APPENDIX A

THE DISPLACEMENT OPERATOR

A useful construct in the analysis of the quantum-mechanical harmonic oscillator is the displacement operator

$$D(\quad) = e^{\quad a^{\dagger} \quad a}, \tag{A.1}$$

where is a dimensionless complex number.[1] The ladder operator a is defined as

$$a = \frac{}{2}^{1/2} \quad x + i\frac{p}{} \quad , \tag{A.2}$$

and satisfies the commutation relation

$$a, a^{\dagger} \quad = 1 . \tag{A.3}$$

The harmonic oscillator Hamiltonian

$$H = \frac{p^2}{2} + \frac{{}^2 x^2}{2} \tag{A.4}$$

can be expressed in terms of the ladder operator as

$$H = \quad a^{\dagger} a + \frac{1}{2} \quad . \tag{A.5}$$

The eigenstates of Eq. (A.5) are defined by

$$H|n\rangle = E(n)|n\rangle \tag{A.6}$$

with $n = 0,1,2...$ the quantum of excitation and

$$E(n) = \quad (n + 1/2) \tag{A.7}$$

the corresponding eigenenergy.

The displacement operator can be factored as

$$D(\quad) = e^{|\ |^2/2} e^{\ a^\dagger} e^{\quad a}, \tag{A.8}$$

owing to the commutation relation of Eq. (A.3) and the Baker-Campbell-Hausdorff Theorem (a.k.a. Glauber's Formula).[1] Then, the action of the displacement operator on the ground state of Eq. (A.5), i.e. $n = 0$, yields

$$\begin{aligned} D(\quad)|0\rangle &= e^{|\ |^2/2} e^{\ a^\dagger} e^{\quad a}|0\rangle \\ &= e^{|\ |^2/2} e^{\ a^\dagger}|0\rangle \\ &= e^{|\ |^2/2} \quad {}_j \frac{\ \ ^j}{\sqrt{j!}}|j\rangle . \end{aligned} \tag{A.9}$$

The final result of Eq. (A.9) is the energy eigenstate representation of the harmonic oscillator coherent state[1]

$$|\quad\rangle = e^{|\ |^2/2} \quad {}_j \frac{\ \ ^j}{\sqrt{j!}}|j\rangle, \tag{A.10}$$

which is also as an eigenstate of the ladder operator,

$$a|\quad\rangle = \quad|\quad\rangle. \tag{A.11}$$

Consequently, the displacement operator transforms the ground state of the harmonic oscillator into the coherent state, i.e.

$$|\quad\rangle = D(\quad)|0\rangle. \tag{A.12}$$

More generally, the unitary transformation

$$a \;= D(\;\;)aD^{\dagger}(\;\;) \tag{A.13}$$

serves to translates the ladder operator a by an amount . Since,

$$D(\;\;)aD^{\dagger}(\;\;)|0\rangle = (a \;\;\;)|0\rangle, \tag{A.14}$$

it follows from Eq. (A.13) that

$$a \;= a \;\;\;\;. \tag{A.15}$$

In particular, for real values of , the displacement operator spatially translates the Hamiltonian. For example, the displaced harmonic Hamiltonian

$$H \;= \frac{p^2}{2} + \frac{\;^2}{2}\left(x \;\;\; x_0\right)^2 \tag{A.16}$$

can be expressed in terms of the undisplaced Hamiltonian, given by Eq. (A.5), as

$$H \;= D(\;\;_0)HD^{\dagger}(\;\;_0) \tag{A.17}$$

for $_0 = (\;\;\;\;/2)^{1/2} x_0$. The eigenstates of H have identical eigenenergies, see Eq. (A.7), but are translated by the amount $_0$

$$|\bar{n}\rangle = D(\;\;_0)|n\rangle. \tag{A.18}$$

Inserting the definition of the ladder operator into $D(\;\;_0)$, we also see that

$$D(\;\;_0) = e^{ipx_0}, \tag{A.19}$$

i.e. $D(\;\;)$ is the spatial translation operator for real values of . The latter result is particularly useful for translating the wave functions of Eq. (A.18);

$$\langle x|\bar{n}\rangle = \langle x|D(\;\;_0)|n\rangle \tag{A.20}$$

$$= \langle x \;\;\; x_0|n\rangle.$$

Other useful properties of the displacement operator include the composition property

$$D(\ \ _1)D(\ \ _2) = D(\ \ _1 + \ \ _2)e^{(\ _1\ _2\ \ _1\ _2)/2} \tag{A.21}$$

and the unitary transformation

$$e^{\ iHt}D(\ \ )e^{iHt} = D\left(\ \ e^{\ i\ \ t}\right), \tag{A.22}$$

both of which follow from Eq. (A.9). See Ref. [1] for additional details.

The overlap between a displaced and undisplaced eigenstate,

$$\langle \bar{m}|n\rangle = \langle m|D^{\dagger}(\ \ )|n\rangle,$$

is calculated using Eq. (A.8). Expanding the exponential operators, we have

$$\langle \bar{m}|n\rangle = e^{\ |\ |^2/2}\langle m|e^{\ \ a^{\dagger}}e^{\ \ a}|n\rangle \tag{A.23}$$

$$= e^{\ |\ |^2/2}\sum_{j=0}\sum_{k=0}\frac{(\ \ \ \ )^j}{j!}\frac{(\ \ \ \ )^k}{k!}\langle m|\left(a^{\dagger}\right)^j\left(a\ \ \right)^k|n\rangle$$

$$= e^{\ |\ |^2/2}\sum_{j=0}^{m\ \ \ n}\sum_{k=0}^{n}\frac{(\ \ \ \ )^j}{j!}\frac{(\ \ \ \ )^k}{k!}\left(\frac{m!}{(m\ \ \ j)!}\frac{n!}{(n\ \ \ k)!}\right)^{1/2}\langle m\ \ \ j|n\ \ \ k\rangle,$$

where in the last line finite upper bounds on the summations are a result of $a|0\rangle = 0$. Due to the orthonormality of the harmonic oscillator eigenstates, we must have

$$m\ \ \ j = n\ \ \ k,$$

which includes two cases of the summation in Eq. (A.23). In the first, $m\ \ \ n$, we substitute $j = k + m\ \ \ n$ and find

$$\langle \bar{m}|n\rangle = e^{\ |\ |^2/2}(\ \ \ \ )^{m\ \ n}\ \ \frac{n!}{m!}\left(\ \ \ \right)^{\frac{1}{2}\ \ n}\sum_{k=0}\frac{\left(\ \ |\ \ |^2\right)^k}{k!}\frac{m!}{(n\ \ \ k)!(k+m\ \ \ n)!} \tag{A.24}$$

$$= e^{|\ |^2/2}(\quad)^{m\ \ n}\ \frac{n!}{m!}(^{\frac{1}{2}}\ L_n^{m\ \ n}\left(|\ \ |^2\right),$$

where in going to the last line we have used the following identity for the associated Laguerre polynomial[2]

$$L_n(x) = {}_{k=0}^{n}\ \frac{(\ \ x)^k}{k!}\frac{(n+\ \ )!}{(n\ \ \ k)!(k+\ \ )!}. \tag{A.25}$$

Similarly, for the second case, $m\ \ \ n$, we let $k = j + n\ \ \ m$ to find

$$\langle \bar{m}|n\rangle = e^{|\ |^2/2}(\quad)^{n\ \ m}\ \frac{m!}{n!}(^{\frac{1}{2}}\ L_m^{n\ \ m}\left(|\ \ |^2\right). \tag{A.26}$$

When $m = n$ both cases coincide and the overlap is given by

$$\langle \bar{n}|n\rangle = e^{|\ |^2/2}L_n^0\left(|\ \ |^2\right), \tag{A.27}$$

where $L_n^0(x) = L_n(x)$ is the usual Laguerre polynomial.

Notes

1. C. Cohen-Tannoudji, B. Diu, and F. Lahoë, *Quantum Mechanics* (Hermann, Paris, 1977).

2. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1994).