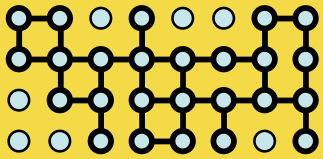
Cluster states and fusion gates

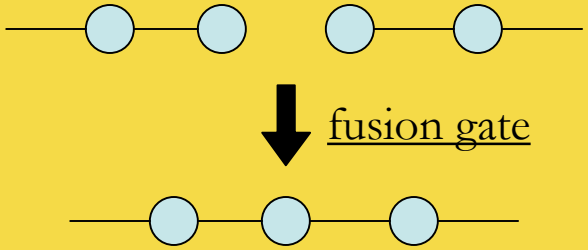
Travis Humble
Oak Ridge National Laboratory internal presentation (21 Nov 2006)

Cluster states - multi-particle entangled states and a medium for performing quantum computations. We consider photonic cluster states.



Fusion gates - an efficient means for preparing photonic cluster states.*

*More efficient means may exist.



Point of the talk - a vignette of cluster states and fusion gates plus my analysis of how spectral entanglement affects cluster state preparation.

Cluster States

Introduced by Briegel and Raussendorf in Phys. Rev. Lett. **86**, 910 (2001)

"Persistent entanglement in arrays of interacting particles".

A distinct class of *N*-particle entangled states motivated by considering the *d*-dimensional spin lattice (quantum Ising model).

$|0\rangle_a$ and $|1\rangle_a$ are eigenstates of $(1-\sigma_z^{(a)})/2$

Initially, all the (unentangled) qubits are prepared in the diagonal state

$$\left(|0\rangle_a + |1\rangle_a\right)/\sqrt{2}$$

The nearest-neighbor interaction is applied (1d case)

$$H = \varphi \sum_a \frac{1+\sigma_z^{(a)}}{2} \frac{1-\sigma_z^{(a+1)}}{2}$$

φ = gt
g - coupling
t - time

Cluster States

For $\varphi = \pi$,

$$|\phi_N\rangle = \frac{1}{2^{N/2}} \bigotimes_{a=1}^{N} \left(|0\rangle_a + |1\rangle_a \sigma_z^{(a+1)} \right) \qquad \sigma_z^{(N+1)} = 1$$

$N = 2$, a Bell state:

$$|\phi_2\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_1 |+\rangle_2 + |1\rangle_1 |-\rangle_2 \right) = \frac{1}{\sqrt{2}} \left(|0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2 \right)$$

$N = 3$, a 3-particle GHZ state:

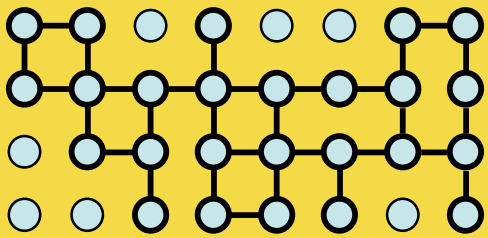
$$|\phi_3\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_1 |0\rangle_2 |0\rangle_3 + |1\rangle_1 |1\rangle_2 |1\rangle_3 \right)$$

$N = 4$, not a 4-particle GHZ state:

$$|\phi_4\rangle = \frac{1}{2} \left(|0\rangle_1 |0\rangle_2 |0\rangle_3 |0\rangle_4 + |0\rangle_1 |0\rangle_2 |1\rangle_3 |1\rangle_4 + |1\rangle_1 |1\rangle_2 |0\rangle_3 |0\rangle_4 + |1\rangle_1 |1\rangle_2 |1\rangle_3 |1\rangle_4 \right)$$

Cluster States

More generally, a cluster state is a pure quantum state of two-level systems located on a *cluster*. The cluster C is a connected subset of a simple cubic lattice.



THE Eigenvalue equation(s)

$$K^{(a)}|\kappa\rangle_C = (-1)^{\kappa_a}|\kappa\rangle_C \qquad \kappa = \{\kappa_a \in \{0,1\} | a \in C\}$$

Complete set of correlation operators (stabilizer generators),

$$K^{(a)} = \sigma_x^{(a)} \bigotimes_{b \in \mathrm{nghb}(a)} \sigma_z^{(b)} \qquad \mathrm{nghb}(a) \text{ set of neighboring lattice sites}$$

The set κ defines the eigenstate. There are $2^{|C|}$ mutually orthogonal eigenstates.

Two measures of cluster states:

Maximal connectedness - any two qubits can be projected into a Bell state.

Persistency - minimum number of measurements to completely disentangle the system. $N/2$

Cluster States and Quantum Computing

Raussendorf and Briegel, Phys. Rev. Lett. **86**, 5188 (2001)
"A one-way quantum computer"

Raussendorf, Browne, and Briegel, Phys. Rev. A **68**, 022313 (2003)
"Measurement-based quantum computation on cluster states"

"The cluster states thereby serves as a universal 'substrate' for any quantum computation.".

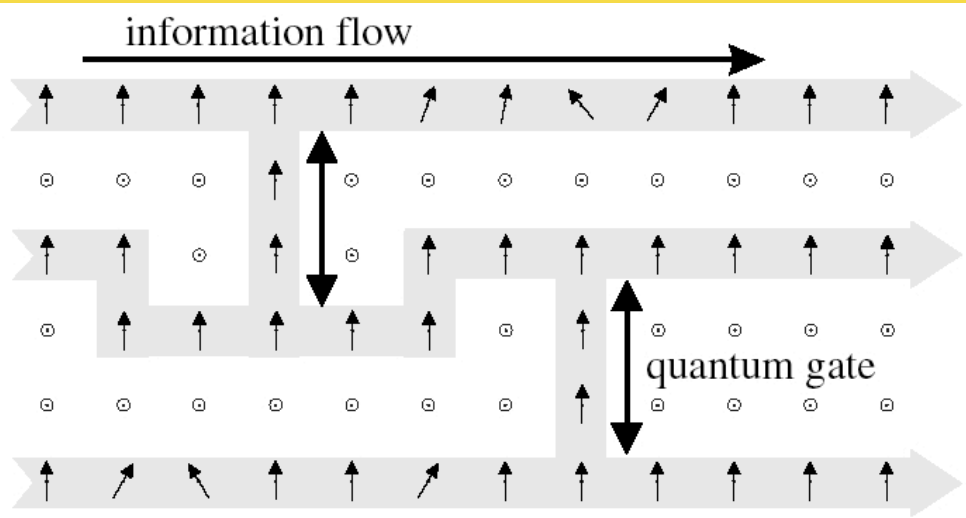


FIG. 1. Quantum computation by measuring two-state particles on a lattice. Before the measurements the qubits are in the cluster state $|\Phi\rangle_C$ of (1). Circles $\odot$ symbolize measurements of σ_z, vertical arrows are measurements of σ_x, while tilted arrows refer to measurements in the x-y plane.

Fusion Gates

Browne and Ralph, Phys. Rev. Lett. **95**, 010501 (2005)

"Resource-efficient linear optical quantum computation"

> **"…we introduce two 'fusion' mechanisms, which allow for the construction of entangled photonic states, known as cluster states."**

Cluster state are prepared non-deterministically, but efficiently (reduce, reuse, recycle).

In contrast to KLM, no photon-number detectors or elaborate interferometers required.

> **"…the interference we make use of is of the simple Hong-Ou-Mandel 'coincidence' form, and thus only requires stability over the coherence length of the photons,…"**

Polarization-encoded qubits, initially in biphoton entangled states, are aggregated into clusters using two types of operations.

Type-I: for making linear chains

Type-II: for making right angles

Type-I Fusion Gate

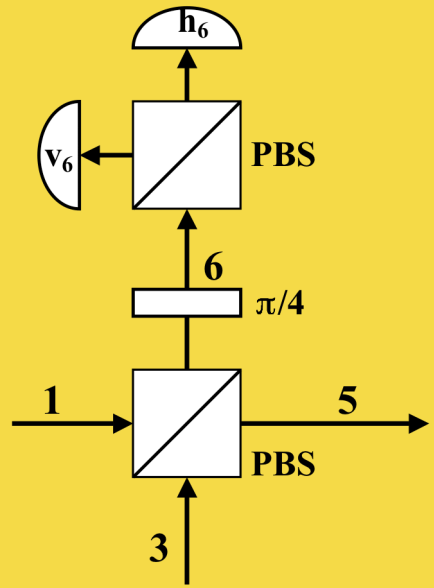
Start with two entangled pairs (1,2) and (3,4)

Send 1 and 3 into PBS

Output is in modes 5 and 6

Rotate mode 6 by $\pi/4$

Detect polarization in 6



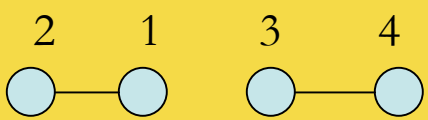
Detection probabilities

0 photons 1/4

1 photon 1/2

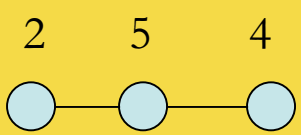
2 photons 1/4

On average, need 4 pairs to get a 3-qbit state



On a single click, h (+) or v (-),

$$\left(\left|v_2,v_4,h_5\right\rangle \pm \left|h_2,h_4,v_5\right\rangle\right)/\sqrt{2}$$

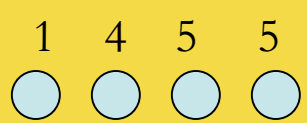


On a double click, hh or vv,

$$\left|h_2,v_4\right\rangle$$

On no clicks,

$$\left|v_2,h_4,h_5,v_5\right\rangle$$



Type-I Fusion Gate

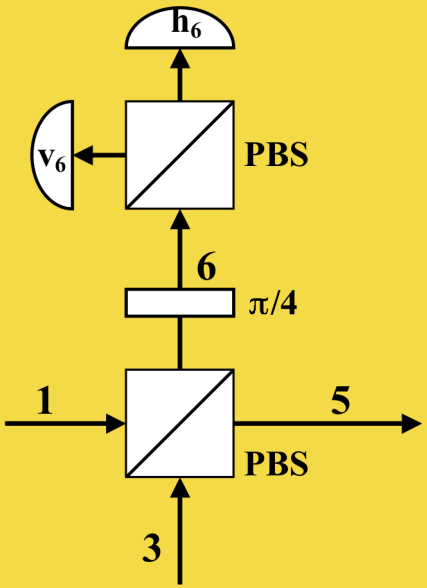
Start with two entangled pairs (1,2) and (3,4)

Send 1 and 3 into PBS

Output is in modes 5 and 6

Rotate mode 6 by $\pi/4$

Detect polarization in 6



Detection probabilities

0 photons 1/4

1 photon 1/2

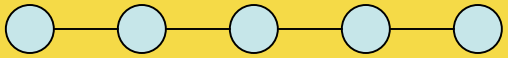
2 photons 1/4

On average, need 16 pairs to get to a 5-qbit state

But failure produces Bell states that are reusable

Next, fuse two 3-qbit states

On a single click, h (+) or v (-),



$$\left(|0,0,0,0,0\rangle \pm |1,1,1,1,1\rangle\right)/\sqrt{2}$$

On a double click,

No click,

Type-II Fusion Gate

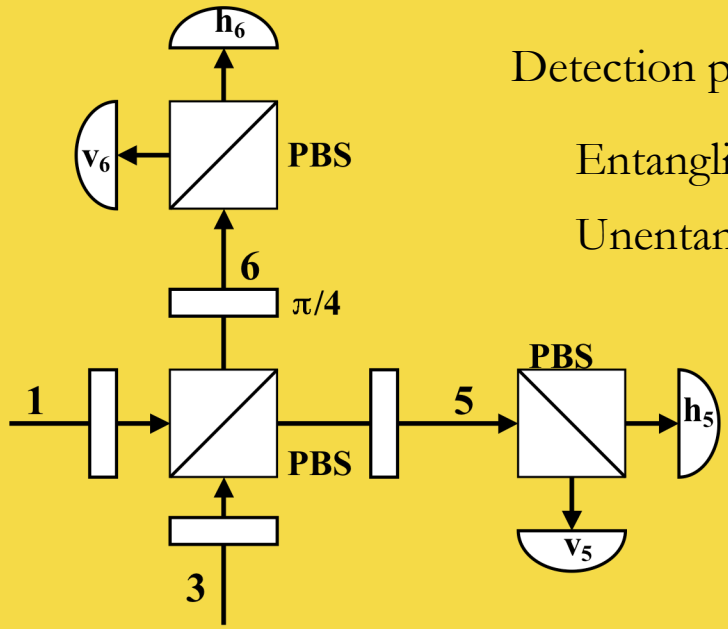
Redundantly encoded pairs (1,2) and (3,4)

Rotate modes 1 and 3 by π/4

Output is in modes 5 and 6

Rotate modes 5 and 6 by π/4

Detect polarization in 5 and 6



Detection probabilities

Entangling 1/2

Unentangling 1/2

Redundant encoding of logical states

$$|\phi_0\rangle|0\rangle + |\phi_1\rangle|1\rangle \rightarrow |\phi_0\rangle|h\rangle^{\otimes n} + |\phi_1\rangle|v\rangle^{\otimes n}$$

These detection schemes do not require photon number resolution

h_3h_4 or v_3v_4 (1/4)

$$\left|\phi_0, h^{\otimes(n-1)}\right\rangle\left|\phi_0', h^{\otimes(n-1)}\right\rangle + \left|\phi_1, v^{\otimes(n-1)}\right\rangle\left|\phi_1', v^{\otimes(n-1)}\right\rangle$$

h_3v_4 or v_3h_4 (1/4),

$$\left|\phi_0, h^{\otimes(n-1)}\right\rangle\left|\phi_1', v^{\otimes(n-1)}\right\rangle + \left|\phi_1, h^{\otimes(n-1)}\right\rangle\left|\phi_0', h^{\otimes(n-1)}\right\rangle$$

Otherwise (1/2)

$$\left(\left|\phi_0, h^{\otimes n-1}\right\rangle + \left|\phi_1, v^{\otimes n-1}\right\rangle\right) \otimes \left(\left|\phi_0', h^{\otimes n-1}\right\rangle + \left|\phi_1', v^{\otimes n-1}\right\rangle\right)$$

Type-II Fusion Gate

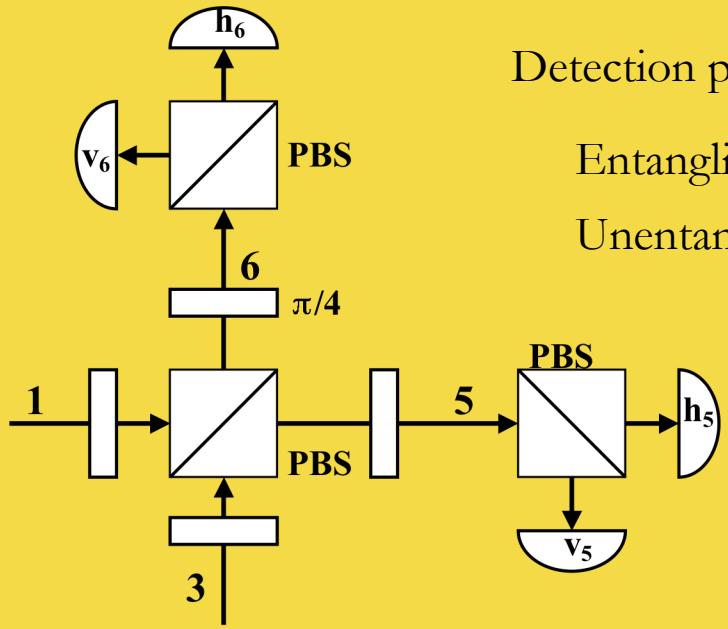
Redundantly encoded pairs (1,2) and (3,4)

Rotate modes 1 and 3 by $\pi/4$

Output is in modes 5 and 6

Rotate modes 5 and 6 by $\pi/4$

Detect polarization in 5 and 6



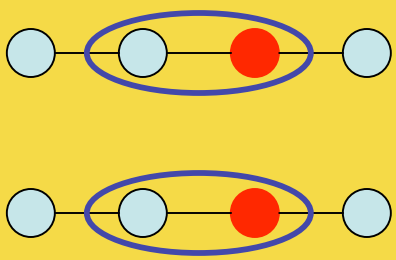
Detection probabilities

Entangling 1/2

Unentangling 1/2

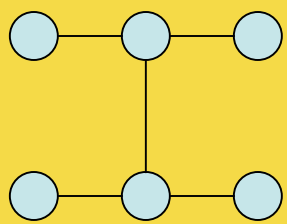
Redundant encoding of logical states

$$|\phi_0\rangle|0\rangle + |\phi_1\rangle|1\rangle \rightarrow |\phi_0\rangle|h\rangle^{\otimes n} + |\phi_1\rangle|v\rangle^{\otimes n}$$

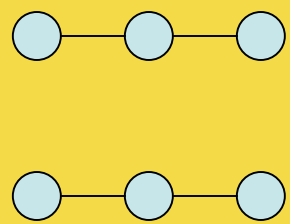


Detachable qubits measured

Entangling (1/2)



Unentangling (1/2)



Type-I Fusion Gate with Multi-mode Photons

$$\left|\Psi\right\rangle = \left|\Psi_{12}\right\rangle\left|\Psi_{34}\right\rangle$$

$$\left|\Psi_{12}\right\rangle = \frac{1}{\sqrt{2}}\int dx\int dy\left[f_{12}(x,y)\left|h_1(x),v_2(y)\right\rangle + g_{12}(x,y)\left|v_1(x),h_2(y)\right\rangle\right], \text{ etc.}$$

$$\left|\Psi'\right\rangle = \mathrm{U}_6^{\pi/4}\mathrm{U}_{1,3}^{\mathrm{PBS}}\left|\Psi\right\rangle$$

$$\begin{aligned}
= \frac{1}{2}\int dx\int dy\int dx'\int dy'\Big\{&\Big(f_{12}(x',y)f_{34}(x,y')\left|h_5(x'),v_2(y),v_4(y')\right\rangle + g(x_{12},y)g_{34}(x',y')\left|v_5(x'),h_2(y),h_4(y')\right\rangle\Big)\left|h_6(x)\right\rangle/\sqrt{2}\\
&+\Big(f_{12}(x',y)f_{34}(x,y')\left|h_5(x'),v_2(y),v_4(y')\right\rangle - g(x_{12},y)g_{34}(x',y')\left|v_5(x'),h_2(y),h_4(y')\right\rangle\Big)\left|v_6(x)\right\rangle/\sqrt{2}\\
&+g_{12}(x,y)f_{34}(x',y')\left|h_2(y),v_4(y')\right\rangle\Big(\left|h_6(x)\right\rangle - \left|v_6(y)\right\rangle\Big)\Big(\left|h_6(x')\right\rangle + \left|v_6(y')\right\rangle\Big)/2\\
&+f_{12}(x,y)g_{34}(x',y')\left|h_5(x),v_2(y),v_5(x'),h_4(y')\right\rangle\Big\}
\end{aligned}$$

Type-I Fusion Gate with Multi-mode Photons

Single-photon detection event, density matrix of remaining three qubits

$$\rho_{245}^{(h_6)} = \int dx \left| \langle h_6(x) | \Psi' \rangle \right|$$

Reduced density matrix of the polarization state (trace over frequencies)

$$\tilde{\rho}_{245}^{(h_6)} = \frac{1}{4} \{ |h_2, h_4, v_5\rangle\langle h_2, h_4, v_5| + G_{245} |h_2, h_4, v_5\rangle\langle v_2, v_4, h_5|$$

$$+ G_{245}^* |v_2, v_4, h_5\rangle\langle h_2, h_4, v_5| + |v_2, v_4, h_5\rangle\langle v_2, v_4, h_5| \}$$

Coherence between 3-qubit polarization states

$$G_{245} = \int dx \int dy \int dx' \int dy' f_{12}(x,y) g_{34}(x,x')^* f_{34}(y',x') g_{12}(y',y)^*$$

Type-I Fusion Gate with Multi-mode Photons

Two expected cases:

1. $f(x,y) = f_{12}(x,y) = g_{12}(y,x) = f_{34}(x,y) = g_{34}(y,x)$

$$G_{245} = \int dx \int dx' \left| \int f(x,y) f(x',y)^* dy \right|^2$$

If the spectrum is separable,

$$f(x,y) = f_a(x) f_b(y) \qquad G_{245} = \int |f_a(x)|^2 dx \int |f_a(x')|^2 dx' \left(\int |f_b(y)|^2 dy \right)^2 = 1$$

Type-I Fusion Gate with Multi-mode Photons

2. $f(x,y) = f_{12}(x,y) = g_{12}(x,y) = f_{34}(x,y) = g_{34}(x,y)$

$$G_{245} = \int dx \int dx' \left| \int f(x,y) f(y,x)^* dy \right|^2$$

If the spectrum is separable,

$$f(x,y) = f_a(x) f_b(y) \qquad G_{245} = \int |f_a(x)|^2 dx \int |f_a(x')|^2 dx' \left| \int f_a(y) f_b(y)^* dy \right|^2$$

$$= \left| \int f_a(y) f_b(y)^* dy \right|^2 \leq 1$$

For a Gaussian joint spectrum

$$G_{245} = \frac{2\sigma\sigma'\sqrt{1-\rho^2}}{\sigma^2 + \sigma'^2} \exp\left[\frac{-\Delta\tau^2 \sigma^2 \sigma'^2 (1-\rho^2)}{\sigma^2 + \sigma'^2} \right]$$

σ and σ' are marginal bandwidths, ρ is the linear correlation, $\Delta\tau$ is temporal delay.