A New Analytical Solution to the Multigroup Diffusion Equation
in One Dimensional Plane Geometry

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+ One-group approximation
+ Multigroup diffusion equation
+ Second order ODEs
+ Homogeneous/particular solution
+ Eigenvectors/eigenvalues
+ Plane/curvilinear geometry
+ Transient solution
ABSTRACT
An analytical solution to the time-independent multigroup diffusion equation in heterogeneous plane media is presented. The solution features the simplicity of the one-group case which is rather remarkable given the generality of the multigroup diffusion equation considered. Beginning with the vector multigroup diffusion equation, the solution is based upon straightforward application of the mathematical principles associated with solving a set of second order ODEs. Once the homogeneous solution is known, the particular solution can be derived by the method of variation of parameters. In this way, the solution is formed in a recursive setting for which, with some effort, an explicit analytical expression can be derived. As a result, a new criticality relation emerges to yield the criticality constant given the slab thicknesses. Based on this analytical solution, it now becomes possible to construct an efficient multidimensional nodal method as well as a 1D transient solution using a numerical Laplace transform inversion. More importantly, however, is the educational principle that this new solution represents, i.e., Even though a problem has been solved in the past, new more efficient solutions should always be sought.
Motivation:

One-group diffusion theory: Homogeneous medium

\[
\begin{align*}
\left[ D_j \frac{d^2}{dx^2} - \Sigma_{aj} \right] \phi_j(x) + v \Sigma_f \phi_j(x) &= -Q_j(x) \\
\phi_{j-1} &= \phi(x_{j-1}) \\
\phi_j &= \phi(x_j) \\
\Rightarrow \left[ \frac{d^2}{dx^2} + B_j^2 \right] \phi_j(x) &= -q_j(x)
\end{align*}
\]
- Standard Solution:

\[ \phi_j(x) = a_j e^{-B_j x} + b_j e^{B_j x} + \phi_{pj}(x) \]

\[ \phi_{pj}^- \equiv \phi_p(x_{j-1}), \quad \phi_{pj}^+ \equiv \phi_p(x_j) \]

\[ a_j e^{-B_j x_{j-1}} + b_j e^{B_j x_{j-1}} = \phi_{j-1} - \phi_{pj}^- \]
\[ a_j e^{-B_j x_j} + b_j e^{B_j x_j} = \phi_j - \phi_{pj}^+ \]

\[
\begin{bmatrix}
    a_j \\
    b_j
\end{bmatrix}
= E_j^{-1}
\begin{bmatrix}
    \phi_{j-1} - \phi_{pj}^- \\
    \phi_j - \phi_{pj}^+
\end{bmatrix}
\]

\[
E_j \equiv
\begin{bmatrix}
    e^{-B_j x_{j-1}} & e^{B_j x_{j-1}} \\
    e^{-B_j x_j} & e^{B_j x_j}
\end{bmatrix}
\]
- Non-standard Solution:

\[
\phi_j(x) = h_j^+(x)(\phi_j - \phi_{pj}^+) + h_j^-(x)(\phi_{j-1} - \phi_{pj}^-) + \phi_{pj}(x)
\]

\[
h_j^+(x) = \left[ \frac{\sin\left(\frac{B_j(x - x_{j-1})}{B_j\Delta_j}\right)}{\sin\left(\frac{B_j\Delta_j}{B_j\Delta_j}\right)} \right]
\]

\[
h_j^-(x) = \left[ \frac{\sin\left(\frac{B_j(x_j - x)}{B_j\Delta_j}\right)}{\sin\left(\frac{B_j\Delta_j}{B_j\Delta_j}\right)} \right]
\]

Basis Functions
- Criticality: First Require $\phi_{pj}(x) \equiv 0$

Choose $x = x^*$, then

$$
\phi_j(x^*) = h_j^+(x^*)\phi_j + h_j^-(x^*)\phi_{j-1}
$$

$$
\phi_j(x^*) = \sin\left(B_j(x^* - x_{j-1})\right)\left[\frac{\phi_j}{\sin(B_j\Delta_j)}\right] + \sin\left(B_j(x_j - x^*)\right)\left[\frac{\phi_{j-1}}{\sin(B_j\Delta_j)}\right]
$$

Also must require: $\phi_{j-1} = \phi_j = 0$

Implies: $\phi_j(x^*) = 0$? Not physically possible for criticality

Implies instead: $\sin(B_j\Delta_j) = 0 \implies B_j(k_{eff}) = \frac{\pi}{\Delta_j}$

Implies: Denominator must vanish also!!
+ One-group diffusion theory: Heterogeneous medium

- Standard Solution:

\[ \phi_j(x) = a_j e^{-B_j^x} + b_j e^{B_j^x} + \phi_{pj}(x) \]

\[ -D_{j-1} \frac{d\phi_{j-1}(x)}{dx} \bigg|_{x_{j-1}} = -D_j \frac{d\phi_j(x)}{dx} \bigg|_{x_j} \quad 2 \leq j \leq n-1 \]

\[ \Rightarrow \left\{ D_{j-1} a_{j-1} e^{-B_j^x_{j-1}} + D_{j-1} b_{j-1} e^{B_j^x_{j-1}} - D_j a_j e^{-B_j^x_j} + D_j b_j e^{B_j^x_j} = 0; \ j = 1, n \right\} \quad \text{2n unknowns} \]

- Non-standard Solution:

3-term recurrence relation with n-2 unknowns

\[ \alpha_j \phi_{j+1} + \beta_j \phi_j + \gamma_j \phi_{j-1} = f_j \]
+ Two-group diffusion theory: Core

\[
\begin{align*}
\left[ D_1 \frac{d^2}{dx^2} - \Sigma_1 \right] \phi_1(x) + \chi_1 \Sigma f_1 \phi_1(x) + \chi_2 \Sigma f_2 \phi_2(x) &= 0 \\
\left[ D_2 \frac{d^2}{dx^2} - \Sigma_2 \right] \phi_2(x) + p \Sigma_{1,2} \phi_1(x) &= 0 \\
\left[ \frac{d^2}{dx^2} + B^2 \right] \Psi_k(x) &= 0, \quad k = 1, 2 \\
\Rightarrow B^2 &= \begin{bmatrix} \mu^2 \\ -\nu^2 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\phi_1(x) &= AX + CY \\
X &= \sin(\mu x) \\
\phi_2(x) &= S_1 AX + S_2 C \\
Y &= \sinh(\nu x)
\end{align*}
\]

- Apply BCs for critical condition

+ Two-group diffusion theory: Core/Reflector

- Interfacial conditions/BCs give 4 simultaneous equations for the critical condition
- Complexity greatly increases with number of regions and groups
The Challenge:

+ Is there a more concise, more theoretically friendly analytical approach to heterogeneous MG diffusion theory?

Consequences:

+ Theory applies equally regardless of the number of regions and groups
+ Educational enrichment: New solution to an old problem
+ Nodal method application (2D/3D)
+ Transient applications
+ Curvilinear geometries
+ $n$-$\gamma$ application
I. **Theory:**
   1. Fundamental Assumptions:
      a. Heterogeneous medium
      b. Fission neutrons to all groups
      c. Fission occurs in all groups
      d. Up/Down scatter to all groups (full stride)
      e. Steady state spatially dependent source

\[ x_0 \quad x_1 \quad x_2 \quad x_{j-1} \quad x_j \quad x_{n-1} \quad x_n \]

Heterogeneous Medium
2. Governing Steady State Diffusion equation for homogeneous region $j$

$$
\begin{bmatrix}
D_{g} \frac{d^2}{dx^2} - \Sigma_{g} \\
\end{bmatrix} \phi_{gj}(x) + \chi_{g} \sum_{g' = 1}^{G} \nu \Sigma_{fg} \phi_{g'}(x) + \\
\sum_{g' = 1}^{G} \Sigma_{g'g} \phi_{g'}(x) = -Q_{gj}(x) \\
\end{bmatrix}
1 \leq j \leq n - 1, \quad 1 \leq g \leq G
$$

a. In vector form for region $j$: \( M_{jG}(x) \phi_{j}(x) = -q_{j}(x) \)

\[
M_{jG}(x) = \begin{bmatrix}
\frac{d^2}{dx^2} + \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1G} \\
\gamma_{21} & \frac{d^2}{dx^2} + \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2G} \\
\gamma_{31} & \gamma_{32} & \frac{d^2}{dx^2} + \gamma_{33} & \cdots & \gamma_{3G} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{G1} & \gamma_{G2} & \cdots & \cdots & \frac{d^2}{dx^2} + \gamma_{GG}
\end{bmatrix}
\]

\[
\phi_{j}(x) = \begin{bmatrix}
\phi_{1}\ (x) \\
\phi_{2}\ (x) \\
\phi_{3}\ (x) \\
\vdots \\
\phi_{G}\ (x)
\end{bmatrix}, \quad q_{j}(x) = \begin{bmatrix}
Q_{1}\ (x)/D_{1} \\
Q_{2}\ (x)/D_{2} \\
Q_{3}\ (x)/D_{3} \\
\vdots \\
Q_{G}\ (x)/D_{G}
\end{bmatrix}
\]

+ For $g = 1, 2, \ldots, G$, $g' = 1, 2, \ldots, G$
The $g$-matrix is defined in each region as (regional subscript is suppressed)

$$
\gamma_{gg} = \frac{\chi_{g} \nu \Sigma_{fg} - (\Sigma_{g} - \Sigma_{gg})}{D_{g}} \quad \gamma_{gg'} = \frac{\chi_{g} \nu \Sigma_{fg'} + \Sigma_{gg'}}{D_{g}}, \quad g \neq g'$$
3. General solution

\[ \phi_j(x) = \Psi_j(x) + \phi_{p,j}(x) \]

\[ \Psi_j(x) = \text{Homogeneous solution} \]

\[ \phi_{p,j}(x) = \text{Particular solution} \]

Straightforward Solution Strategy:

(A) Solve the following without regard to BCs

\[ M_{j,G}(x) \Psi_j(x) = 0 \]

\[ M_{j,G}(x) \phi_{p,j}(x) = -q_j(x) \]

(B) Apply BCs to general solution above
4. First consider the homogeneous solution for \( M_{j,G}(x) \Psi_j(x) = 0 \)

   a. Seek solution in terms of eigenvalues \( B_j^2 \)

   \[
   \left[ \nabla^2 + B_j^2 I \right] \Psi_j(x) = 0
   \]

   \[
   \nabla^2 \equiv \begin{bmatrix}
   \frac{d^2}{dx^2} & 0 & \ldots & \ldots & 0 \\
   0 & \frac{d^2}{dx^2} & 0 & \ldots & 0 \\
   \ldots & \ldots & \ldots & \ldots & \ldots \\
   0 & \ldots & \ldots & 0 & \frac{d^2}{dx^2}
   \end{bmatrix} = \frac{d^2}{dx^2} I
   \]

   b. Original operator becomes

   \[
   M_{j,G}(x) = \frac{d^2}{dx^2} I + \gamma_j
   \]

   \[
   \gamma_j \equiv \begin{bmatrix}
   \gamma_{1,1} & \gamma_{1,2} & \ldots & \gamma_{1,G} \\
   \gamma_{2,1} & \ldots & \ldots & \ldots \\
   \ldots & \ldots & \ldots & \ldots \\
   \gamma_{G,1} & \ldots & \gamma_{G,G}
   \end{bmatrix}
   \]

   \[
   M_{j,G}(B_j^2) \Psi_j(x) = 0
   \]

   \[
   M_{j,G}(B_j^2) \equiv \begin{bmatrix}
   \gamma_{1,1} - B_j^2 & \gamma_{1,2} & \ldots & \gamma_{1,G} \\
   \gamma_{2,1} & \ldots & \ldots & \ldots \\
   \ldots & \ldots & \ldots & \ldots \\
   \gamma_{G,1} & \ldots & \gamma_{G,G} - B_j^2
   \end{bmatrix}
   \]
\[ \therefore \text{Det} \left[ M_{j,G} \left( B_j^2 \right) \right] = 0 \implies B_{jk}^2, \ k = 1,2,\ldots G \]

+ Note: \( B_{jk}^2 \) are assumed to be distinct but may be complex in conjugate pairs

c. For each \textit{k-mode} therefore \( \left[ \frac{d^2}{dx^2} + B_{jk}^2 \right] \Psi_{j,k}(x) = 0 \)

+ Each (group) component will have two independent solutions

\[ \Psi_{jgk}(x) = C_{jgk}^+ h^+_{jk}(x) + C_{jgk}^- h^-_{jk}(x) \]

\[ \left[ \frac{d^2}{dx^2} + B_{jk}^2 \right] h_{jk}^\pm(x) = 0 \]

Note: Since \( B_{jk}^2 \) can be complex, the coefficients \( C_{jgk}^\pm \) can also be complex
d. General solution representation for group \( g \) is a sum of all possible solution modes

\[
\Psi_{gj}(x) = \sum_{k=1}^{G} \left[ C_{jk}^+ h_{jk}^+(x) + C_{jk}^- h_{jk}^-(x) \right]
\]

\( \Psi_{jk}(x) \) must be real

+ To solve:

\[
\left[ \frac{d^2}{dx^2} + B_{jk}^2 \right] h_{jk}^\pm (x) = 0
\]

subject to the convenient conditions

\[
\begin{align*}
 h_{jk}^+(x_j) &\equiv 1 & h_{jk}^-(x_j) &\equiv 0 \\
 h_{jk}^+(x_{j-1}) &\equiv 0 & h_{jk}^-(x_{j-1}) &\equiv 1
\end{align*}
\]

Considering a 2D set of basis solutions with above conditions

\[
\begin{aligned}
 h_{jk}^+(x) &= \frac{\sin(B_{jk}(x-x_{j-1}))}{\sin(B_{jk}\Delta_j)} \\
 h_{jk}^-(x) &= \frac{\sin(B_{jk}(x_j-x))}{\sin(B_{jk}\Delta_j)}
\end{aligned}
\]

Note: The argument of the sine can be complex
+ Representation of $C_{jgk}^\pm$

From original equations by group written as

$$\frac{d^2 \Psi_{gj}(x)}{dx^2} + \sum_{g'=1}^{G} \gamma_{gg'} \Psi_{gj'}(x) = 0$$

and the general solution by group-- implies

$$B^2_{jk} C_{jgk}^\pm - \sum_{g'=1}^{G} \gamma_{gg'} C_{jg'k}^\pm = 0, \quad k = 1, 2, ... G$$

Set of homogeneous equations of rank $G-1$ for each $k$. Thus, there is a one-parameter family of solutions that can be expressed in terms of an arbitrary constant. We choose that constant such that

$$C_{jgk}^\pm = \alpha_{gk} C_{j1k}^\pm, \quad g = 2, 3, ... G$$

For consistency: $\alpha_{1k} \equiv 1, \quad k = 1, 2, ... G$

$$\Rightarrow \sum_{g'=2}^{G} \left[ B^2_{jk} \delta_{gg'} - \gamma_{gg'} \right] \alpha_{g'k} = \gamma_{g1}, \quad g = 2, 3, ... G, \quad k = 1, 2, ... G$$
e. Intermediate expression for homogeneous solution:

\[
\Psi_{gj}(x) = \sum_{k=1}^{G} \left[ \alpha_{gk} h_{jk}^+(x) C_{j1k}^+ + \alpha_{gk} h_{jk}^-(x) C_{j1k}^- \right]
\]

or in vector form

\[
\Psi_j(x) = \alpha_j h_j^+(x) C_{j1}^+ + \alpha_j h_j^-(x) C_{j1}^-
\]

\[
\alpha_j = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \alpha_{2,1} & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{G,1} & \ldots & \ldots & \alpha_{G,G} \end{bmatrix} \quad h_j^+(x) = \begin{bmatrix} h_{j1}^+(x) & 0 & \ldots & 0 \\ 0 & h_{j2}^+(x) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & h_{jG}^+(x) \end{bmatrix} \quad C_{j1}^\pm = \begin{bmatrix} C_{j11}^\pm \\ C_{j12}^\pm \\ \ldots \\ C_{j1G}^\pm \end{bmatrix}
\]
f. Determination of $C_{j1}^\pm$

+ The general solution is of the form

$$\phi_j(x) = \alpha_j h_j^+(x) C_{j1}^+ + \alpha_j h_j^-(x) C_{j1}^- + \phi_{p,j}(x)$$

+ If we define $\phi_j \equiv \phi_j(x_j)$

$$\phi_{j-1} = \phi_{j-1}(x_{j-1}) = \phi_j(x_{j-1})$$  (from flux continuity)

then at $x = x_j$ and $x = x_{j-1}$

$$\phi_j(x_j) - \phi_{p,j}(x_j) = \alpha_j h_j^+(x_j) C_{j1}^+ + \alpha_j h_j^-(x_j) C_{j1}^-$$

$$\phi_j(x_{j-1}) - \phi_{p,j}(x_{j-1}) = \alpha_j h_j^+(x_{j-1}) C_{j1}^+ + \alpha_j h_j^-(x_{j-1}) C_{j1}^-$$

giving

$$C_{j1}^+ = \alpha_j^{-1} [\phi_j - \phi_{p,j}^+]$$  \hspace{1cm} \phi_{p,j}^+ \equiv \phi_{p,j}(x_j)$$

$$C_{j1}^- = \alpha_j^{-1} [\phi_{j-1} - \phi_{p,j}^-]$$  \hspace{1cm} \phi_{p,j}^- \equiv \phi_{p,j}(x_{j-1})$$
+ The final form of the solution then becomes

\[
\phi_j(x) = \left[ \alpha_j h_j^+(x) \alpha_j^{-1} \right] \left( \phi_j - \phi_{p,j}^+ \right) + \left[ \alpha_j h_j^-(x) \alpha_j^{-1} \right] \left( \phi_{j-1} - \phi_{p,j}^- \right) + \phi_{p,j}(x)
\]

- Now same form as the one-group case
- For convenience let

\[
A_j(x) \equiv \alpha_j h_j^+(x) \alpha_j^{-1} \quad \text{Need to be real}
\]
\[
B_j(x) \equiv \alpha_j h_j^-(x) \alpha_j^{-1}
\]

\[
\phi_j(x) = A_j(x) \left( \phi_j - \phi_{p,j}^+ \right) + B_j(x) \left( \phi_{j-1} - \phi_{p,j}^- \right) + \phi_{p,j}(x)
\]

Note: \( \phi_j \) are not known
5. Current continuity and recurrence

a. Current continuity at interfaces requires

\[-D_{j-1} \left. \frac{d\phi_{j-1}(x)}{dx} \right|_{x_{j-1}} = -D_{j} \left. \frac{d\phi_{j}(x)}{dx} \right|_{x_{j}}, \quad 2 \leq j \leq n\]

gives

\[M_{j} \phi_{j} - N_{j} \phi_{j-1} - P_{j} \phi_{j-2} = f_{j}, \quad 2 \leq j \leq n\]

\[M_{j} \equiv D_{j} \left. \frac{dA_{j}(x)}{dx} \right|_{x_{j-1}}\]

\[N_{j} \equiv D_{j-1} \left. \frac{dA_{j-1}(x)}{dx} \right|_{x_{j-1}} - D_{j} \left. \frac{dB_{j}(x)}{dx} \right|_{x_{j-1}}\]

\[P_{j} \equiv D_{j-1} \left. \frac{dB_{j}(x)}{dx} \right|_{x_{j-1}}\]

\[f_{j} \equiv D_{j} \left[ \left. \frac{dA_{j}(x)}{dx} \right|_{x_{j-1}} \phi_{p,j}^+ + \left. \frac{dB_{j}(x)}{dx} \right|_{x_{j-1}} \phi_{p,j}^- - \left. \frac{d\phi_{p,i}(x)}{dx} \right|_{x_{j-1}} \right] - \]

\[-D_{j-1} \left[ \left. \frac{dA_{j-1}(x)}{dx} \right|_{x_{j-1}} \phi_{p,j-1}^+ + \left. \frac{dB_{j-1}(x)}{dx} \right|_{x_{j-1}} \phi_{p,j-1}^- - \left. \frac{d\phi_{p,j-1}(x)}{dx} \right|_{x_{j-1}} \right].\]
6. Free surface boundary conditions: Closure
   a. Zero flux
      \[ \phi_0 = \phi_n = 0 \]
   b. Zero current:
      \[ \left. \frac{d\phi_0}{dx} \right|_{x_0} = \left. \frac{d\phi_i}{dx} \right|_{x_n} = 0 \]
      + Can reformulate conditions on \( \phi_0 \) and \( \phi_i \) such that
        \( \phi_0 = \phi_i = 0 \)
   c. To solve:
      \[ M_j \phi_j - N_j \phi_{j-1} - P_j \phi_{j-2} = f_j, \quad 2 \leq j \leq n \]
      with \( \phi_0 = \phi_i = 0 \)
7. Solution to recurrence relation (primarily for criticality)

\[ M_j \phi_j - N_j \phi_{j-1} - P_j \phi_{j-2} = f_j, \quad 2 \leq j \leq n \]

a. Assume form of solution is (for general condition at \( j = 0 \))

\[ \phi_j = g_j \phi_0 + \rho_j \phi_1 + \sum_{l=2}^{j} \beta_{j,l} f_l \]

where the complementary solutions are

\[ M_j \begin{bmatrix} g_j \n \rho_j \end{bmatrix} - N_j \begin{bmatrix} g_{j-1} \n \rho_{j-1} \end{bmatrix} - P_j \begin{bmatrix} g_{j-2} \n \rho_{j-2} \end{bmatrix} = 0 \]

\[ \begin{bmatrix} g_0 \\
\rho_0 \end{bmatrix} = \begin{bmatrix} I \\
0 \end{bmatrix}, \quad \begin{bmatrix} g_1 \\
\rho_1 \end{bmatrix} = \begin{bmatrix} 0 \\
I \end{bmatrix} \]

b. Satisfaction of BC at \( x = x_n \)

\[ \phi = -\rho_n^{-1} \left[ g_n \phi_0 + \sum_{l=2}^{n} \beta_{n,l} f_l \right] \]

\[ \phi_j = \left[ g_j - \rho_n^{-1} \rho_j g_n \right] \phi_0 - \rho_j \rho_n^{-1} \sum_{l=2}^{n} \beta_{j,l} f_l + \sum_{l=2}^{j} \beta_{j,l} f_l \]
c. Criticality:
  + Obvious $n$ by $G$ determinant condition

\[
\begin{vmatrix}
-N_1 & M_1 & 0 & \ldots & 0 \\
0 & -P_2 & -N_2 & M_2 & \ldots \\
\ldots & 0 & \ldots & \ldots & \ldots \\
\ldots & \ldots & 0 & \ldots \\
0 & \ldots & 0 & -P_n & -N_n
\end{vmatrix} = 0
\]

+ New criticality condition:
  Require $f_i$ and $\phi_0$ to vanish
  Require any interface flux vector to be nonzero
  - In particular for $j = 1$
    \[
    \rho_n \phi = -\left[ g_n \phi_0 + \sum_{l=2}^{n} \beta_{j,l} \phi \right] = 0
    \]
    - Implies $\rho_n$ must be singular $\Rightarrow \text{Det} \left[ \rho_n \left( k_{\text{eff}} \right) \right] = 0$
      (like the denominator for the one-group case )
8. Explicit solution to recurrence relation

+ Consider: $M_j\rho_j - N_j\rho_{j-1} - P_j\rho_{j-2} = 0$

$\rho_{j-1} - \rho_{j-1} = 0$

+ Reformulate with $y_j \equiv \begin{bmatrix} \rho_j \\ \rho_{j-1} \end{bmatrix}$

$\Rightarrow A_j y_j - B_j y_{j-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A_j = \begin{bmatrix} M_j & 0 \\ 0 & I \end{bmatrix}$

$B_j = \begin{bmatrix} N_j & P_j \\ I & 0 \end{bmatrix}$

$\rho_0 = 0 \quad \rho_1 = I \quad \Rightarrow y_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}$

$y_j - A_j^{-1}B_j y_{j-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$G_j = A_j^{-1}B_j$

$y_j = \begin{bmatrix} \rho_j \\ \rho_{j-1} \end{bmatrix} = \prod_{l=2}^{j} G_l 

y_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}$

+ Criticality: $\text{Det}[\rho_n] = \text{Det}\left\{\begin{bmatrix} I & 0 \\ \prod_{l=2}^{n} G_l & I \end{bmatrix}\right\}$
9. The particular solution

\[
\left[ \nabla^2 + \gamma_j \right] \phi_{p,j}(x) = -q_j(x)
\]

+ Apply variation of parameters

\[
\phi_{p,j}(x) = \alpha_j h_j^+(x) u_1(x) + \alpha_j h_j^-(x) u_2(x)
\]

to find

\[
\phi_{p,j}(x) = \alpha_j W^{-1} h_j^+(x) \int_x^{x_j} dx h_j^- (x) \alpha_j^{-1} q_j (x) +
\]

\[
+ \alpha_j W^{-1} h_j^-(x) \int_x^{x_{j-1}} dx h_j^+ (x) \alpha_j^{-1} q_j (x)
\]

\[
W \equiv \text{diag} \left( \begin{array}{c} 
\frac{B_{jk}}{\sin(B_{jk} \Lambda_j)} \end{array} \right)
\]
II Some Results

a. ADS steady State for 4-group xsecs of 10 September 2003 for uniform source in first region
- By the \( n \) by \( G \) determinant

<table>
<thead>
<tr>
<th>Mode</th>
<th>( k_{\text{eff}} )</th>
</tr>
</thead>
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<td>9.8825122560660 E-01</td>
</tr>
<tr>
<td>Quad Precision</td>
<td>9.8825122560341 E-01</td>
</tr>
<tr>
<td>Mathematica</td>
<td>9.88251225603411E-01</td>
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</tbody>
</table>

- By the \( G \) by \( G \) determinant but also a recurrence for \( \rho_n \)

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</tr>
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</tr>
<tr>
<td>Quad Precision</td>
<td>9.8825122560341 E-01</td>
</tr>
<tr>
<td>Mathematica</td>
<td>9.88251225603411E-01</td>
</tr>
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Comparison of Benchmarks
(2-D Stationary)

Relative Error

Group 1
Group 2

X
0 20 40 60 80 100 120

Y
1e-16 1e-15 1e-14 1e-13 1e-12
III Some Discussion and Comments
  + Curvilinear
  + Basis for 2D/3D Nodal algorithm
  + Transients via LTI
  + Other?