u_D + e_{A1} \\
\partial_t U_{A2} &= \partial_{\vec{r}}(\hat{D}_{A2} \partial_{\vec{r}} u_{A2}) \\
&\quad + \partial_{\vec{r}}(\vec{C}_{A2} u_{A2}) + u_{A2}(a_{A2} + b_{A2} u_{A2} + c_{A2} * u_D) + d_{A2} u_D + e_{A2} \\
\partial_t u_D &= \partial_{\vec{r}}(\hat{D}_D \partial_{\vec{r}} u_D) \\
&\quad + \partial_{\vec{r}}(\vec{C}_D u_D) + u_D(a_D + b_D u_D + c_D * u_A) + d_D u_A + e_D
\end{aligned} \tag{2.2}$$

The equation to describe the defending force remains the same as in Eqs. (2.1), but now $u_A = u_{A1} + u_{A2}$. In equations (2.1) and (2.2), the terms have the following meanings

— $\partial_{\vec{r}}(\hat{D}_i \partial_{\vec{r}} u_i)$ is a (Fickian) diffusion term that models the natural tendency of any force, ancient or modern, to spread out from its initial configuration as it moves, fights, etc., or simply as just time elapses, due to fatigue, loss of concentration, loss of motivation, etc.

— $\partial_{\vec{r}}(\vec{C}_i u_i)$ is the advection term describing the large–scale, ordered "flow" of troops on the battlefield as opposed to the "chaotic," small–scale movement represented by diffusion.

— $a_i u_i$ represents re–supply of the force u_i at the rate $a_i > 0$.

— $b_i u_i^2$ models (for $b_i < 0$), self–repressing effects due to crowding, saturation, etc.

— $u_i c_i * u_j$ describes typical interaction between opposing forces and is given by

$$u_i c_i * u_j = u_i(\vec{r}, t) \int c_i(\vec{r} - \vec{r}\,')u_j(\vec{r}\,', t)d\vec{r}\,' \ .$$

— $d_i u_j + e_i$ reproduces the linear and nonhomogeneous terms in the classical Lanchester form (1.1).

For the case of two opposing forces, $i, j \in \{A, D\}$, $i \neq j$ and for two attacking forces against a single defender, $i = A1$ or $A2$, $j = D$. The kernel c_i represents a space dependent generalization of the area-fire attrition inflicted on force u_i by force u_j during the engagement. Of course, the aimed fire term $d_i u_j$ can be made nonlocal too, in the form $d_i * u_j = \int d_i(\vec{r} - \vec{r}\,')u_j(\vec{r}\,', t)d\vec{r}\,'$.

Equations (2.1) and (2.2) are supplemented with the initial conditions (I.C.)

$$u_i(\vec{r}, t = 0) = u_{i_0}(\vec{r}) \tag{2.3}$$

and the boundary conditions (B.C.)

$$\alpha_i u_i + \vec{\beta}\partial_{\vec{r}} u_i|_{\vec{r} \in \partial\Omega} = h_i(\vec{r}) \tag{2.4}$$

where the subscript i denotes (A, D) or $(A1, A2, D)$ depending on the number of forces engaged in the battle.

3. MANEUVER MODELING

Together with attrition, maneuver is an essential element of combat. The forms of maneuver include envelopment, the turning movement, infiltration, penetration, and frontal attack. Depending on the tactical situation, these may be used alone or in combination, and each poses a very different command and control challenge to the commander of the attacking forces.

In this section, the capability of the PDE model to replicate both *attrition and maneuver* is demonstrated for the engagement of homogeneous forces. The principal purpose of this study is to demonstrate that the model can describe movement during combat thus representing a significant improvement over the ODE models which can describe only attrition. Although for some cases, the sensitivity of the attrition rate is calculated as a function of the force concentration, the results remain still qualitative since realistic quantitative results depend on further refinements to the model. To provide a quantitative and detailed description of maneuver, real data regarding attritions, speeds, diffusions, etc. need to be taken into account. These data are usually not readily available or easily derived from engagement histories.[4] Moreover, the data are almost exclusively given in terms of outcomes (number of troops); finding the input parameters (attrition rates, advancement speeds) from these outcomes requires a difficult and uncertain process of parameter identification.[5]

For all the cases studied, the battlefield is taken to be a square having sides of unit length. The initial force distribution, u_{i_0}, is a bivariate Gaussian. This shape was chosen to simplify the numerical analysis since flat rectangular distributions yielded oscillations in the tails of the distribution as the battle evolved. The global force concentrations are calculated from the distributions $u_i(\vec{r},t)$ by integration over the spatial domain

$$U_i(t) = \int u_i(\vec{r},t)d\vec{r} \qquad i = 1,2\,. \tag{3.1}$$

Similarly, one computes the initial strengths of the forces

$$U_{i_0} = \int u_{i_0}(\vec{r})d\vec{r} \qquad i = 1,2\,. \tag{3.2}$$

Since the purpose here was only to demonstrate maneuver to contact, the battle time was chosen to carry the problem to engagement. No attempt was made to disengage the forces as a function of troop losses even though the capability does exist.[6,7] For the subsequent analysis, it was assumed that 15% global losses represented the unacceptable casualties criterion used for battle termination.

3.1 FRONTAL ATTACK

The frontal attack generally occurs over a wide front and along the axis of the most direct approach. It is the simplest and least economical form of maneuver since the attacking force is subjected to the concentrated forces of the defender while its firepower is most constrained. The frontal attack is used principally to overwhelm a lightly defended position or to disorganize the enemy. It is also used by a subordinate force to an attacking element or larger force carrying out an envelopment or an infiltration.

The temporal evolution of two opposing forces during a frontal attack when the global attacking force A has a 1:1 troop ratio, i.e., no advantage over the global defending force, D, is shown in Fig. 1. The defender maintains his position as the attacking force advances. During the engagement, both forces spread out on the battle plane at the same rate. The diffusion of the defending force occurs since the value chosen for its diffusion coefficient was in this, and all cases studied here, taken to be equal with the diffusion coefficient of the attacking force. This was done since there is, at present, insufficient experience and/or evidence for choosing more realistic values.[4]

Figs. 2a and 2b show the change in the total concentration, U_i/U_{i_0}, of the attacking and defending forces, respectively, as a function of the duration of the battle. Results are given for the case shown in Fig. 1* and also for the cases when U_{A_0}/U_{D_0} is 2:1 and 4:1. For all of the cases, the attacking-to-defending force attrition rate ratio was taken to be 1.5:1 in order to give the defensive forces a firepower advantage.

The curves show that concentration is an advantage in a frontal attack maneuver. For example, when parity exists, the attacking force loses 15% of its troops at 0.55 time units into the battle compared to 6% for the defender. The superior firepower of the defense has the advantage. However, when the offense forces have troop superiority, as should be the case in this form of maneuver, the defending force is eventually overwhelmed and its losses are too great to sustain the battle. When the offense has a 2:1 superiority, ~15% losses occur at 0.85 time units and are slowly varying at times greater than 0.85. On the other hand, the defensive force loss rates show a much higher loss rate after the opponents become fully engaged increasing from ~15% to 30% between 0.85 and 0.95 time units. The results are even more dramatic when the attacking-to-defending force ratio is 4:1. The defending force suffers 15% casualties at 0.65 time units compared to only 6% for the attacking force. As the battle continues, the offensive force loss

* In plots showing the frontal attack for force ratios of 2:1 and 4:1, the larger attacking force shadows the defending force as the battle proceeds. The 1:1 case was selected for Fig. 1 in the interest of clarity.

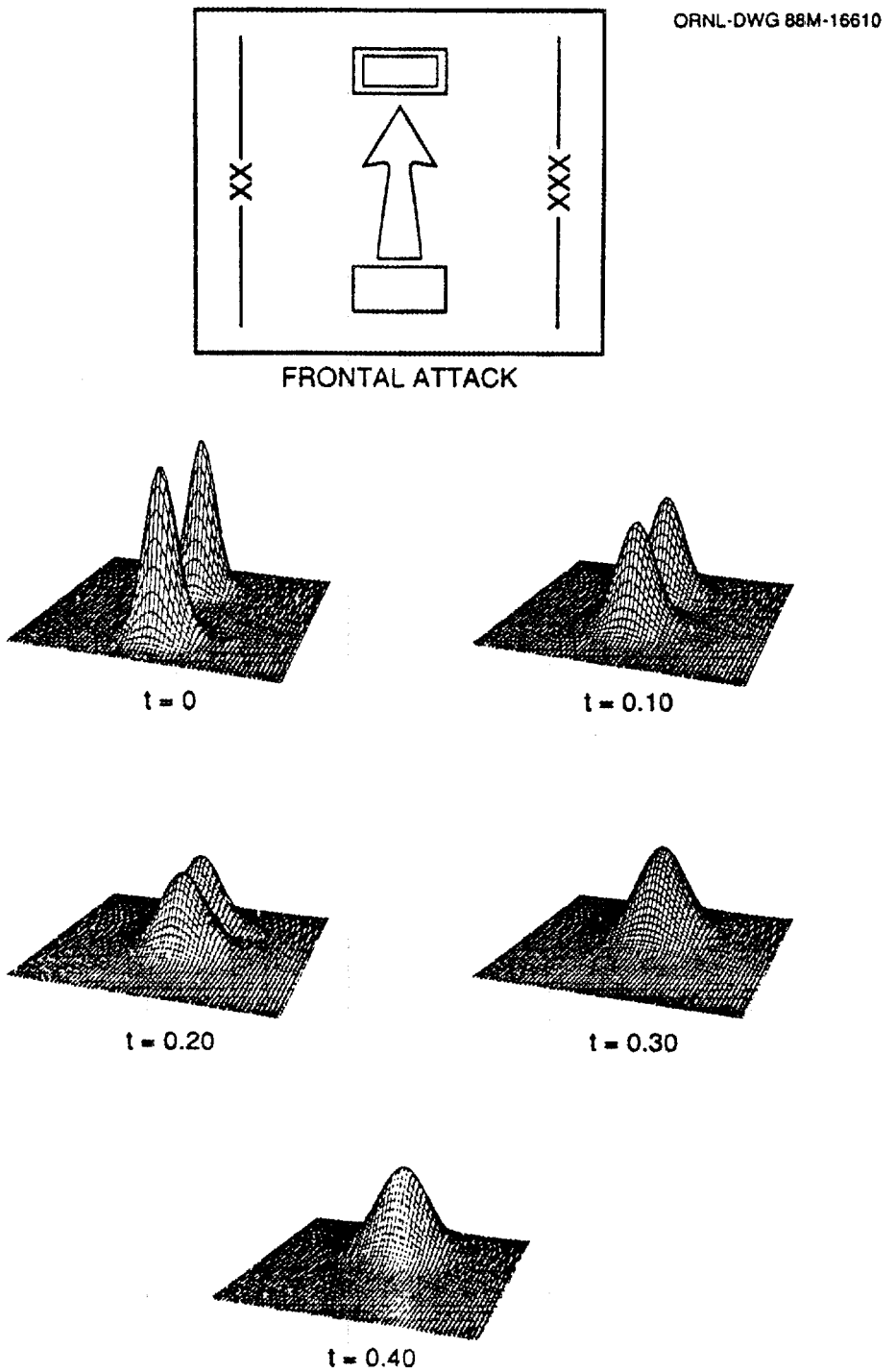

Fig. 1. Time evolution of a frontal attack maneuver for the case $U_{A_0}/U_{D_0} = 1{:}1$.

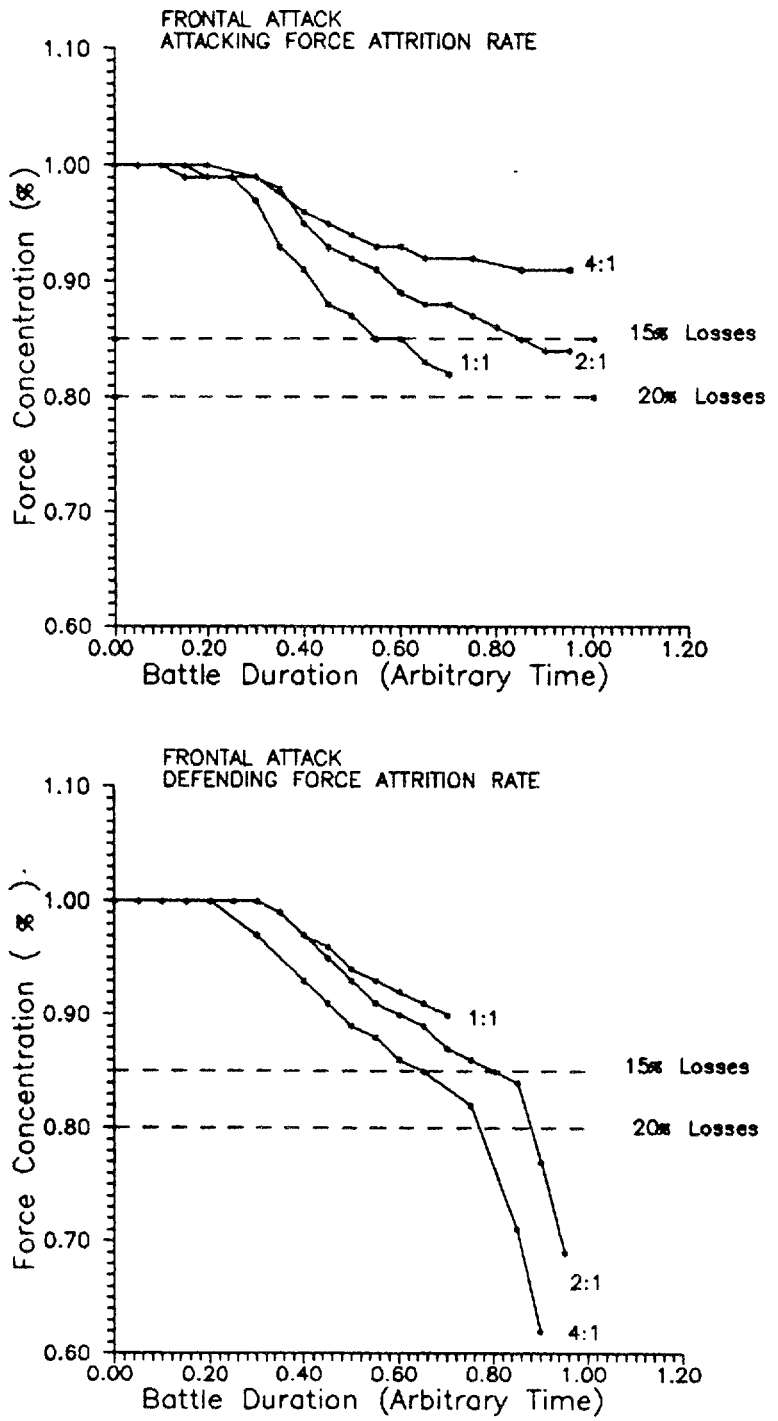

Fig. 2. Global force concentration as a function of battle duration for attacking (a) and defending (b) forces when the offensive force is carrying out a frontal attack maneuver for the cases U_{A_0}/U_{D_0} = 1:1, 2:1, and 4:1.

rate remains essentially the same while the defensive force sustains casualties at a rapidly increasing rate.

3.2 TURNING MOVEMENT

The turning movement, which is a variant of the envelopment, is a tactical maneuver wherein the attacker attempts to by-pass a heavily defended position to assault a lightly defended position or secure an undefended objective. The representation of this maneuver is shown in Fig. 3 for the case when U_{A_0}/U_{D_0} is 4:1 and the attacking-to-defending force attrition rate ratio is 1.5:1. For the purpose of this analysis, the firepower advantage was given to the defending element and the troop advantage to the offense. For a very lightly defended objective this might not actually occur.

The numerical simulation did clearly demonstrate the movement of troops on the field, the active phase of the battle when the direct contact is realized and the mutual attrition decreases the number of troops engaged in combat, and the retreat of the defeated force (not shown in Fig. 3). If the defeated force is the one that made the attack, this last phase of the battle takes place on the same track as the one used for engagement but in reversed direction. If the entrenched force is the defeated one its retreat is made on some new track conveniently chosen. The loss of the total number of troops of one combatant that triggered the retreat was arbitrarily set at 15%.

Fig. 4 compares the troop losses as a function of battle duration for the case given in Fig. 3. The offensive force superiority results in a victory by inflicting unacceptable casualties on the defending troops. The defending force suffers 15% losses at 0.75 time units while the attacking force losses are 7%. However, as the attacker expends 1.87 troops for each defender, victory is achieved, but at a heavy price. If the offense was modeled with both force and firepower superiority for the attacker, the outcome would be considerably different and, perhaps, a more realistic representation of this maneuver.

3.3 ENVELOPMENT

Envelopment is the form of maneuver that pits strength against weakness. The main element of the attacking force avoids the enemy front where his forces are most heavily defended and where his firepower is most concentrated. The defender's attention is fixed forward by the diversionary assault of a small force while the main attacking body moves around the enemy defenses to strike at his flanks. A single envelopment is directed against one flank and the double envelopment is used to assault both of the enemy's flanks. If the enemy forces move forward to repel the frontal attack, the enveloping maneuver can result in an encirclement that severs lines of communication and prevents escape or retreat and blocks the arrival of reinforcements. The envelopment places a priority on speed and agility since

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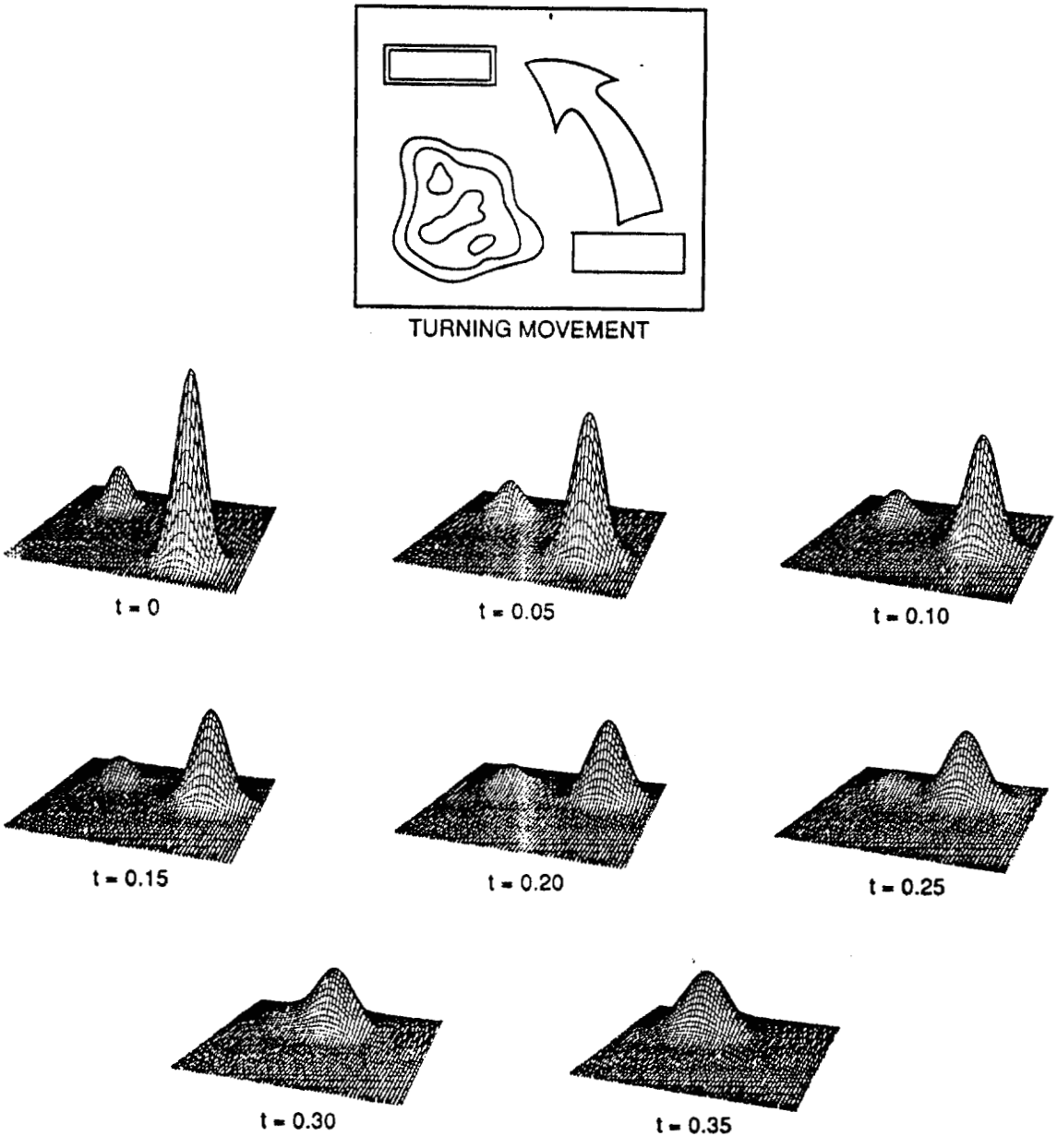

Fig. 3. Time evolution of a turning movement for the case $U_{A_0}/U_{D_0} = 4{:}1$.

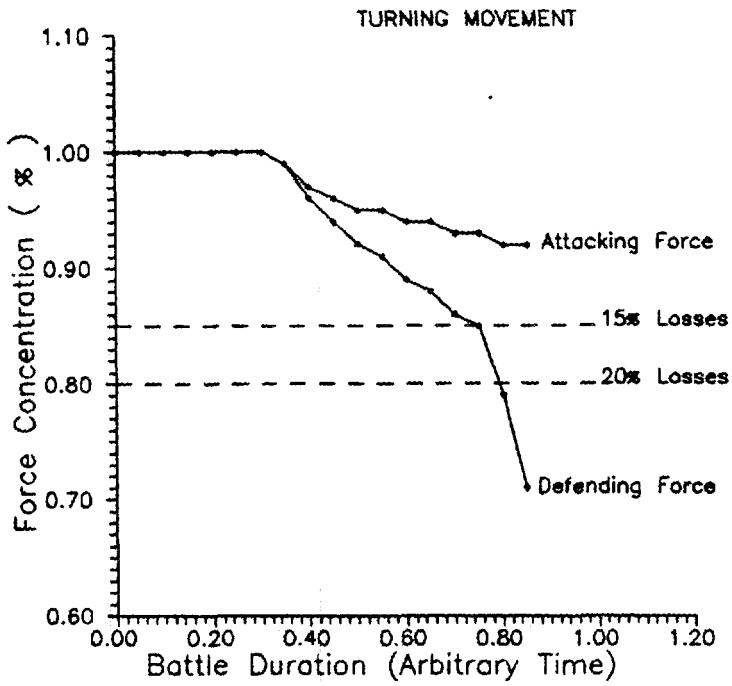

Fig. 4. Global force concentration as a function of battle duration when the offensive force is performing a turning movement for the case $U_{A_0}/U_{D_0} =$ 4:1.

success depends on reaching the enemy's vulnerable flanks before he can shift his forces and fire power.

The scenario considered for this maneuver consists of a small attacking force having initial strength u_{A1_0} carrying out a frontal attack on a defensive initial force u_{D_0} while a larger force with initial strength u_{A2_0} makes an envelopment and attacks the enemy flank. Three variations on this engagement were calculated corresponding to total attacking-to-defending force ratios, $(U_{A1_0} + U_{A2_0})/U_{D_0}$, of 1:1, 2:1, and 4:1. For all of the cases, U_{A2_0}/U_{A1_0} is 3:1. Since this maneuver is used to pit strength against weakness, firepower superiority was given to the offensive forces in the ratios 2:1 and 4:1 for attacking forces $A1$ and $A2$, respectively. The movement of the offensive forces was adjusted so that both offensive elements reached the objective at the same time.

The evolution of the engagement when $(U_{A1_0} + U_{A2_0})/U_{D_0}$ is 4:1 is shown in the sequence of topographical plots in Fig. 5. As in the case of the turning maneuver, the envelopment is completed by the attacking force making a right-angle turn to engage the enemy force.

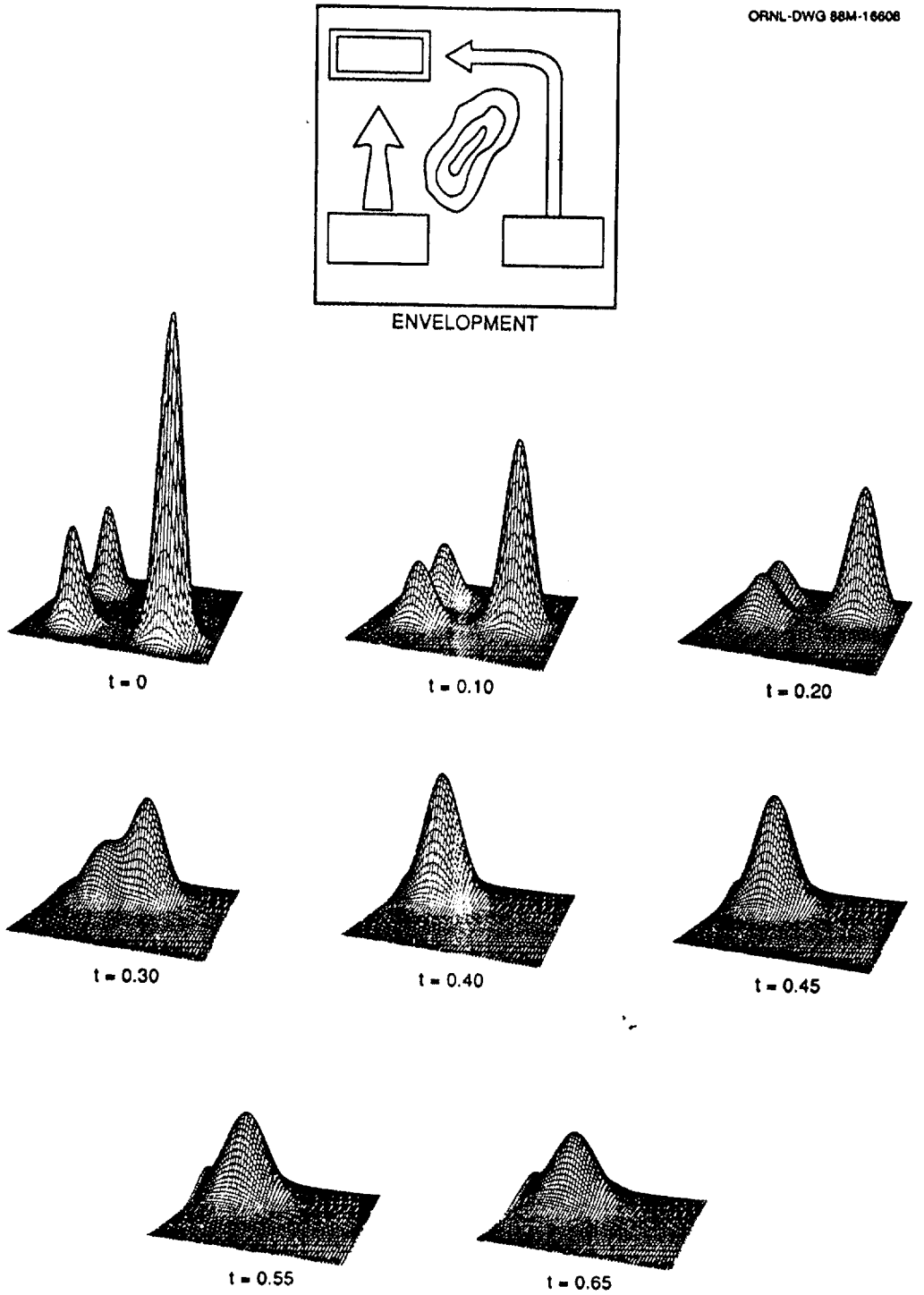

Fig. 5. Time evolution of an envelopment maneuver for the case $(U_{A1_0} + U_{A2_0})/\ U_{D_0} = 4{:}1$. Force $A1$ is on the left and force $A2$ is on the right.

Fig. 6 compares the troop loss rates as a function of the duration of the battle for the three engagement scenarios. For the 1:1 troop ratio case, the defensive forces suffer 15% casualties in 0.62 time units into the battle compared to 11% and 7% for attacking forces 1 and 2, respectively. When the attacking force has a 2:1 advantage, 15% casualties occur among the defending troops at 0.45 time units while the attacking force losses are 6% and 2%. For the case shown in Fig. 5, the same defensive force losses occur at 0.38 time units and the attacking elements lose 3% and 2% of their initial strength. As the offensive force superiority increases, the casualty rate of defensive force increases rapidly as expected with the high offensive firepower advantage.

The casualties suffered by attack force 1 are, in all cases, greater than those of attack force 2. This occurs for several reasons. Since the velocities of the assaulting troops are set to effect simultaneous contact with the enemy, force 1 is exposed to area fire for a longer time than the enveloping force. The casualty rate depends on the ratio U_{A1_0}/U_{D_0} which for the total force ratios cases of 1:1 and 2:1 is 1:4 and 1:2, respectively. For these cases, the enemy force advantage reduces the attacking force 1 firepower advantage. For the 4:1 total troop ratio, U_{A1_0}/U_{D_0} is 1:1 but the attacking force maintains the advantage in area and aimed fire.

3.4 INFILTRATION

The infiltration maneuver is one of the means for reaching the enemy's rear without fighting through prepared defenses. It is a covert movement where all or part of the attacking forces cross the enemy lines to secure a favorable position in the rear. A successful infiltration requires that the initial movement of forces go undetected so the attacking force is generally limited in size. This maneuver is used in rough terrain where visibility is limited or in areas poorly covered by observation and fire. It may be used to attack a lightly defended position or to assail a stronger position by attacking the enemy's flank.

The cases analyzed here combine infiltration with a frontal attack on a lightly defended position. The attacking force is initially split into two equal size forces ($U_{A1} = U_{A2}$) to carry out the infiltration. These elements then combine to complete the frontal attack. In the first case, the offensive-to-defensive force ratio, U_A/U_D, is 1:1 with the offense having a 3:1 firepower advantage. The second case demonstrates the assault on a heavily defended position by an attacking force having a 1:2 troop disadvantage but a 5:1 firepower advantage.

The time evolution of the infiltration maneuver for the case when $(U_{A1} + U_{A2})/U_{D_0}$ is 1:1 is shown in Fig. 7. Shown in Fig. 8 are the rates of losses for the attacking and defending forces in this maneuver. Since each of the two attacking force elements have the same size, the attrition rates are identical and appear as one curve in the figure. For the case when the troop ratio is 1:1, the defense loses 15% of its troops in 0.60 time units compared to 5% per attacking element for the

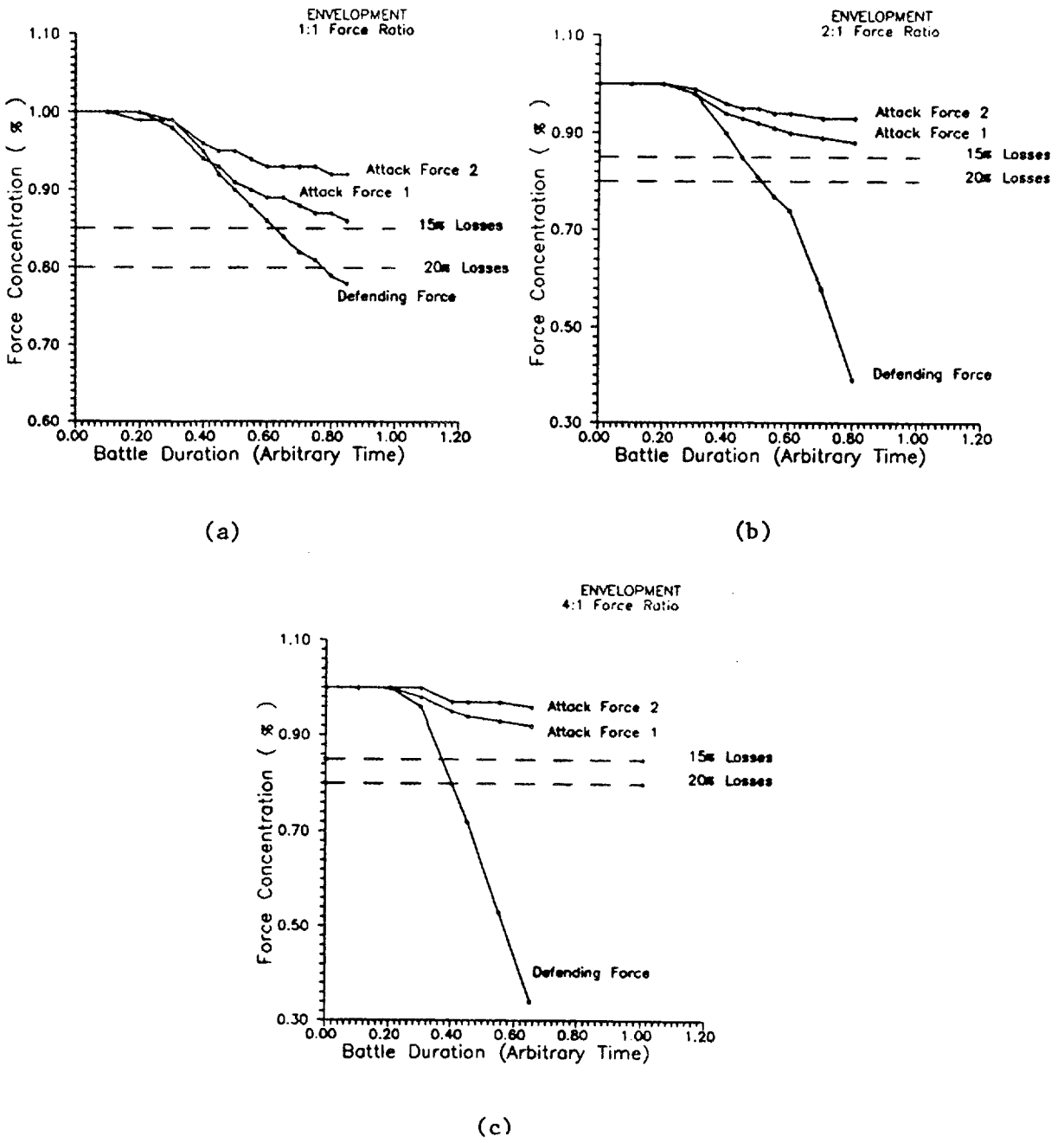

Fig. 6. Global force concentration as a function of battle duration during an envelopment maneuver for the cases when $(U_{A1_0} + U_{A2_0})/U_{D_0} = 1{:}1$ (a), 2:1 (b), and 4:1 (c).

offensive troops for the 1:2 troop ratio case, the defense suffers 15% loses in 0.46 time units while the attacking forces endure 5% casualties per attacking force. Even though the defense has troop superiority in this case, the firepower advantage of the assaulting forces secures a rapid victory.

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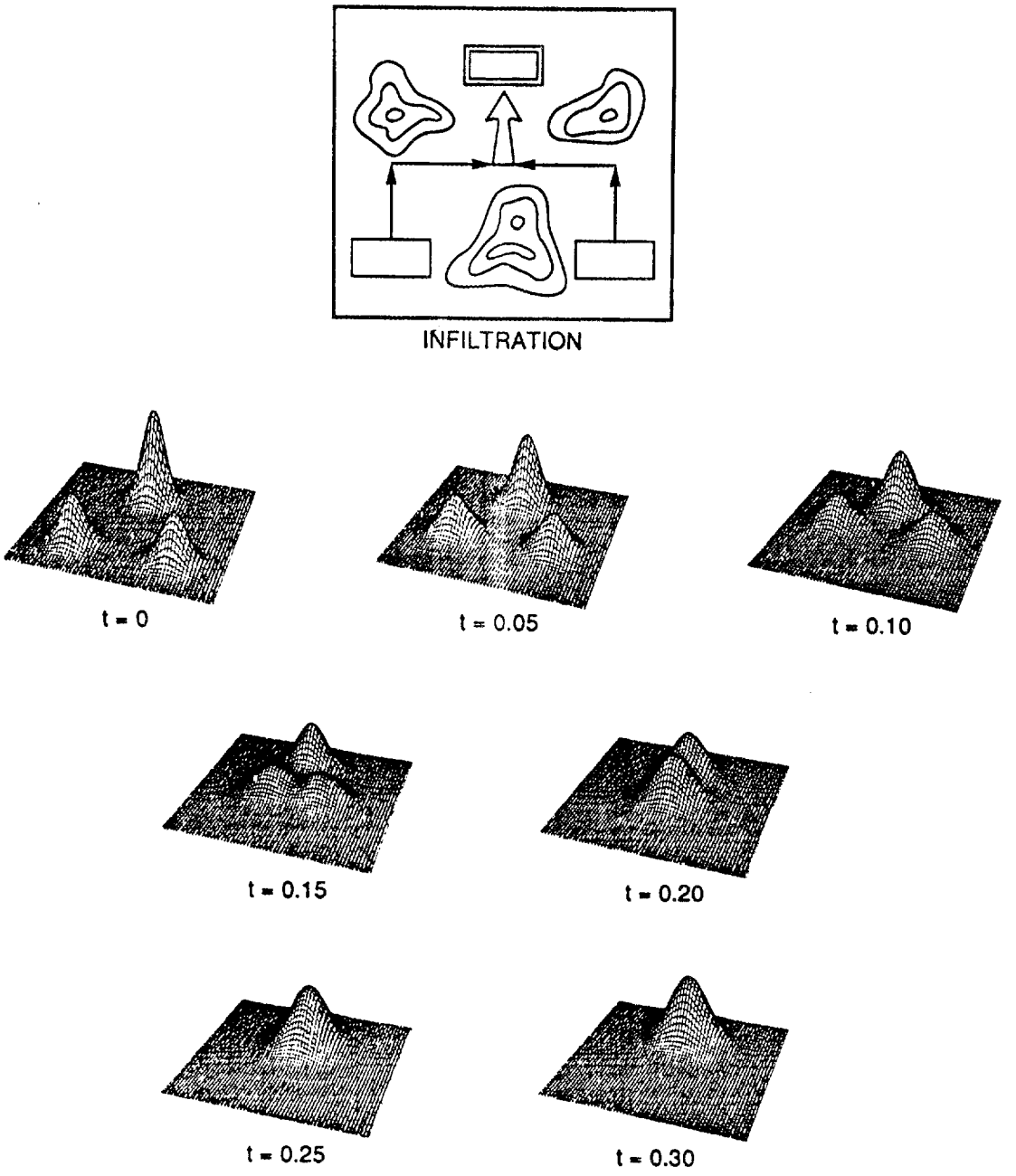

Fig. 7. Time evolution of an infiltration maneuver for the case $(U_{A1_0} + U_{A2_0})/U_{D_0} = 1{:}1$. Force $A1$ is initially on the left and force $A2$ is initially on the right.

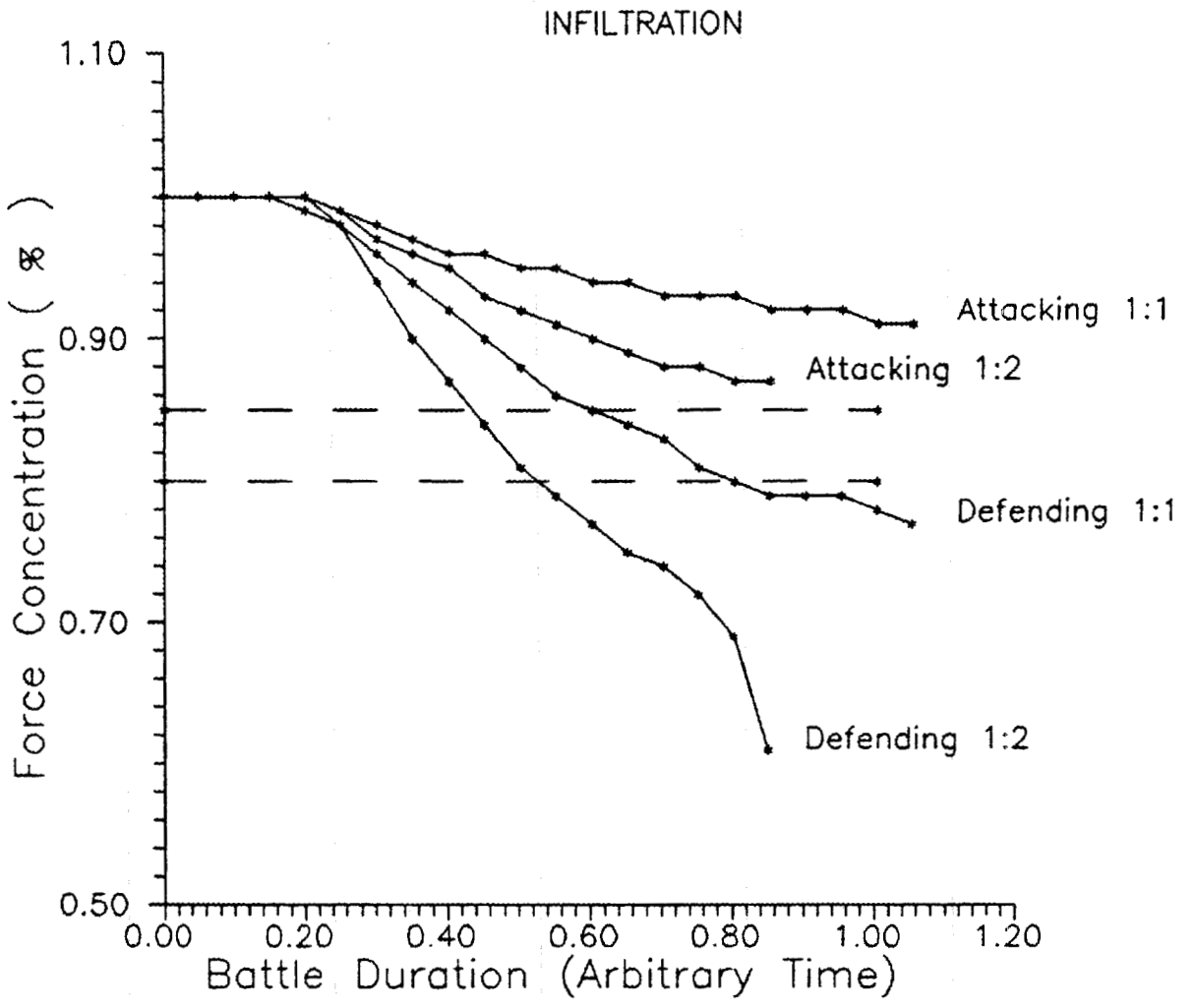

Fig. 8. Global force concentration as a function of battle duration during an infiltration maneuver for the cases $(U_{A1_0} + U_{A2_0})/U_{D_0} = 1{:}1$ and $1{:}2$.

4. NUMERICAL COMPUTATIONS

The approach taken in solving the combat modeling PDEs in one dimension[3] was extended to two dimensions by discretizing the PDEs in both spatial dimensions. The integrator chosen to solve this problem is based on the Method of Lines, designed[8,9] to solve a large class of PDE systems with different I.C. and B.C. Driver programs for these integrators were written to include the tactical information needed to simulate the chosen maneuvers. Thus it was necessary to investigate and assess the reliability and precision of the software for the specific class of problems to be solved here. Extensive testing of the program was carried out on a large set of test cases. This preliminary work involved different versions of the integrator code, as well as analytic work that produced exact solutions for comparison with the numerical results. The main purpose of this testing was to determine numerical values for the coefficients in Eqs. (2.1) and (2.2) to be used in the modeling of the nonanalytically solvable cases.

4.1 COMPUTATIONAL METHODS AND COMPUTER PROGRAMS

The Method of Lines on which the integrator is based consists of two parts. The first one is the discretization of the spatial derivative terms in the systems (2.1)-(2.2); it generates a large ODE system for the time evolution of the troop densities. The second part consists of the integration of this ODE system using the powerful numerical techniques that have been independently developed to solve this kind of problem.

Let NPDE be the number of original PDEs, NX the number of nodal points in the x-direction, and NY the number of nodal points in the y-direction. Upon discretization in the spatial variables, the original system of NPDE PDEs in the three independent variables x, y, and t is converted to a system of (NEQ = NPDE * NX * NY) ordinary differential equations (ODEs) in the single independent variable t. Relatively recent developments include software for previously difficult-to-solve stiff ODEs and reliable, efficient algorithms for dynamically changing time step size and method order to maintain mathematical stability and a user-specified accuracy during the course of the integration.

The particular implementation of the Method of Lines used to solve the combat modeling equations in two dimensions is that embodied in the FORTRAN package PDETWO/PSETM/GEARB.[9] The actual integration of the system of ODEs is performed by the code GEARB[10] which is a variable step/variable order solver designed to handle both stiff and nonstiff ODE systems. The user's program directly calls the GEARB code which, in turn, invokes PDETWO to generate the right hand sides of the ODEs. Given a description of the original PDE problem, including the computational mesh, I.C., and B.C., PDETWO constructs the corresponding semidiscrete approximation system of time-dependent ODEs using second-order

centered difference approximations for the spatial derivatives. The routine PSETM is designed to allow efficient computation of the Jacobian matrix (the first derivative matrix for the right-hand side of the ODE system with respect to the dependent variables) for those stiff problems which require the use of the Jacobian matrix. The elements of the Jacobian matrix are given by

$$J_{mn} = \frac{\partial f_m}{\partial v_n} \qquad \text{for } m, n = 1, \ldots, \text{ NEQ}$$

where f_m = the right-hand side of the mth equation in the ODE system and $v_n = u_{k,i,j} \equiv u_k(x_i, y_j)$ for $k = 1, \ldots,$ NPDE; $i = 1, \ldots,$ NX; $j = 1, \ldots,$ NY.

Generating the entries in the Jacobian matrix in a straightforward manner using finite difference approximations to the first partial derivatives would require NEQ evaluations of the right-hand side of the ODE system as each of the variables v_n was perturbed in turn. However, a close examination of the structure (the location of the nonzeros) of the Jacobian leads to the recognition that more than one v_n can be perturbed simultaneously if these v_n are chosen judiciously so that each f_m is a function of only one of the sets of selected v_n's. Indeed, as a consequence of the special block-tridiagonal structure (arising from the five-point difference formulas used to approximate the spatial derivatives in the original PDEs) of the Jacobian, only five right-hand side evaluations are required to completely determine the Jacobian elements corresponding to a particular PDE and hence 5 * NPDE evaluations to define the entire Jacobian.

For checking and testing reasons, we investigated first the conservation properties of the solution for the one-dimensional pure diffusion-convection equation

$$\partial_t u = \partial_x(D\partial_x u) + \partial_x(Cu) \ ,$$

with homogeneous mixed B.C.

$$D\partial_x u + Cu = 0 \ .$$

When the initial condition $u_0(x)$ used in the integration of the above PDE satisfies the same B.C., the solution u that conserves the initial concentration in the exact analytic sense; namely, $U(t) = U_0$. However, from the numerical viewpoint, the situation is very different. Whatever precautions are taken, the numerical solution will yield global concentrations that vary slowly with time. Yet, the rate of change can be decreased if the spatial grid is made finer and special care is taken in the treatment of the B.C. This investigation clarified the relation between different choices for B.C. and the accuracy of the numerical solution provided by the software. Thus, with some numerical imprecision unavoidable, we were able to find reasonable values for the parameters such that the total loss through the boundaries due to

diffusion and convection in the attrition free problem be very small compared to the expected attrition losses in the evolution problems (2.1)–(2.4). This ensures that the combating armies are confined to the battlefield during the engagement, instead of escaping through the boundaries via spurious numerical artifacts.

We considered next the following version of (2.2):

$$\partial_t u_i = \partial_{\vec{r}}[\hat{D}_i \partial_{\vec{r}} u_i + \vec{C}_i u_i] + \sum_{j \neq i} a_{ij} u_i u_j \qquad i = 1, 2, 3\,, \tag{4.1}$$

in which diffusion, convection, and local attrition terms are accounted for, as a basic model of tactical maneuvers. This system of equations have been supplemented with the I.C.

$$u_i|_{t=t_0} = u_{i0}(\vec{r}) \qquad i = 1, 2, 3, \tag{4.2}$$

and the mixed B.C.

$$(\hat{D}_i \partial_{\vec{r}} u_i + \vec{C}_i u_i)|_{\vec{r} \in \partial\Omega} = 0 \qquad i = 1, 2, 3. \tag{4.3}$$

Here, u_1 and u_3 are two force densities that represent either two distinct components of the same army or two allied armies. For this reason, in the matrix of the attrition coefficients we have $a_{31} = a_{13} = 0$. The diffusion tensor is diagonal $\hat{D}_{ij} = D\delta_{ij}$ and the magnitude of its nonzero elements has been chosen to be $D = 0.01$. The convection velocities $\vec{C}_i$ vary with t, their magnitudes taking values between 0 and 2.5. We need to point out that the choice of a local attrition term in Eq. (4.1) was made to simplify the numerical computation. This choice is not so restrictive as it may seem, if only a qualitative study of the problem is intended.

The knowledge of a typical time for the dynamical process associated to each combat situation is very important for an intuitive approach to our investigation. We considered this time to be the hypothetical time needed by the troops to travel across the battlefield, when they move with their average speed and no engagement happens. Taking the battlefield length as the unit for distances, the value of the typical time τ is numerically equal to the inverse of the average speed of the moving troops. Now we required from our code that the conservation of the total number of troops be satisfied within about 2% for intervals of time Δt that are of the same order of magnitude as τ. This requirement was met by imposing a perfect conservation at the boundaries of the battlefield. To this end we chose mixed B.C. which ensured the cancelation of normal components of the individual diffusion-convection currents $\vec{j}_i = \hat{D}_i \partial_{\vec{r}} u_i + \vec{C}_i u_i$ on $\partial\Omega$, the boundary of the domain Ω. Thus, in (2.4) we have $h_i = 0$, $\alpha_i = \vec{C}_i \cdot \vec{n}$, $\vec{\beta}_k = \hat{D}_i \vec{n}$, where $\vec{n}$ is the unit normal to $\partial\Omega$. The knowledge gained from the one-dimensional testing proved to be useful for this two-dimensional modeling.

For the purpose of the present study we decided that "quantitatively good results" mean boundary generated losses smaller than 2-2.5% in the total number

of troops for evolutions lasting times that are $\mathcal{O}(\tau)$. The conclusion of our testing is that "good quantitative results" can be obtained only if $NX \sim \mathcal{O}(100)$. When $NPDE = 2$ this means $NEQ \sim \mathcal{O}(10^4)$ which is too large a number of equations to allow a numerical treatment of the problem that could be termed as reasonable with regard to the computing resources used.

The parameter controlling the numerical stability and the accuracy of the solution for the simulations with the attrition turned off is the ratio $\rho = \frac{|\vec{C}|}{D}$. In our simulations, a good value for ρ was found to be 2.5×10^2. This value insured sufficient stability during evolution for a Gaussian shaped initial distribution of forces. Square type initial distributions have also been tried. In this case the combination between the small number of points per direction and the value of ρ given above proved to be unsatisfactory. During the evolution the square distribution of troops developed oscillating tails. This was unacceptable because the density of troops can never take negative values. For this reason we adopted the Gaussian as the initial distribution of forces in all subsequent runs.

The correct qualitative description of the combat maneuvers is visible in our modeling based on the Eq. (2.1), even when 32 points per direction are chosen. Although the losses due to numerical leaks through the boundary are larger in this case, parallel runs of the code with the attrition turned on and off can distinguish between the losses due to engagement and those due to numerical imprecision. For this rough grid and NPDE= 2 it follows that NEQ= 2048 in the case a two-forces combat, and NEQ= 3072 for the cases when three forces are involved. This last case is in fact a combat in which only two armies are engaged, with one of them made up of two distinguishable components.

The use of the published software for integrating the PDE system required the writing of a driver program which generates the specific equations to be integrated, the I.C. and the B.C. attached to the problem. Detailed description of the driver programs can be found in the papers presenting the integrator.[8,9] We point out that the essential subroutine of the driver which has built in it all the features allowing for the modeling of a specific combat situation is the subroutine defining the convection field. This subroutine returns upon its call all the values of the convection velocities $\vec{C}_i$ for any values of the independent variables t and x. It works in conjunction with the main program which analyzes the situation at any moment of time and triggers changes in the tactical approach of the situation based on the results of it analysis.

We tried different approaches for modeling the troop movements with the convection field $\vec{C}_i$. A point dependent convection field with little or no time dependence has shown that sensible losses through the boundary cannot be avoided. A simple uniform convection field with a strongly time-dependent direction proved to be the best choice; it ensured an acceptable conservation of the total number of troops when the attrition was turned off.

The boundary conditions have been imposed according to the instantaneous convection velocities which are entered as coefficients within the B.C. This approach

proved to be at the same time the simplest and the most efficient. It allows a reasonably simple version of the numerical code, and permits a good numerical conservation of the number of troops during the zero attrition runs.

An example of combat situation which was successfully modeled is the turning maneuver. This simulation required the integration of a simple version of the system (4.1) with only two coupled equations. The numerical simulation did clearly demonstrate the movement of troops on the field, the active phase of the battle when the direct contact is realized and the mutual attrition decreases the number of troops engaged in combat, and the retreat of the defeated force (not shown in Fig. 3). If the defeated force is the one that made the attack, this last phase of the battle takes place on the same track as the one used for engagement but in reversed direction. If the entrenched force is the defeated one its retreat is made on some new track conveniently chosen. The loss of the total number of troops of one combatant that triggered the retreat was arbitrarily set at 15%. The specific results obtained through numerical simulation of offensive combat maneuvers are well illustrated in a paper dedicated solely to this subject.[7]

There arises a natural question about the complexity of the combat situations the model could handle, and we need to address this issue here. From the theoretical point of view it is clear that the PDE model can cover any situation, whatever complex it may be, as long as the individuality of the elements participating in the combat can be ignored. This is when only spatial averages giving the field distribution of the combatants need to be considered, i.e., when only the "big picture" is of concern. From this follows that, in principle, one can consider as many components for combating armies as it is necessary to describe the tactical situation. However, from the numerical viewpoint, there are clear limitations on the complexity of the combat situations that can be handled. Complex situations can be modeled only if the spatial grid is made fine enough to account for the the details of the battlefield activity. A finer grid implies the increase of NEQ, and this generates an increase in the computational time needed to solve the problem. On the other hand, there are indications that the increase in NEQ, through the increase of NX and NY, brings in some stiffness to the ODE system attached to a specific scenario. The integration of a stiff system is a difficult task, and, while the situation can be still handled numerically, one finds out that the computational time has increased beyond reasonable limits.

4.2 COMPUTATIONAL COMPLEXITIES

In calculating force levels and, more significantly, in evaluating the attrition integrals, a very large number of integral computations of the form

$$u_i(x,y) \int_X \int_Y \phi_i[(\xi,\eta),\ (x,y)]u_j(\xi,\eta)\, d\xi\, d\eta$$

were required where $i,j \in \{1,2\}$, $i \neq j$. In determining force levels, ϕ_i and u_i were taken equal to one, while in calculating attrition integrals ϕ_i was normally taken to

be a function of the distance $[(\xi - x)^2 + (\eta - y)^2]^{1/2}$ between the nodes (ξ, η) and (x, y) in concert with the expected decrease in effectiveness of fire with increase in distance between troop units. Because of the large fraction of the total computing time which was consumed in evaluating such integrals, it was highly desirable to explore ways in which to minimize the computational times in evaluating them. Several approaches were used.

The integration was avoided altogether if in the case of attrition integrals the multiplier $u_i(x, y)$ was below a threshold value, on the basis that the product of the multiplier and the integral would be negligibly small. Secondly, the integral itself was evaluated applying Simpson's rule in two dimensions. But rather than using a double summation, the Simpson's rule evaluation was unrolled to produce a single summation, thereby significantly improving vectorization (and hence speed) on the CRAY computer because of a single long loop rather than two much shorter nested loops. Finally, rather than using every point on the original nodal grid in evaluating the integrals, every fourth point, for example, in each direction might be used; an approach which was particularly effective when a much finer grid was required to accurately solve the underlying PDE's than was needed to accurately approximate the attrition or force level integrals.

Since the number of ODEs to be solved is NEQ = NPDE * NX * NY, the size of the ODE system to be solved quadruples when the number of nodes in both the x and y direction is doubled, imposing a stiff price in computation time for greater resolution in the spatial dimensions. To alleviate the necessity for a fine mesh simply to allow accurate representation of initially steep profiles (as, for example, the triangular, rectangular, and spike initial troop distributions used in the one-dimensional investigation[1]), initial force density profiles were taken to be much less steep bivariate Gaussian distributions.

The limit on the coarseness of the grid, aside from ensuring reasonably accurate representation of the initial and developing force profiles, was set by the need to avoid numerical instabilities in the solution. These numerical instabilities, long recognized in convection-diffusion problems, manifest themselves most visibly as nonphysical oscillations in the solution and are accentuated when convection is strongly dominant in the model relative to diffusion. For the range of velocities and diffusion coefficients studied, it was found that on the order of 65 nodes in each direction were required to yield acceptably stable solutions, resulting in a total of $2 * 65 * 65 = 8450$ ODEs to be integrated.

Another effect of concern in performing the calculations was possible loss of conservation of the total forces when forces encountered the boundaries. Ref. 6 contains discussion of the theoretical boundary conditions which should be prescribed to ensure strict conservation (no net gain or loss at the boundaries), but notes the failure of the numerical algorithm to achieve perfect conservation, although errors decreased as a finer mesh was used. In practice, the force velocities and diffusion coefficients, the size of the "battlefield," and the battle scenario were

chosen to minimize interaction of the forces with the boundaries by effectively keeping the forces away from the boundaries as much as practicable without having to introduce additional grid points. Clearly, boundary effects warrant further study in order to avoid ambiguity between losses due to attrition and losses (or gains) due to numerical errors at the boundaries.

For the values of the PDE coefficients and mesh spacing studied, it was feasible to specify the nonstiff option (set via a flag passed to GEARB) in solving the ODE system. This was a decided advantage (versus having to specify the use of stiff methods in the GEARB code) because of the much greater overhead associated with stiff methods as a consequence of having to generate the Jacobian matrix and to solve nonlinear systems of equations at each time step. For the two-dimensional problems studied here, the number of ODEs almost precludes the use of stiff methods because of the computational times involved in forming and solving large systems of nonlinear equations at each time step and because of the tremendous memory requirements (approximately 3,300,000 additional words of storage needed for NX = NY = 65). To date, none of the test problems have evidenced the need for very small integrator time steps to maintain numerical stability and accuracy, which is the primary symptom arising when a stiff problem is tackled using nonstiff methods.

Although a definitive exposition of what is meant by stiffness is well beyond the scope of this document (see Ref. 11 for an excellent review of this subject), a practical measure of stiffness is indicated by the size of the time steps used by the integrator relative to the rate of change of the solution for a given accuracy, with the problem being more stiff the smaller the time step required. Very small time steps which are obviously inconsonant with the time scale of the problem are a clear indicator that a problem is stiff. The difficulty is that the size of the time steps is being governed by stability criteria rather than by accuracy requirements. Thus, for example, the stability criterion may demand that a certain ratio of the time step size and the spatial step sizes not exceed a fixed value. It can well be that the fineness of the mesh required to accurately solve the problem necessitates the use of such small time steps as to make a nonstiff solver impractical, thereby compelling the use of stiff solvers which employ methods which permit much longer time steps. Further, it is clear that method of lines formulations which are nonstiff for a coarse mesh may become stiff for a finer spatial mesh since as the grid is made finer, the time steps need to be taken smaller (often much smaller because the stability criterion is of the form $\Delta t/(\Delta x \cdot \Delta y) <$ constant) in order to maintain stability.

Mesh sizes were governed by the need to ensure numerical stability and acceptable accuracy without requiring inordinate computational time or computer memory. A relative error tolerance of 1×10^{-4} was specified for the time integration of the ODE system ensuring that the weighted single step error estimates were kept less than this value in the root-mean-square norm (the Euclidean norm of the

dependent variable vector divided by the square root of the number of variables). With this error criterion and with 33 to 65 nodes in each of the x and y directions, computational times were generally of the order of one to five minutes on a CRAY X-MP computer, with the times a function of the PDE coefficients and the specific nonzero terms included in the model PDE.

5. CONCLUSIONS

The numerical study of the PDE combat model has proved its value in dealing with some of the complex aspects of modern warfare. The results presented here clearly show that the tactical aspects of certain forms of maneuver are accurately described by the two-dimensional version of the model. The results obtained here are for very idealized engagement scenarios. Opposing force ratios, attrition rates, diffusion coefficients, velocities of moving forces, and boundary conditions were arbitrarily chosen and do not correspond to actual tactical conditions since the purpose of this work was only to demonstrate the capability of the model to replicate offensive combat maneuvers. This has been accomplished. The numerical results, however, must be treated cautiously because of the parameter values that were chosen, and since only homogeneous forces having two-dimensional Gaussian distributions were used in the description of the engagements. The point to note is that the PDE model does represent a significant departure from the Lanchester's ODE model and is the foundation for a more sophisticated approach to analytically modeling combat.

Further developments are in order. The PDE solver must be improved to minimize losses at the edges of the battlefield. The capability to represent more realistic spatial distributions of opposing forces must be implemented in conjunction with the representation of heterogeneous force structures. The limitation of a square battle area must be eliminated in favor of wide versus narrow geometries and vice versa where the depth of the battlefield would logically include close, deep, and rear operations separately or simultaneously. To achieve these requirements, a new two-dimensional PDE code called WAR specifically designed for combat modeling has been developed[12] that already provides the following features:

1. Flexibility in defining the input data, with an input module with on-line instructions that enables creating or modifying the input file interactively.
2. Dynamic allocation of the memory (using the container array strategy) provides greater flexibility in executing a wider variety of battles without modifying the code.
3. Interactive control of the model parameters (speed, diffusion coefficient, attrition rates, etc.) is permitted every one or more time steps, as determined by the user.
4. Interactive color graphics using the DISSPLA Library representing forces as contour curves, with a variable frequency of plotting as determined by the user.

Additional features that are currently being developed or installed in the code include:

1. Rewriting the equations in the PDE solver routine in conservative form to avoid "numerical dissipation."
2. Modifying the PDE solver routine to reduce the effect of numerical diffusion.
3. Modeling non-local interactions in an efficient way.
4. Installing a balance-table-routine to account for all sources/sinks of forces at each time step and at conclusion of the battle.
5. Refine interactive control to include redeployment of forces, undoing previous time step, etc.

Significant sensitivity and parametric studies are also necessary to determine ranges of parameter values to reproduce realistic or historical confrontations and lend more credibility to analysis of potential conflict situations. Stronger coupling of intelligence data to govern force movements and firepower requirements also remain.

However, the general outlook is good and conducive to optimism. The model is sound and even though the cases studied are idealized, meaningful data have ensued. The numerical results would provide guidance on force dispositions, firepower requirements, and tactical effectiveness. A better knowledge of the software that can be used generated a more realistic view on the expectations one can place from combat models. The problems encountered are of such a nature that, based on the past experience with numerical problems, we can be almost certain they will find a solution in a not too distant future.

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