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Parameter Identification for Generalized Lanchester's Equations

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Parameter Identification for Generalized Lanchester's Equations

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ABSTRACT

The validation of analytic combat models is hindered by the almost complete lack of a priori knowledge about the aggregated attrition coefficients serving as inputs in such models. To overcome this difficulty we discuss the applicability of two methods that use empirical data (combat outcomes) to identify the unknown attrition coefficients. Both methods are based on the minimization of an optimization criterion (equation error criterion or output error criterion).

1. INTRODUCTION

Lanchester's equations were proposed in 1914 as a first attempt to describe combat situations with an analytic model based on ordinary differential equations.[1] Various developments and refinements of the original model as well as an exhaustive list of references can be found in Taylor's monograph.[2] Recently, alternative analytic combat models, based on partial differential equations[3,4] and on discrete maps,[5–7] have been proposed. The primary role of these models is to provide an empirical basis to describe, assess, understand, and predict the development and outcome of a combat situation. In particular, the emphasis is put on: (i) reproducing attrition, aggregation, and maneuver, (ii) estimating the enemy's capabilities, (iii) structuring of the friendly forces to counter enemy's threat, (iv) assessing the effective technical performance of new weapons, and (v) measuring the effect of these factors on casualties, termination time, tactical decisions, etc.

In general, the adequacy and usefulness of a model can be determined only within a certain well-defined paradigm, as a consistent part of the epistemologic process. Usually, one has in mind here well formalized mathematical models. However, due to the enormous complexity of the questions arising in connection with military situations, the shear size of information, and the inherent presence of intricate and elusive ingredients such as political and economical considerations, weather aspects, human factors, etc., combat models contain a lot of intuitive/judgmental elements that are present in nonanalytic form in people's mind. The heart of the matter here is the lack and/or inapplicability of the usual consensual criteria* to gauge combat models. More disturbing are other factors that create unease in the combat modeling community:

- the ambiguous (fuzzy, non quantifiable, unpredictable) part played by judgment and, in general, by human factors;
- the lack of complete, relevant, reliable, and, after all, *usable* data, due to the fact that on one hand experiments (as processes taking place under controlled, repeatable circumstances) are impossible, and on the other hand the existing corpus of gathered information is very fragmented, inconsistently collected, and not pertinent for the new weapons to be used in future battles;
- the uncertainty regarding the basic assumptions on the individual and global processes taking place in the battlefield;
- the dichotomy the decision-maker faces, split between field operational realities on one hand, and—in the luckiest of cases—the results of studies based on analytical models, Monte-Carlo simulations, and war gaming, on the other hand.

Thus it is unreasonable to expect phenomena as complex and uncertain as battles and wars could be accurately described by a simplistic model like Lanchester's equations (or any other analytical model for that matter). These type of models can only aspire at producing a low resolution, highly aggregated picture of

* something along the lines: within a well received paradigm, does the model agree with the facts as determined from independent observations and from experiments

the events in which hints, trends, patterns, and perspectives are more relevant than actual numbers and rates and their use should be limited to education, training, and qualitative assessment for command decision-making.

Even within such simple models, the verification, validation, and accreditation questions are crucial and may remain unsolved when the data input is inadequate. Then faulty results may be due either to faulty data or to faulty structure (or to both) and it is very difficult to disentangle the two aspects and measure their relative magnitude and importance. Usually, validating a model comprises: (i) experimentally measuring data inputs under controlled repeated runs of the same process, (ii) statistically evaluating the output, and (iii) testing this output against independent experimental or observational evidence. If the model is validated, it can then be used for predictions within the same class of phenomena. In the case of combat modeling, this process is still very far from being completed in a satisfactory way.[8] The "input data" for a given combat model are initial forces (usually well known) and aggregated attrition rates (usually very poorly known). In estimating aggregated attrition rates, lethality, aiming accuracy, vulnerability, weapon reliability, weather conditions, operational degradation, human factors, etc., play an important role. Some of these variables are technical, others physical, others tactical. Some of them can be (partially) quantified by *ad hoc* or systematic sensitivity analysis, others have to be guessed, or judgmentally assigned. Moreover, a quantitative formula has to be designed (usually based on empirical considerations) to compute the aggregated attrition rates from these quantified factors.

Techniques to infer a model from measured data typically contains two steps: (i) the identification of a class of candidate models and (ii) the parameter estimation for the particular member of the class that fits the data best. In this paper, we shall be concerned with the second step for the generalized Lanchester's model. In Section 2, we review Lanchester's models and some of their generalizations. Section 3 is devoted to developing two methods of parameter identification for Lanchester's equations. The paper ends with a short section that summarizes our conclusions.

2. THE GENERALIZED LANCHESTER MODEL

Generalized Lanchester's equations read

$$\frac{dx_i}{dt} = -a_i x_i \; x_j - b_i \, x_j + c_i, \quad i, \, j = 1, \, 2, \, i \neq j \tag{1}$$

where $x_1(t)$, $x_2(t)$ denote the number of opposing troops at time t, $a_i > 0$ represent the area-fire attrition coefficients, $b_i > 0$ are the aimed-fire attrition coefficients, and $c_i > 0$ describe autonomous sources.

Eqs. (1) are to be supplemented with initial conditions

$$x_i(0) = x_i^0 > 0, \;\; i = 1, \, 2 \; . \tag{2}$$

In general, a_i, b_i, c_i are functions of time, but in the following we shall consider them constant. For $b_i = c_i = 0$, Eqs. (1) reduce to pure area-fire case, while for $a_i = c_i = 0$, Eqs. (1) describe the pure aimed-fire situation.[1,2]

By combining the two forms into one equation, one could treat cases when both types of interactions are present in combat. However, in the 1960's, it was felt in the combat modeling community that only area-fire and aimed-fire combat could not cover all the situations encountered on a battlefield.[9,10] The generic situation was perceived to be more a weighted combination of the two. Accordingly, Helmbold[9,10] suggested to describe the combat law in the form

$$\frac{dx_i}{dt} = -a_i \, x_i^{\alpha_i} \;\; x_j^{\beta_i} \;\; i, \, j = 1, \, 2, \, i \neq j \; , \tag{3}$$

where the attrition coefficients a_i and the exponents α_i, $\beta_i \in [0, 1]$, $i = 1, \, 2$ are to be determined in the process of validation. In principal, determining the exponents α_i, β_i is a judgmental operation, while the aggregated attrition coefficients a_i, b_i should be an input of the problem. Unfortunately, all these inputs can be quantified only within very approximate limits. Moreover, for past battles, most of which are extremely poorly documented, what is actually recorded is the output of the problem, i.e., number of troops and thus we face the problem of determining the attrition coefficients from the measured outcomes of the battle.

Thus selecting the most suitable model is of utmost importance in any identification problem. The choice is application-dependent, but sometimes may be influenced by extraneous factors such as the availability of particular optimization programs. Other factors that may influence the choice of the model are flexibility, formal simplicity, algorithm complexity, and properties of the criterion function.[11]

3. PARAMETER IDENTIFICATION FOR THE LANCHESTER'S EQUATIONS

Parameter identification belongs to a class of inverse problems which are, in general, ill-posed.[12] Namely, the solutions need not exist, be unique, nor depend continuously on the data. These mathematical aspects have very important consequences for the practitioner and military user. In order to reduce the non-uniqueness measure, additional information is often required to choose among several solutions.

In order to estimate the variable parameters a_i, b_i, c_i, α_i, and β_i, $i = 1, 2$ for the combat models (1) and (3) from the previous section, we consider two classical approaches which consist of minimizing either the *equation error criterion* or the *output error criterion*. Both methods have pros and cons which will be discussed along with the implementation. Since model (1) is linear in the parameters, we will directly use the associated discrete-time equation. For the treatment of model (3), which is non-linear in α_i, and β_i, we will derive another continuous-time equation (linear in all parameters) that will be better suited for discretization and further generalizations.

3.1 PARAMETER IDENTIFICATION FOR MODEL (1)

We consider the following discrete-time version of Eq. (1)

$$x_i(t+1) = x_i(t) - a_i x_i(t) - b_i x_j(t) + c_i, \quad i,j = 1,2, \; i \neq j, \; t \in \{0, 1, \ldots, T\} \quad . \tag{4}$$

If we are given two observed trajectories $\hat{x}_i(t)$, $i = 1, 2$, $t \in \{0, 1, ..., T\}$ for which the original parameters a_i, b_i and c_i are to be identified, we can use two approaches. The first one involves the minimization of the *equation error*:

$$\mathcal{E}^e(a_i, b_i, c_i) = \sum_{\substack{i,j=1, \\ i \neq j}}^{2} \sum_{t=0}^{T} [\hat{x}_i(t+1) - \hat{x}_i(t) + a_i \hat{x}_i(t) \hat{x} j(t) + b_i \hat{x}_j(t) - c_i]^2 \quad . \tag{5}$$

Since $\mathcal{E}^e$ is, in general, strictly convex (because it is quadratic in a_i, b_i and c_i), it has only one minimum that can be computed easily with a Newton method. The minimization of the equation error is analogous to a Least-Mean Square fit.

The other approach, more closely related to adaptive control methods, considers the *output error*

$$\mathcal{E}^o = \sum_{i,j=1, \; i \neq j}^{2} \sum_{t=0}^{T} [\hat{x}_i(t+1) - x_i(t+1)]^2 \tag{6}$$

where $x_i(t)$ is a trajectory generated by some parameters a_i, b_i and c_i. The reader can convince himself easily that $\mathcal{E}^o$ is convex *in the trajectory* $x_i(t)$ parameterized by (a_i, b_i, c_i) but not necessarily in the parameters themselves. One can thus expect $\mathcal{E}^o$ to have local minima. Its minimization also requires that at each step k of a

descent algorithm, we generate the trajectories $x_1^k(t)$ and $x_2^k(t)$ with the current values a_i^k, b_i^k and c_i^k of the parameters.

In the following, we describe the implementation of both methods, their application to various test problems, and we discuss their sensitivity with respect to measurement errors on $\hat{x}_i$. In order to assess the identification programs, we have generated two trajectories $\hat{x}_i$, $i = 1, 2$ with known parameters and $x_i(0) = 1$. Since we wanted to render the test significant from an application point of view (real-life difficulty of having enough reliable data), we limited ourselves to 12 data per trajectory ($T = 11$).

Since real empirical data are not available, we generate the data $\hat{x}_i(t)$, $i = 1, \ldots, T$ by using known values of the parameters $a_i, \ldots, c_i$. Then, by considering $\hat{x}_i(t)$ as given points, we use them to determine the "unknown" parameters $a_i, \ldots, c_i$. In Section 3.3, we perturb the model generated data by a multiplicative noise.

3.1.1 Equation Error Method

The Equation Error Method (EEM) was straightforward to implement on a Macintosh II with *Mathematica*^{TM}, S. Wolfram's mathematical package for personal computers.[13] The advantage of using such a symbolic manipulation package was the immediate availability of the Hessian of $\mathcal{E}^e$, $\mathcal{H}^e$, which enabled us to verify the strict convexity condition. We used the built-in Newton algorithm to find the minimum of $\mathcal{E}^e$. In Table 1, we give the results of two typical tests. In all cases, the criterion was strictly convex (see the eigenvalues in the table), the convergence took less than 15 iterations and the exact and unique solution was found.

	a_1	b_1	c_1	a_2	b_2	c_2
Exact values (1)	0.084	0.081	0.028	0.056	0.053	0.016
Initial values	0.00	0.00	0.00	0.00	0.00	0.00
Solutions obtained	0.084	0.081	0.028	0.056	0.053	0.016
Eigenvalues of $\mathcal{H}^e$	41.02	1.40	$7\ 10^{-4}$	36.01	0.012	1.916
Exact values (2)	0.040	0.032	0.010	0.15	0.062	0.03
Initial values	0.00	0.00	0.00	0.00	0.00	0.00
Solutions obtained	0.040	0.032	0.010	0.15	0.062	0.03
Eigenvalues of $\mathcal{H}^e$	32.62	$7\ 10^{-3}$	2.69	44.34	1.60	$9\ 10^{-4}$

Table 1. Performance of the Equation Error Method for Model (1)

Remark: Due to the large differences in the eigenvalues of $\mathcal{H}^e$ (typically from 10^{-4} to 50), one can see that the problem is badly conditioned and should not behave well either to a simple gradient descent or to perturbation of the data. This aspect will be emphasized in Section 3.3.

3.1.2 Output Error Method

The Output Error Method (OEM) was slightly more involved and we implemented it on a Vax 8600, using a subroutine E04FDF of the *FORTRAN NAG Library*.[14] This particular subroutine uses the additive structure (sum of squares of non-linear functions) of $\mathcal{E}^o$. It requires no derivatives (they are computed by finite-differences), and can be called directly in the main program. The user must provide a subroutine (called LSFUN1) that computes, for a given set of parameters, the vector of entries $[\hat{x}_i(t+1) - x_i(t+1)]^2$, $t = 0, T$, $i = 1, 2$. In order to enforce the positivity constraints, we have projected the iterates of the parameters onto $\mathbf{R}^6_+$. This projection does not change the original problem, but should affect somewhat the performance of the algorithm.

Table 2 illustrates the non-convexity of $\mathcal{E}^o$ (different "minima" for different starting points).

	a_1	b_1	c_1	a_2	b_2	c_2
Exact values (1)	0.084	0.081	0.028	0.056	0.053	0.016
Initial values (a)	0.00	0.00	0.00	0.00	0.00	0.00
Solutions obtained (a)	$1.5\ 10^{-4}$	0.233	0.089	0.172	0.00	0.030
Initial values (b)	0.00	0.01	0.01	0.01	0.02	0.02
Solutions obtained (b)	0.0840	0.0809	0.0279	0.0560	0.0529	0.0160
Exact values (2)	0.040	0.032	0.010	0.15	0.062	0.03
Initial values (a)	0.00	0.00	0.00	0.00	0.00	0.00
Solutions obtained (a)	0.0400	0.0319	0.0100	0.1499	0.0620	0.030
Initial values (b)	0.08	0.07	0.00	0.00	0.00	0.00
Solutions obtained (b)	0.149	0.00	0.0523	0.0156	0.5469	0.3825

Table 2. Performance of the Output Error Method for Model (1)

Remark: In both cases when the algorithm converged to a "wrong minimum", one of the positivity constraints (e.g. $b_i > 0$) was hit. Also, the subroutine E04FDF returned a warning flag indicating that the minimum obtained was not a "sure" minimum. Indeed, the corresponding values of $\mathcal{E}^o$ were small (less than 10^{-6})

but much larger than for the true minimum (about 10^{-20}). However, most of the starting points led to the true minimum.

3.2 PARAMETER IDENTIFICATION FOR MODEL (3)

3.2.1 Equation Error Method

If one wanted to apply the *equation error criterion* to the following discrete-time equation,

$$x_i(t+1) = x_i(t) - a_i x_i^{\alpha_i}(t)\, x_j^{\beta_i}(t)\; i,j = 1,2,\;\; i \neq j,\;\; t \in \{0,1,\ldots,\, T\}$$

one would get a highly non-linear expression in α_i and β_i. In order to obtain a quadratic criterion, we perform the following operations on the original continuous-time equation

$$\frac{dx_i}{dt} = -a_i\, x_i^{\alpha_i}\;\; x_j^{\beta_i} \qquad\qquad i,\, j = 1,\, 2,\;\; i \neq j, \tag{7}$$

We differentiate Eq. (7) with respect to time

$$\frac{d^2x_i}{dt^2} = -\alpha_i\, [a_i\, x_i^{\alpha_i} x_j^{\beta_i}]\, \frac{1}{x_i}\, \frac{dx_i}{dt} - \;\beta_i [a_i x_i^{\alpha_i}\; x_j^{\beta_i}]\, \frac{1}{x_j}\, \frac{dx_j}{dt}\;,$$

and replace the quantities in square brackets by the left hand sides of Eq. (7):

$$\frac{d^2x_i}{dt^2} = -\alpha_i\, \frac{1}{x_i}\, (\frac{dx_i}{dt})^2 - \;\beta_i\, \frac{1}{x_j}\, \frac{dx_i}{dt}\, \frac{dx_j}{dt}\;\;. \tag{8}$$

We now discretize Eq. (8)

$$\begin{aligned} x_i(t+2) - 2x_i(t+1) + x_i(t) = &-\alpha_i \frac{(x_i(t+1) - x_i(t))^2}{x_i(t)} \\ &- \beta_i \frac{(x_i(t+1) - x_i(t))(x_j(t+1) - x_j(t))}{x_j(t)} \end{aligned}$$

and rewrite the error criterion in the form

$$\begin{aligned} \mathcal{E}^{e0} = \sum_{i \neq j}^{2} \sum_{t=0}^{T} \Bigg[& x_i(t+2) - 2x_i(t+1) + x_i(t) + \alpha_i \frac{(x_i(t+1) - x_i(t))^2}{x_i(t)} \\ & + \beta_i\, \frac{(x_i(t+1) - x_i(t))(x_j(t+1) - x_j(t))}{x_j(t)} \Bigg]^2 \end{aligned}$$

which is quadratic in α_i and β_i.

Once α_i and β_i have been estimated, one can obtain the parameters a_i through the minimization of the direct equation error (again quadratic)

$$\mathcal{E}^{e1} = \sum_{i,j=1,\ i\neq j}^{2} \sum_{t=0}^{T} [x_i(t+1) - x_i(t) + a_i\, x_i^{\alpha_i}(t)\, x_j^{\beta_i}(t)]^2$$

where the parameters a_i only are optimization variables.

This procedure gives excellent estimations of a_i. We then re-use these values to refine the values obtained for α_i and β_i. To do this, we minimize the error criterion

$$\mathcal{E}^{e2} = \sum_{i,j=1,\ i\neq j}^{2} \sum_{t=0}^{T} (\log[\frac{x_i(t+1) - x_i(t)}{a_i}] - \alpha_i \log[x_i(t)] - \beta_i \log[x_j(t)])^2 \ .$$

Note that we could have used $\mathcal{E}^{e2}$ from the beginning instead of $\mathcal{E}^{e1}$, but we would have taken the risk of loosing the convexity in a_i.

The results obtained with this procedure are summarized in Table 3.

	a_1	α_1	β_1	a_2	α_2	β_2
Exact values (1)	0.05	0.7	0.5	0.08	0.3	0.9
Initial values		0.5	0.5		0.5	0.5
Solutions with $\mathcal{E}^{e0}$		0.679	0.508		0.285	0.903
Solutions with $\mathcal{E}^{e1}$	0.05			0.08		
Solutions with $\mathcal{E}^{e2}$		0.7	0.5		0.3	0.9
Exact values (2)	0.04	0.9	0.2	0.08	0.3	0.6
Initial values		0.5	0.5		0.5	0.5
Solutions with $\mathcal{E}^{e0}$		0.884	0.208		0.282	0.614
Solutions with $\mathcal{E}^{e1}$	0.04			0.08		
Solutions with $\mathcal{E}^{e2}$		0.9	0.2		0.3	0.6

Table 3. Performance of the Equation Error Method for Model (3)

The minimum value obtained for $\mathcal{E}^{e1}$ and $\mathcal{E}^{e2}$ were consistently about 10^{-8} and 10^{-19} respectively.

3.2.2 Output Error Method

The output error method was less accurate with the Helmbold equation, except for the a_i parameters. In general, we had to try several initial values before finding a true minimum. The results are summarized in Table 4.

	a_1	α_1	β_1	a_2	α_2	β_2
Exact values (1)	0.05	0.7	0.5	0.08	0.3	0.9
Initial values	0.03	0.03	0.03	0.03	0.03	0.03
Solutions	0.0499	0.7002	0.4998	0.0799	0.3001	0.8998
Exact values (2)	0.04	0.9	0.2	0.08	0.3	0.6
Initial values	0.02	0.5	0.1	0.05	0.1	0.2
Solutions	0.0399	0.9897	0.1682	0.0742	0.315	0.607

Table 4. Performance of the Output Error Method for Model (3)

3.3 PARAMETER IDENTIFICATION IN THE PRESENCE OF UNCERTAINTY

In this subsection, we assume that the trajectory $\hat{x}_i$ is "noisy", i.e. it is not known with very good precision, which is the case for almost every combat data. We model this imprecision by a uniform multiplicative noise

$$\hat{x}_i^a(t) = (1 + \sigma y)\hat{x}_i(t)$$

where $\hat{x}_i^a(t)$ is the data really available, σ is a range of uncertainty (e.g., $\sigma = 1\%,\ 5\%,\ etc.$) and y is a random variable with uniform law on $[-1, +1]$.

Both identification methods performed very poorly on both perturbed problems, even when the amount of noise was negligible (e.g., 0.01%), and even when the starting values were the exact parameters used to generate the trajectories $\hat{x}_i$. This phenomenon, as illustrated by Tables 5-8, does not question the optimization methods used here, but the inherent ill-posedness of the two models. Reducing the number of unknowns improves somewhat the situation. For instance, if in Model 1 we consider some of the parameters exactly known and try to recover only the remaining ones, we can increase the level of noise up to 0.1 and still get reasonable answers. (See Tables 9-12). This improvement is independent of the values assigned to the exactly known parameters.

	a_1	b_1	c_1	a_2	b_2	c_2
Exact values	0.084	0.081	0.028	0.056	0.053	0.016
Solutions with $\sigma = 0.0001$	0.0839	0.0810	0.0280	0.0561	0.0528	0.0159
Solutions with $\sigma = 0.001$	0.0874	0.0746	0.0249	0.0532	0.0560	0.0164
Solutions with $\sigma = 0.01$	0.0564	0.1299	0.0516	0.0421	0.0665	0.0179

Table 5. Performance of the Equation Error Method for Model (1) with Noise

	a_1	b_1	c_1	a_2	b_2	c_2
Exact values	0.084	0.081	0.028	0.056	0.053	0.016
Solutions with $\sigma = 0.0001$	0.0813	0.0862	0.0305	0.0603	0.0487	0.0154
Solutions with $\sigma = 0.001$	0.0497	0.1486	0.0618	0.1072	$1.25\ 10^{-9}$	0.0085
Solutions with $\sigma = 0.01$	0.1033	0.0282	$2.\ 10^{-9}$	0.0449	0.0499	0.0097

Table 6. Performance of the Output Error Method for Model (1) with Noise

	a_1	α_1	β_1	a_2	α_2	β_2
Exact values	0.05	0.7	0.5	0.08	0.3	0.9
Solutions with $\sigma = 0.0001$	0.0499	0.7269	0.4842	0.0799	0.2890	0.9063
Solutions with $\sigma = 0.001$	0.0500	0.7852	0.4524	0.0799	0.3719	0.8583
Solutions with $\sigma = 0.002$	0.0492	0.6190	0.5100	0.0800	0.5210	0.7731

Table 7. Performance of the Equation Error Method for Model (3) with Noise

	a_1	α_1	β_1	a_2	α_2	β_2
Exact values	0.04	0.9	0.2	0.08	0.3	0.6
Solutions with $\sigma = 0.0001$	0.0399	0.0452	0.5127	0.0964	2.1813	0.1666
Solutions with $\sigma = 0.001$	0.0398	1.2077	0.0959	0.0624	0.7055	10^{-5}
Solutions with $\sigma = 0.002$	0.0397	1.317	0.0589	0.0552	0.2424	10^{-5}

Table 8. Performance of the Output Error Method for Model (3) with Noise

	a_1	b_1	c_1	a_2	b_2	c_2
Exact values	0.054	0.00	0.00	0.045	0.00	0.00
Solutions with $\sigma = 0.001$	0.0539	0.00	0.00	0.0449	0.00	0.00
Solutions with $\sigma = 0.01$	0.0544	0.0006	0.00	0.0457	0.00	0.00
Solutions with $\sigma = 0.02$	0.0516	0.00	0.00	0.0448	0.00	0.00
Solutions with $\sigma = 0.05$	0.0569	0.00	0.00	0.0445	0.00	0.00
Solutions with $\sigma = 0.10$	0.0490	0.00	0.00	0.0559	0.00	0.00

Table 9. Performance of the Equation Error Method for Model (1) with Noise $b_1 = c_1 = b_2 = c_2 = 0$

	a_1	b_1	c_1	a_2	b_2	c_2
Exact values	0.00	0.081	0.00	0.00	0.053	0.00
Solutions with $\sigma = 0.001$	0.00	0.0809	0.00	0.00	0.0529	0.00
Solutions with $\sigma = 0.01$	0.00	0.0809	0.00	0.00	0.0519	0.00
Solutions with $\sigma = 0.02$	0.00	0.0804	0.00	0.00	0.0555	0.00
Solutions with $\sigma = 0.05$	0.00	0.0828	0.00	0.00	0.0557	0.00
Solutions with $\sigma = 0.10$	0.00	0.0949	0.00	0.00	0.0493	0.00

Table 10. Performance of the Equation Error Method for Model (1) with Noise $a_1 = a_2 = c_1 = c_2 = 0$

	a_1	b_1	c_1	a_2	b_2	c_2
Exact values	0.00	0.081	0.028	0.000	0.053	0.016
Solutions with $\sigma = 0.001$	0.00	0.0810	0.0280	0.00	0.0528	0.0159
Solutions with $\sigma = 0.01$	0.00	0.0917	0.0368	0.00	0.0622	0.0228
Solutions with $\sigma = 0.02$	0.00	0.0971	0.0409	0.00	0.0531	0.0167
Solutions with $\sigma = 0.05$	0.00	0.0651	0.0147	0.00	0.0329	0.0021

Table 11. Performance of the Equation Error Method for Model (1) with Noise $a_1 = a_2 = 0$

	a_1	b_1	c_1	a_2	b_2	c_2
Exact values	0.054	0.00	0.028	0.045	0.00	0.016
Solutions with $\sigma = 0.001$	0.0544	0.00	0.0283	0.0453	0.00	0.0162
Solutions with $\sigma = 0.01$	0.0563	0.00	0.0291	0.0485	0.00	0.0189
Solutions with $\sigma = 0.02$	0.0521	0.00	0.0261	0.0366	0.00	0.0095
Solutions with $\sigma = 0.05$	0.0273	0.00	0.0074	0.0412	0.00	0.0143

Table 12. Performance of the Equation Error Method for Model (1) with Noise $b_1 = b_2 = 0$

4. DISCUSSION

Data acquisition, processing, and interpreting is one of the most difficult and important parts of combat model validation. Since we are able to (approximately) quantify only a limited number of factors, while all the others, in particular those related to human factors, lay beyond the reach of quantification, it is legitimate to question the validity of combat models that could predict the course of the events with any reasonable likelihood. Moreover, it is legitimate to ask what would be the value of such a model for future battles with new weapons whose performances can only be guessed at.

Real tests for the analytic models are difficult to come by: historical data are scarce, operational testing is very costly, while simulations and wargames provide a comparison, but not with the real world. Comparisons between analytic models (e.g., Lanchester vs. partial differential equations) are even less convincing.

In the absence of real data, model generated data have been used for estimating the parameters of a generalized Lanchester model supposed to describe highly aggregated mid-intensity classical combat. Indeed, parameter identification is a necessary step towards actual validation, and thus, it is of considerable conceptual and practical interest. The results can be viewed as a consistency test for the model.

In this paper, we have shown that the model has a certain consistency and a reasonable predictive capability. The conclusions are the following:

— When data are model generated, results are good up to six parameters; more parameters produce instabilities.

— When noise is added, results worsen dramatically; this is due to the fact that the convexity properties of the error functional vary wildly according to different directions. Slight perturbations may alter convexity in some directions and/or introduce local minima, or flat plateaus.

— Results are worse when the number of parameters to be determined is larger and the number of given points smaller. When noise is absent, we noticed a certain insensitivity of the results to these factors.

— The chronic lack of precise combat data clearly does not make the task easier. The predictive power of such models – even if otherwise accurate – seems questionable.

5. ACKNOWLEDGMENTS

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