ANALYSIS OF LONG TIME STABILITY AND ERRORS OF TWO PARTITIONED METHODS FOR UNCOUPLING EVOLUTIONARY GROUNDWATER - SURFACE WATER FLOWS

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Abstract. The most effective simulations of the multi-physics coupling of groundwater to surface water must involve employing the best groundwater codes and the best surface water codes. Partitioned methods, which solve the coupled problem by successively solving the sub-physics problems, have recently been studied for the Stokes-Darcy coupling with convergence established over bounded time intervals (with constants growing exponentially in t). This report analyzes and tests two such partitioned (non-iterative, domain decomposition) methods for the fully evolutionary Stokes-Darcy problem. Under a modest time step restriction of the form $\Delta t \leq C$ where $C = C$*(physical parameters)* we prove unconditional asymptotic *(over* $0 \leq t < \infty$*)* stability of both partitioned methods. From this we derive an optimal error estimate that is *uniform in time* over $0 \leq t < \infty$.

1. Introduction. Many important applications require the accurate solution of multi-domain, multi-physics coupling of groundwater flows with surface flows (the Stokes-Darcy problem). The essential problems of estimation of the penetration of a plume of pollution into groundwater and remediation after such a penetration are that (i) the coupled problem in both sub-regions are inherently time dependent, (ii) the different physical processes suggest that codes optimized for each sub-process need to be used for solution of the coupled problem and (iii) the large domains plus the need to compute for several turn-over times for reliable statistics *require calculations over long time intervals*. With these issues in mind, we give a complete analysis of the stability and errors over long time intervals of two partitioned methods (which require only one uncoupled Stokes and one Darcy subdomain solve per time step) for the coupled, fully time dependent Stokes-Darcy problem. This builds upon recent studies of partitioned methods over bounded time intervals (with constants $C(T) \sim e^{aT}$) of Mu and Zhu [MZ10] who studied the first (first order) partitioned method for the Stokes-Darcy coupling. The method was extended to allow different time steps in the two subregions [SZL11]. This work used Gronwall's inequality in an essential way for analyzing both stability and convergence. Thus, the stability and error behaviour over the required long time intervals is important for both algorithm development and analysis.

In this report we analyze the *stability and error behaviour over long time intervals* ($0 \leq t < \infty$) of two partitioned methods for uncoupling the evolutionary Stokes-Darcy problem. The first method we study is the first order method of [MZ10] consisting of first order implicit discretization of subdomain terms and explicit discretization of coupling terms. The stability regions of the explicit method used for the anti-symmetric coupling terms suggest that exponential growth of errors and perturbations is inevitable for the combination. Surprisingly, we show that this is not the case: the method is stable and optimally convergent uniformly over $0 \leq t < \infty$. The second method we study is a new, two step partitioned scheme motivated by the

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form of the coupling. It involves first order implicit discretization of the subdomain terms and leap frog discretization of the exactly skew symmetric coupling terms. We prove that this combination has superior stability properties and is optimally accurate and convergent uniformly over $0 \leq t < \infty$. We also present numerical experiments verifying the numerical analysis and comparing the accuracy of the two approaches.

To specify the problem considered, let the two domains be denoted by Ω_f, Ω_p and lie across an interface I from each other. The fluid velocity and porous media piezometric head (Darcy pressure) satisfy

$$\begin{aligned}
u_t - \nu\Delta u + \nabla p = \mathbf{f}_f(x,t), \nabla \cdot u = 0, \quad \text{in} \quad \Omega_f,\\
S_0\phi_t - \nabla \cdot (\mathcal{K}\nabla\phi) = f_p, \ \text{in} \ \ \Omega_p,\\
\phi(x,0) = \phi_0, \ \text{in } \Omega_p \text{ and } u(x,0) = u_0, \ \text{in } \Omega_f,\\
\phi(x,t) = 0, \ \text{in } \partial\Omega_p \backslash I \text{ and } u(x,t) = 0, \ \text{in } \partial\Omega_f \backslash I,\\
+ \text{ coupling conditions across } I.
\end{aligned}$$

The exact boundary conditions chosen above on the exterior boundaries $(\partial\Omega_{f/p}\backslash I)$ are not essential to either the analysis or algorithms studied herein. Let $\widehat{n}_{f/p}$ denote the indicated, outward pointing, unit normal vector on I. The coupling conditions are conservation of mass and balance of forces on I

$$\begin{aligned}
u \cdot \widehat{n}_f + \mathbf{u}_p \cdot \widehat{n}_p = 0, \text{ on I } \Leftrightarrow \ u \cdot \widehat{n}_f - \mathcal{K}\nabla\phi \cdot \widehat{n}_p = 0, \text{ on I},\\
p - \nu \ \widehat{n}_f \cdot \nabla u \cdot \widehat{n}_f = g\phi \ \text{ on I.}
\end{aligned}$$

The last condition needed is a tangential condition on the fluid region's velocity on the interface. The most correct condition is not completely understood (possibly due to matching a pointwise velocity in the fluid region with an averaged or homogenized velocity in the porous region). We take the Beavers-Joseph-Saffman (-Jones) interfacial coupling

$$-\nu \ \widehat{\tau}_i \cdot \nabla u \cdot \widehat{n}_f = \alpha_{BJ}\sqrt{\frac{\nu g}{\widehat{\tau}_i \cdot \mathcal{K} \cdot \widehat{\tau}_i}} u \cdot \widehat{\tau}_i, \text{ on } I \text{ for any } \widehat{\tau}_i \text{ tangent vector on } I.$$

See [BJ67], [Saf71], [JM00]. This is a simplification of the original and more physically realistic Beavers-Joseph conditions (in $u \cdot \widehat{\tau}_i$ which is replaced by $(u - \mathbf{u}_p) \cdot \widehat{\tau}_i$), studied in [CGHW10b], [CGH$^+$10]. Here

$$\begin{aligned}
\phi &= \text{Darcy pressure + elevation induced pressure = piezometric head},\\
\mathbf{u}_p &= \text{fluid velocity in porous media region}, \ \ \Omega_p,\\
p &= \text{kinematic pressure in fluid region}, \Omega_f,\\
u &= \text{fluid velocity in Stokes region}, \Omega_f,\\
\mathbf{f}_f, f_p &= \text{body forces in fluid region and source in porous region},\\
\mathcal{K} &= \text{hydraulic conductivity tensor},\\
\nu &= \text{kinematic viscosity of fluid},\\
S_0 &= \text{specific mass storativity coefficient},\\
g &= \text{gravitational acceleration constant}.
\end{aligned}$$

We shall assume that all material and fluid parameters above are positive and $O(1)$, in particular, $\mathcal{K}$ is a symmetric positive definite matrix with $k_{min} > 0$ is the smallest eigenvalue.

1.1. Previous work. The literature on numerical analysis of methods for the Stokes-Darcy coupled problem has grown extensively since [DMQ02], [LSY02]. See [DQ09] for a recent survey and [BJ67], [CGHW10b], [PSS99], [PS98], [Saf71], [Wan10] and [LSY02] for theory of the continuum model. There is less work on the fully evolutionary Stokes-Darcy problem. One approach is monolithic discretization by an implicit method followed by iterative solution of the non-symmetric system where subregion uncoupling is attained by using a domain decomposition preconditioner; see, e.g., [CGHW10b], [CGH$^+$10], [CGHW10a], [MX07], [CMX09], [Dis04], [DQ04], [DQ03], [HPV07], [Jia09], [MQS03], [VY11]. Partitioned methods allow parallel, non-iterative uncoupling into one (SPD) Stokes and one (SPD) Darcy system per time step. The first such partitioned method was studied in 2010 by Mu and Zhu [MZ10]. This has been followed by an asynchronous (allow different time steps in the two subregions) partitioned method in [SZL11] and higher order partitioned methods in [CGHW11], [LT11]. In most of these works, stability and convergence were studied over bounded time intervals $0 \le t \le T < \infty$ and the estimates included $e^{\alpha T}$ multipliers. Partitioned methods also have the key large advantage that the subdomain problems can be solved by legacy codes, each optimized for the physics of the individual subprocess.

Alternate approaches for coupling surface water flows with groundwater flows include Brinkman one-domain models, Angot [Ang99], Ingram [Ing10], which are a more accurate description of the physical processes. One-domain Brinkman models are also more computationally expensive. Monolithic quasi-static models (one domain evolutionary and the other assumed to instantly adjust back to equilibrium) have also been studied, e.g., [CR]. While they are not considered herein in detail, the methods considered also give non-iterative, domain decomposition schemes for quasi-static models (e.g., set $S_0 \equiv 0$ in (3.1), (3.2) below).

Partitioned methods employ implicit discretizations of the sub-physics/ subdomain problems and explicit time discretizations of the coupling terms, e.g., [VCC08], [MZ10], [BF07], [BF09], [CGN05], [CM10], [CHL09a], [CHL09b]. Thus there is a very strong connection between application-specific partitioned methods and more general IMEX (IMplicit - EXplicit) methods; the latter developed in, e.g., [Ver09], [Var80], [ARW95], [Cro80], [FHV96], [HV03], [FHV96], [APL04], [CM10] and [Ver10]. On the other hand, application-specific partitioned methods are often motivated by available codes for subproblems, [VCC08]. Examples of partitioned methods include ones designed for fluid-structure interaction [BF07], [BF09], [CGN05], Maxwell's equations, [Ver10] and atmosphere-ocean coupling, [CM10], [CHL09a].

Long time stable numerical schemes have also been introduced for related problems, especially for 2D Navier-Stokes equations. For such works, we refer to [TW06], [GTW$^+$12], [Ton09] and [Ju02].

2. The continuous problem and semi-discrete approximation. We denote the $L^2(I)$ norm by $\|\cdot\|_I$ and the $L^2(\Omega_{f/p})$ norms by $\|\cdot\|_{f/p}$, respectively; the corresponding inner products are denoted by $(\cdot,\cdot)_{f/p}$. Define

$$
\begin{aligned}
X_f &= \{v \in \left(H^1(\Omega_f)\right)^d : v = 0 \text{ on } \ \partial\Omega_f \backslash I\}, \\
X_p &= \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \ \partial\Omega_p \backslash I\}, \\
Q_f &= L_0^2(\Omega_f) := \left\{ q \in L^2(\Omega_f) : \int_{\Omega_f} q dx = 0 \right\}.
\end{aligned}
$$

Throughout this paper, we will use C_0 to denote a generic positive constant whose value may be different from place to place but which is independent of mesh size, time step and final time. We will use the combined trace, interpolation and Poincaré inequality

$$\|\phi\|_I \leq C(\Omega_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p} \text{ and } \|u\|_I \leq C(\Omega_f)\sqrt{\|u\|_f\|\nabla u\|_f} \tag{2.1}$$

where, by a scaling argument, $C(\Omega_{f/p}) = O(\sqrt{L_{f/p}}), L_{f/p} = diameter(\Omega_{f/p})$.

Define the bilinear forms

$$\begin{aligned}
a_f(u,v) &= (\nu\nabla u, \nabla v)_f + \sum_i \int_I \alpha_{BJ}\sqrt{\frac{\nu g}{\widehat{\tau}_i \cdot \mathcal{K} \cdot \widehat{\tau}_i}}(u\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)ds,\\
a_p(\phi,\psi) &= g(\mathcal{K}\nabla\phi, \nabla\psi)_p,\\
c_I(u,\phi) &= g\int_I \phi u\cdot\widehat{n}_f ds.
\end{aligned}$$

The bilinear forms $a_{f/p}(\cdot,\cdot)$ are continuous and coercive. The key to uncoupling the problem is the treatment of coupling term through the bilinear form $c_I(u,\phi)$. Also, we define three parameter-dependent constants

$$\begin{aligned}
C_1 &= \frac{g^{3/2}\left[C(\Omega_f)C(\Omega_p)\right]^2}{4\sqrt{\nu k_{\min}}}, \qquad C_2 = \frac{1}{\nu^2}C_{P,f}^2[gC(\Omega_f)C(\Omega_p)]^4,\\
C_3 &= \frac{1}{k_{\min}^2}C_{P,p}^2 g^2[C(\Omega_f)C(\Omega_p)]^4.
\end{aligned}$$

Lemma 2.1. *We have for $(u,\phi) \in X_f \times X_p$,*

$$|c_I(u,\phi)| \leq \frac{\nu}{4\varepsilon_1}\|\nabla u\|_f^2 + \frac{gk_{\min}}{4\varepsilon_1}\|\nabla\phi\|_p^2 + \varepsilon_1 C_1\left(\|u\|_f^2 + \|\phi\|_p^2\right), \tag{2.2}$$

$$|c_I(u,\phi)| \leq \frac{1}{4\varepsilon_2}\|\phi\|_p^2 + \varepsilon_2 C_2\|\nabla\phi\|_p^2 + \frac{\nu}{4}\|\nabla u\|_f^2, \tag{2.3}$$

$$|c_I(u,\phi)| \leq \frac{1}{4\varepsilon_3}\|u\|_f^2 + \varepsilon_3 C_3\|\nabla u\|_f^2 + \frac{gk_{\min}}{4}\|\nabla\phi\|_p^2. \tag{2.4}$$

for every ε_1, ε_2, $\varepsilon_3 > 0$, where $C_{P,f}$ and $C_{P,p}$ are Poincaré constants of the indicated domains.

Proof. Using basic estimates and the arithmetic geometric mean inequality twice we obtain

$$\begin{aligned}
c_I(u,\phi) &= g\int_I \phi u\cdot\widehat{n}ds\\
&\leq gC(\Omega_f)C(\Omega_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p}\cdot\sqrt{\|u\|_f\|\nabla u\|_f}\\
&\leq gC(\Omega_f)C(\Omega_p)\sqrt{\|\nabla u\|_f\|\nabla\phi\|_p}\cdot\sqrt{\|u\|_f\|\phi\|_p}\\
&\leq \frac{\nu}{4\varepsilon_1}\|\nabla u\|_f^2 + \frac{gk_{\min}}{4\varepsilon_1}\|\nabla\phi\|_p^2 + \varepsilon_1 C_1\left(\|u\|_f^2 + \|\phi\|_p^2\right).
\end{aligned}$$

For (2.3), observe that

$$c_I(u,\phi) = g\int_I \phi u\cdot\widehat{n}ds \le gC(\Omega_f)C(\Omega_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p}\cdot\sqrt{\|u\|_f\|\nabla u\|_f}$$
$$\le \left(\frac{1}{\varepsilon_2^{1/4}}\|\phi\|_p^{1/2}\right)\left(gC(\Omega_f)C(\Omega_p)\varepsilon_2^{1/4}\left(\frac{2}{\nu}\right)^{1/2}C_{P,f}^{1/2}\|\nabla\phi\|_p^{1/2}\right)\left(\left(\frac{\nu}{2}\right)^{1/2}\|\nabla u\|_f\right)$$
$$\le \frac{1}{4\varepsilon_2}\|\phi\|_p^2 + \varepsilon_2 C_2\|\nabla\phi\|_p^2 + \frac{\nu}{4}\|\nabla u\|_f^2.$$

Finally, (2.4) comes from a similar argument. □

A (monolithic) variational formulation of the coupled problem is to find (u,p,ϕ) : $[0,\infty)\to X_f\times Q_f\times X_p$ satisfying the given initial conditions and, for all $v\in X_f, q\in Q_f, \psi\in X_p$

$$\begin{aligned}(u_t,v)_f + a_f(u,v) - (p,\nabla\cdot v)_f + c_I(v,\phi) &= (\mathbf{f}_f,v)_f,\\ (q,\nabla\cdot u)_f &= 0,\\ gS_0(\phi_t,\psi)_p + a_p(\phi,\psi) - c_I(u,\psi) &= g(f_p,\psi)_p.\end{aligned} \tag{2.5}$$

Note that, setting $v=u,\psi=\phi$ and adding, the coupling terms exactly cancel in the monolithic sum yielding the energy estimate for the coupled system.

To discretize the Stokes-Darcy problem in space by the finite element method, we select finite element spaces

$$\text{velocity: } X_f^h\subset X_f,\ \text{Darcy pressure: } X_p^h\subset X_p,\ \text{Stokes pressure: } Q_f^h\subset Q_f$$

based on a conforming FEM triangulation with maximum triangle diameter denoted by "h". No mesh compatibility or interdomain continuity at the interface I between the FEM meshes in the two subdomains is assumed. The Stokes velocity-pressure FEM spaces are assumed to satisfy the usual discrete inf-sup condition for stability of the discrete pressure, e.g., [GR86], and X_f^h, X_p^h, Q_f^h satisfy approximation properties of piecewise polynomials on quasi-uniform meshes of local degrees $k,k,k-1$ respectively. We denote the discretely divergence free velocities by

$$V^h := X_f^h\cap\{v_h : (q_h,\nabla\cdot v_h)_f = 0,\ \text{for all } q_h\in Q_f^h\}.$$

The semi-discrete approximations are maps $(u_h,p_h,\phi_h):[0,\infty)\to X_f^h\times Q_f^h\times X_p^h$ satisfying the given initial conditions and, for all $v_h\in X_f^h$, $q_h\in Q_f^h, \psi_h\in X_p^h$

$$\begin{aligned}(u_{h,t},v_h)_f + a_f(u_h,v_h) - (p_h,\nabla\cdot v_h)_f + c_I(v_h,\phi_h) &= (\mathbf{f}_f,v_h)_f,\\ (q_h,\nabla\cdot u_h)_f &= 0,\\ gS_0(\phi_{h,t},\psi_h)_p + a_p(\phi_h,\psi_h) - c_I(u_h,\psi_h) &= g(f_p,\psi_h)_p.\end{aligned} \tag{2.6}$$

Note in particular the exactly skew symmetric coupling between the Stokes and the Darcy sub-problems.

3. The Partitioned Methods. The first method we analyzed is **BEFE = Backward Euler - Forward Euler**, the original method of Mu and Zhu [MZ10]. Since we focus on the long time error and stability, we shall use the same time step in both subdomains. Let $t^n := n\Delta t$ and let superscripts denote the time level of the approximation. The BEFE partitioned approximations are maps $(u_h^n, p_h^n, \phi_h^n) \in X_f^h \times Q_f^h \times X_p^h$ for $n \geq 1$ satisfying, for all $v_h \in X_f^h, q_h \in Q_f^h, \psi_h \in X_p^h$

$$
\begin{aligned}
(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h)_f + a_f(u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla \cdot v_h)_f + c_I(v_h, \phi_h^n) = (\mathbf{f}_f^{n+1}, v_h)_f,& \\
(q_h, \nabla \cdot u_h^{n+1})_f = 0,& \qquad (3.1)\\
gS_0(\frac{\phi_h^{n+1} - \phi_h^n}{\Delta t}, \psi_h)_p + a_p(\phi_h^{n+1}, \psi_h) - c_I(u_h^n, \psi_h) = g(f_p^{n+1}, \psi_h)_p.&
\end{aligned}
$$

The second method we consider is **BELF = Backward Euler Leap Frog**, a combination of the three level implicit method with the coupling terms treated by the explicit Leap-Frog method. We shall use the same time step, Δt, in both sub domains; extension to asynchronous time stepping, e.g. [SZL11], is an important open problem. The BELF partitioned approximations are maps $(u_h^n, p_h^n, \phi_h^n) \in X_f^h \times Q_f^h \times X_p^h$ for $n \geq 2$ satisfying, for all $v_h \in X_f^h, q_h \in Q_f^h, \psi_h \in X_p^h$

$$
\begin{aligned}
(\frac{u_h^{n+1} - u_h^{n-1}}{2\Delta t}, v_h)_f + a_f(u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla \cdot v_h)_f + c_I(v_h, \phi_h^n) = (\mathbf{f}_f^{n+1}, v_h)_f,& \\
(q_h, \nabla \cdot u_h^{n+1})_f = 0,& \qquad (3.2)\\
gS_0(\frac{\phi_h^{n+1} - \phi_h^{n-1}}{2\Delta t}, \psi_h)_p + a_p(\phi_h^{n+1}, \psi_h) - c_I(u_h^n, \psi_h) = g(f_p^{n+1}, \psi_h)_p.&
\end{aligned}
$$

BELF is a 3 level method and approximations are needed at the first two time steps to begin. We shall suppose these are computed to appropriate accuracy, such as by BEFE (the first method above). See Verwer [Ver09] for subtle effects that depend on the starting values. The stability region of the usual Leap-Frog time discretization for $y' = \lambda y$ is exactly the interval of the imaginary axis $-1 \leq Im(\Delta t\lambda) \leq +1$. Thus, LF is unstable for every problem except for ones which are exactly skew symmetric such as the coupling herein. For them, as with any explicit scheme, it inherits a time step restriction.

3.1. Asymptotic stability of the two partitioned methods. In this section we analyze the *asymptotic stability*, *uniform in t^n stability* over $0 \leq t^n < \infty$. We derive the restriction needed as the time step, of the form $\Delta t \leq C(\textit{physical parameter})$ under which

1. the approximate solutions are uniform in time stable and convergent,
2. if $\mathbf{f}_f = f_p = 0$, $u^n, \phi^n \to 0$ as $t^n \to \infty$, and
3. if $\|\mathbf{f}_f(t)\|$, $\|f_p(t)\|$ are uniformly bounded in time then $\sup_{t^n} (\|u^n\| + \|\phi^n\|) < \infty$.

3.1.1. BEFE Stability. The analysis of Mu and Zhu includes (inside the error estimation) a proof (which also extends to BELF as well) of stability over bounded time intervals of the form: for any Δt and $0 \leq t^n < T < \infty$

$$
\|u_h^n\|_f^2 + \|\phi_h^n\|_p^2 \leq C(T) \left[\sup_{0 \leq t_n \leq T} \left\{ \|\mathbf{f}_f(t_n)\|_f^2 + \|f_p(t_n)\|_p^2 \right\} + \|u_h^0\|_{H^1(\Omega_f)}^2 + \|\phi_h^0\|_{H^1(\Omega_p)}^2 \right]
$$

where $C(T)$ arises from Gronwall's inequality and thus grows exponentially with T. We prove a strong, asymptotic and uniform in time stability of the BEFE partitioned approximation (3.1) under the time step restriction

$$
\Delta t \leq \Delta t_{BEFE} := \min \left\{ \frac{k_{\min}}{C_{P,p}^2}, \frac{S_0 \nu}{C_{P,f}^2} \right\} \frac{\nu k_{\min}}{8g^2 (C(\Omega_f) C(\Omega_p))^4}. \qquad (3.3)
$$

Note that the RHS of (3.3) is independent of h so that, in the usual terminology of numerical PDEs, BEFE is unconditionally stable. Our experiments in Section 5 with $k_{\min} = 10^{-6}$ show that there is some dependence of Δt_{BEFE} on $k_{\min}$ but the dependence indicated in (3.3) is likely not sharp.

THEOREM 3.1. *(BEFE asymptotic stability) Consider BEFE method* (3.1). *Under the time step condition* (3.3) *it is uniformly in time and asymptotically stable*

$$\begin{aligned} &\frac{1}{2}\|u_h^N\|_f^2 + \frac{gS_0}{2}\|\phi_h^N\|_p^2 + \Delta t\sum_{n=0}^{N-1}\left(\frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 + \frac{gk_{\min}}{4}\|\nabla\phi_h^{n+1}\|_p^2\right) \\ &\quad + \frac{\Delta t\nu}{8}\|\nabla u_h^N\|_f^2 + \frac{\Delta t g k_{\min}}{8}\|\nabla\phi_h^N\|_p^2 \\ &\quad \le \frac{1}{2}\|u_h^0\|_f^2 + \frac{gS_0}{2}\|\phi_h^0\|_p^2 + \Delta t\sum_{n=0}^{N-1}\left(\frac{C_{P,f}^2}{\nu}\|\mathbf{f}_f^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{k_{\min}}\|f_p^{n+1}\|_p^2\right) \\ &\quad + \frac{\Delta t\nu}{8}\|\nabla u_h^0\|_f^2 + \frac{\Delta t g k_{\min}}{8}\|\nabla\phi_h^0\|_p^2. \end{aligned} \tag{3.4}$$

There is a $C_0 < \infty$ *such that if* $\mathbf{f}_f \in L^\infty(L^2(\Omega_f))$, $f_p \in L^\infty(L^2(\Omega_p))$ *then*

$$\begin{aligned} \sup_{0\le N\le\infty}\left\{\|u_h^N\|_f^2 + gS_0\|\phi_h^N\|_p^2\right\} \le & C_0\Bigg(\sup_{0\le N\le\infty}\left\{\|\mathbf{f}_f^{N+1}\|_f^2 + \|f_p^{N+1}\|_p^2\right\} \\ & + \|u_h^0\|_{H^1(\Omega_f)}^2 + \|\phi_h^0\|_{H^1(\Omega_p)}^2\Bigg), \end{aligned} \tag{3.5}$$

and if $\mathbf{f}_f \equiv 0$, $f_p \equiv 0$ *then*

$$u_h^N \to 0,\ \ \phi_h^N \to 0 \tag{3.6}$$

in $H^1(\Omega_f)$ *and* $H^1(\Omega_p)$ *respectively as* $N \to \infty$.

Proof. In (3.1), set $v_h = u_h^{n+1}$ and $\psi_h = \phi_h^{n+1}$ and add. This gives

$$\begin{aligned} &\frac{1}{2\Delta t}\|u_h^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|u_h^n\|_f^2 + \frac{1}{2\Delta t}\|u_h^{n+1} - u_h^n\|_f^2 + \frac{gS_0}{2\Delta t}\|\phi_h^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_h^n\|_p^2 \\ &+ \frac{gS_0}{2\Delta t}\|\phi_h^{n+1} - \phi_h^n\|_p^2 + a_f(u_h^{n+1}, u_h^{n+1}) + a_p(\phi_h^{n+1}, \phi_h^{n+1}) + c_I(u_h^{n+1} - u_h^n, \phi_h^{n+1}) \\ &- c_I(u_h^{n+1}, \phi_h^{n+1} - \phi_h^n) = (\mathbf{f}_f^{n+1}, u_h^{n+1}) + g(f_p^{n+1}, \phi_h^{n+1}). \end{aligned}$$

Applying (2.3) and (2.4) with $\varepsilon_2 = \frac{\Delta t}{2gS_0}$ and $\varepsilon_3 = \frac{\Delta t}{2}$ we have

$$\begin{aligned} & c_I(u_h^{n+1} - u_h^n, \phi_h^{n+1}) - c_I(u_h^{n+1}, \phi_h^{n+1} - \phi_h^n) \\ \ge\ & -\frac{gS_0}{2\Delta t}\|\phi_h^{n+1} - \phi_h^n\|_p^2 - \frac{\Delta t}{2gS_0}C_2\|\nabla(\phi_h^{n+1} - \phi_h^n)\|_p^2 - \frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 \\ & -\frac{1}{2\Delta t}\|u_h^{n+1} - u_h^n\|_f^2 - \frac{\Delta t}{2}C_3\|\nabla(u_h^{n+1} - u_h^n)\|_f^2 - \frac{gk_{\min}}{4}\|\nabla\phi_h^{n+1}\|_p^2 \\ \ge\ & -\frac{gS_0}{2\Delta t}\|\phi_h^{n+1} - \phi_h^n\|_p^2 - \frac{\Delta t}{gS_0}C_2(\|\nabla\phi_h^{n+1}\|_p^2 + \|\nabla\phi_h^n\|_p^2) - \frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 \\ & -\frac{1}{2\Delta t}\|u_h^{n+1} - u_h^n\|_f^2 - \Delta t C_3(\|\nabla u_h^{n+1}\|_f^2 + \|\nabla u_h^n\|_f^2) - \frac{gk_{\min}}{4}\|\nabla\phi_h^{n+1}\|_p^2 \\ \ge\ & -\frac{gS_0}{2\Delta t}\|\phi_h^{n+1} - \phi_h^n\|_p^2 - \frac{gk_{\min}}{8}(\|\nabla\phi_h^{n+1}\|_p^2 + \|\nabla\phi_h^n\|_p^2) - \frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 \\ & -\frac{1}{2\Delta t}\|u_h^{n+1} - u_h^n\|_f^2 - \frac{\nu}{8}(\|\nabla u_h^{n+1}\|_f^2 + \|\nabla u_h^n\|_f^2) - \frac{gk_{\min}}{4}\|\nabla\phi_h^{n+1}\|_p^2. \end{aligned}$$

Furthermore, a combination of Schwarz inequality and Poincaré inequality yields

$$
\begin{aligned}
&(\mathbf{f}_f^{n+1}, u_h^{n+1}) + g(f_p^{n+1}, \phi_h^{n+1}) \\
&\leq \frac{C_{P,f}^2}{\nu}\|\mathbf{f}_f^{n+1}\|_f^2 + \frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{k_{\min}}\|f_p^{n+1}\|_p^2 + \frac{gk_{\min}}{4}\|\nabla \phi_h^{n+1}\|_p^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|u_h^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|u_h^n\|_f^2 + \frac{gS_0}{2\Delta t}\|\phi_h^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_h^n\|_p^2 \\
&+ \frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 + \frac{gk_{\min}}{4}\|\nabla \phi_h^{n+1}\|_p^2 + \frac{\nu}{8}\left(\|\nabla u_h^{n+1}\|_f^2 - \|\nabla u_h^n\|_f^2\right) \\
&+ \frac{gk_{\min}}{8}\left(\|\nabla \phi_h^{n+1}\|_p^2 - \|\nabla \phi_h^n\|_p^2\right) \leq \frac{C_{P,f}^2}{\nu}\|\mathbf{f}_f^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{k_{\min}}\|f_p^{n+1}\|_p^2.
\end{aligned} \tag{3.7}
$$

Summing this up from $n = 0$ to $n = N-1$ results in

$$
\begin{aligned}
&\frac{1}{2}\|u_h^N\|_f^2 + \frac{gS_0}{2}\|\phi_h^N\|_p^2 + \frac{\Delta t\nu}{8}\|\nabla u_h^N\|_f^2 + \frac{\Delta t g k_{\min}}{8}\|\nabla \phi_h^N\|_p^2 \\
&\qquad + \Delta t\sum_{n=0}^{N-1}\left(\frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 + \frac{gk_{\min}}{4}\|\nabla \phi_h^{n+1}\|_p^2\right) \\
&\leq \frac{1}{2}\|u_h^0\|_f^2 + \frac{gS_0}{2}\|\phi_h^0\|_p^2 + \frac{\Delta t\nu}{8}\|\nabla u_h^0\|_f^2 + \frac{\Delta t g k_{\min}}{8}\|\nabla \phi_h^0\|_p^2 \\
&\qquad + \Delta t\sum_{n=0}^{N-1}\left(\frac{C_{P,f}^2}{\nu}\|\mathbf{f}_f^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{k_{\min}}\|f_p^{n+1}\|_p^2\right).
\end{aligned}
$$

For the second part, let

$$
Q(\Delta t) = \min\left\{\frac{2\nu\Delta t}{4C_{P,f}^2 + \nu\Delta t}, \frac{2k_{\min}\Delta t}{4S_0C_{P,p}^2 + k_{\min}\Delta t}\right\}.
$$

After simple calculations and applying Poincaré inequality

$$
\begin{aligned}
\frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 &\geq Q(\Delta t)\left(\frac{1}{2\Delta t}\|u_h^{n+1}\|_f^2 + \frac{\nu}{8}\|\nabla u_h^{n+1}\|_f^2\right), \\
\frac{gk_{\min}}{4}\|\nabla \phi_h^{n+1}\|_p^2 &\geq Q(\Delta t)\left(\frac{gS_0}{2\Delta t}\|\phi_h^{n+1}\|_p^2 + \frac{gk_{\min}}{8}\|\nabla \phi_h^{n+1}\|_p^2\right).
\end{aligned} \tag{3.8}
$$

Denote

$$
\begin{aligned}
s_{n+1} &= \frac{1}{2\Delta t}\|u_h^{n+1}\|_f^2 + \frac{\nu}{8}\|\nabla u_h^{n+1}\|_f^2 + \frac{gS_0}{2\Delta t}\|\phi_h^{n+1}\|_p^2 + \frac{gk_{\min}}{8}\|\nabla \phi_h^{n+1}\|_p^2 \\
\text{and}\quad P &= \frac{C_{P,f}^2}{\nu}\sup_{0\leq N\leq\infty}\|\mathbf{f}_f^{N+1}\|_f^2 + \frac{gC_{P,p}^2}{k_{\min}}\sup_{0\leq N\leq\infty}\|f_p^{N+1}\|_p^2.
\end{aligned}
$$

From (3.7) and (3.8), we have

$$
(1 + Q(\Delta t))s_{n+1} - s_n \leq P,
$$

which yields

$$
s_{n+1} \leq \frac{P}{Q(\Delta t)} + \frac{1}{(1 + Q(\Delta t))^{n+1}}s_0.
$$

Plugging the expression defining s_{n+1} in, it gives

$$\frac{1}{2\Delta t}\|u_h^{n+1}\|_f^2 + \frac{gS_0}{2\Delta t}\|\phi_h^{n+1}\|_p^2 \le \frac{P}{Q(\Delta t)} + \frac{1}{(1+Q(\Delta t))^{n+1}} s_0.$$

Hence

$$\begin{aligned}
&\|u_h^{n+1}\|_f^2 + gS_0\|\phi_h^{n+1}\|_p^2 \le \frac{2\Delta t P}{Q(\Delta t)} + 2\Delta t s_0 \\
&\le \max\left\{\frac{4C_{P,f}^2}{\nu} + \Delta t, \frac{4C_{P,p}^2 S_0}{k_{\min}} + \Delta t\right\} P + 2\Delta t s_0 \\
&\le \max\left\{\frac{4C_{P,f}^2}{\nu} + \Delta t_{BEFE}, \frac{4C_{P,p}^2 S_0}{k_{\min}} + \Delta t_{BEFE}\right\} P \\
&\quad + \|u_h^0\|_f^2 + \frac{\nu \Delta t_{BEFE}}{4}\|\nabla u_h^0\|_f^2 + gS_0\|\phi_h^0\|_p^2 + \frac{gk_{\min}\Delta t_{BEFE}}{4}\|\nabla \phi_h^0\|_p^2,
\end{aligned}$$

which proves (3.5).

Finally, if $\mathbf{f}_f \equiv 0$, $f_p \equiv 0$, from (3.4), the series

$$\sum_{n=0}^{\infty}\left(\frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 + \frac{gk_{\min}}{4}\|\nabla \phi_h^{n+1}\|_p^2\right)$$

is convergent and conclusion (3.6) follows. ▯

3.1.2. BELF Stability. As noted above, stability over bounded time intervals (allowing exponential growth in time) follows for BELF as for BEFE without any time step restriction. We thus turn to long time stability. Define

$$\Delta t_{BELF} := \frac{2\sqrt{\nu k_{\min}}\min\{1, gS_0\}}{\left[C(\Omega_f)C(\Omega_p)\right]^2 g^{3/2}}.$$

We prove unconditional (no relation needed coupling Δt and h) *uniform in time* stability of BELF (3.2) over $0 \le t < \infty$ under the time step condition

$$\Delta t < \Delta t_{BELF}. \tag{3.9}$$

Our experiments in Section 5 with $k_{\min} = 10^{-6}$ also suggest that the dependence of Δt_{BELF}, while better than for Δt_{BEFE}, is too pessimistic. However, the experiments are consistent with the analysis in that BELF is stable for smaller values of $k_{\min}$ than BEFE.

THEOREM 3.2. *(BELF uniform in t stability) Consider BELF method (3.2). Under the time step condition (3.9) above, it is uniformly in t stable over $0 \le t^n < \infty$:*

$$\begin{aligned}
&\|u_h^N\|_f^2 + \|u_h^{N-1}\|_f^2 + gS_0\|\phi_h^N\|_p^2 + gS_0\|\phi_h^{N-1}\|_p^2 \qquad (3.10)\\
&+ \Delta t \sum_{n=1}^{N-1}\left[\nu\|\nabla\left(u_h^{n+1} + u_h^{n-1}\right)\|_f^2 + gk_{\min}\|\nabla\left(\phi_h^{n+1} + \phi_h^{n-1}\right)\|_p^2\right] \\
&\le 4\Delta t \sum_{n=1}^{N-1}\left(\frac{C_{P,f}^2}{\nu}\left\|\mathbf{f}_f^{n+1}\right\|_f^2 + \frac{gC_{P,p}^2}{k_{\min}}\left\|f_p^{n+1}\right\|_p^2\right) + 2\|u_h^1\|_f^2 + 2\|u_h^0\|_f^2 \\
&+ 2gS_0\|\phi_h^1\|_p^2 + 2gS_0\|\phi_h^0\|_p^2 + 2\Delta t\left(a_f(u_h^1, u_h^1) + a_p(\phi_h^1, \phi_h^1) + a_f(u_h^0, u_h^0) + a_p(\phi_h^0, \phi_h^0)\right) \\
&+ 4\Delta t g\int_I \left(\phi_h^0 u_h^1 \cdot \widehat{n}_f - \phi_h^1 u_h^0 \cdot \widehat{n}_f\right) ds.
\end{aligned}$$

Proof. Define

$$E^n := \frac{1}{2}\|u_h^n\|_f^2 + \frac{gS_0}{2}\|\phi_h^n\|_p^2.$$

In (3.2) set $v_h = u_h^{n+1} + u_h^{n-1}, q_h = p_h^{n+1}$, $\psi_h = \phi_h^{n+1} + \phi_h^{n-1}$ respectively and add[1]. This gives

$$\begin{aligned}\frac{1}{\Delta t}\left(E^{n+1} - E^{n-1}\right) + a_f(u_h^{n+1}, u_h^{n+1} + u_h^{n-1}) + a_p(\phi_h^{n+1}, \phi_h^{n+1} + \phi_h^{n-1})\\ + c_I(u_h^{n+1} + u_h^{n-1}, \phi_h^n) - c_I(u_h^n, \phi_h^{n+1} + \phi_h^{n-1})\\ = (\mathbf{f}_f^{n+1}, u_h^{n+1} + u_h^{n-1})_f + g(f_p^{n+1}, \phi_h^{n+1} + \phi_h^{n-1})_p.\end{aligned}$$

Since $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$ are symmetric we have

$$\begin{aligned}a_f(u_h^{n+1}, u_h^{n+1} + u_h^{n-1}) &= \frac{1}{2}a_f(u_h^{n+1}, u_h^{n+1}) - \frac{1}{2}a_f(u_h^{n-1}, u_h^{n-1})\\ &\quad+\frac{1}{2}a_f(u_h^{n+1} + u_h^{n-1}, u_h^{n+1} + u_h^{n-1}),\\ a_p(\phi_h^{n+1}, \phi_h^{n+1} + \phi_h^{n-1}) &= \frac{1}{2}a_p(\phi_h^{n+1}, \phi_h^{n+1}) - \frac{1}{2}a_p(\phi_h^{n-1}, \phi_h^{n-1})\\ &\quad+\frac{1}{2}a_p(\phi_h^{n+1} + \phi_h^{n-1}, \phi_h^{n+1} + \phi_h^{n-1}).\end{aligned} \tag{3.11}$$

Let us denote

$$\begin{aligned}A^n &= \frac{1}{2}a_f(u_h^n, u_h^n) + \frac{1}{2}a_p(\phi_h^n, \phi_h^n),\\ B^n &= \frac{1}{2}a_f(u_h^{n+1} + u_h^{n-1}, u_h^{n+1} + u_h^{n-1}) + \frac{1}{2}a_p(\phi_h^{n+1} + \phi_h^{n-1}, \phi_h^{n+1} + \phi_h^{n-1}),\\ C^{n+1/2} &= c_I(u_h^{n+1}, \phi_h^n) - c_I(u_h^n, \phi_h^{n+1}).\end{aligned}$$

Adding and subtracting E^n and $\Delta t A^n$ in the first two terms below and rearranging the remainder gives

$$\begin{aligned}&\left[E^{n+1} + E^n + \Delta t A^{n+1} + \Delta t A^n + \Delta t C^{n+1/2}\right]\\ &-\left[E^{n-1} + E^n + \Delta t A^n + \Delta t A^{n-1} + \Delta t C^{n-1/2}\right]\\ &+\Delta t B^n = \Delta t\Big((\mathbf{f}_f^{n+1}, u_h^{n+1} + u_h^{n-1})_f + g(f_p^{n+1}, \phi_h^{n+1} + \phi_h^{n-1})_p\Big).\end{aligned}$$

Summing this up from $n = 1$ to $n = N - 1$ results in

$$\begin{aligned}&E^N + E^{N-1} + \Delta t(A^N + A^{N-1}) + \Delta t C^{N-1/2} + \Delta t\sum_{n=1}^{N-1} B^n\\ &= E^1 + E^0 + \Delta t(A^1 + A^0) + \Delta t C^{1-1/2}\\ &\quad+\Delta t\sum_{n=1}^{N-1}(f_f^{n+1}, u_h^{n+1} + u_h^{n-1})_f + \Delta t\sum_{n=1}^{N-1} g(f_p^{n+1}, \phi_h^{n+1} + \phi_h^{n-1})_p.\end{aligned}$$

[1] This first step already diverges from the normal method of analyzing stability of the implicit method.

We have already shown that

$$A^n \geq \frac{\nu}{2}\|\nabla u_h^n\|_f^2 + \frac{gk_{\min}}{2}\|\nabla \phi_h^n\|_p^2,$$
$$B^n \geq \frac{\nu}{2}\|\nabla\left(u_h^{n+1}+u_h^{n-1}\right)\|_f^2 + \frac{gk_{\min}}{2}\|\nabla\left(\phi_h^{n+1}+\phi_h^{n-1}\right)\|_p^2.$$

Applying (2.2) with $\varepsilon_1 = \frac{1}{2}$ yields

$$C^{n-1/2} \geq -\frac{\nu}{2}\|\nabla u_h^n\|_f^2 - \frac{\nu}{2}\|\nabla u_h^{n-1}\|_f^2 - \frac{gk_{\min}}{2}\|\nabla \phi_h^n\|_p^2 - \frac{gk_{\min}}{2}\|\nabla \phi_h^{n-1}\|_p^2$$
$$-\frac{1}{2}C_1\left(\|u_h^n\|_f^2 + \|u_h^{n-1}\|_f^2 + \|\phi_h^n\|_p^2 + \|\phi_h^{n-1}\|_p^2\right).$$

From Schwarz inequality and Poincaré inequality

$$(\mathbf{f}_f^{n+1}, u_h^{n+1}+u_h^{n-1})_f \leq \frac{C_{P,f}^2}{\nu}\|\mathbf{f}_f^{n+1}\|_f^2 + \frac{\nu}{4}\|\nabla\left(u_h^{n+1}+u_h^{n-1}\right)\|_f^2,$$
$$g(f_p^{n+1}, \phi_h^{n+1}+\phi_h^{n-1})_p \leq \frac{gC_{P,p}^2}{k_{\min}}\left\|f_p^{n+1}\right\|_p^2 + \frac{gk_{\min}}{4}\|\nabla(\phi_h^{n+1}+\phi_h^{n-1})\|_p^2.$$

Applying the last five inequalities into the energy estimate, summing and rearranging terms yield the result. ☐

Interestingly, our proof of asymptotic stability of BELF requires a different additional time step condition. We impose the following time step condition

$$\Delta t \leq \min\left\{\frac{k_{\min}}{C_{P,p}^2}, \frac{S_0\nu}{C_{P,f}^2}\right\}\frac{\nu k_{\min}}{4g^2(C(\Omega_f)C(\Omega_p))^4} \tag{3.12}$$

to prove asymptotic stability over $0 \leq t < \infty$.

THEOREM 3.3. *(BELF asymptotic stability) Consider BELF method* (3.2). *Under the time step conditions* (3.9) *and* (3.12)*,*

$$\|u_h^N\|_f^2 + gS_0\|\phi_h^N\|_p^2 + \Delta t\sum_{n=1}^{N-1}\nu\|\nabla u_h^{n+1}\|_f^2 + \Delta t\sum_{n=1}^{N-1}gk_{\min}\|\nabla\phi_h^{n+1}\|_p^2 \tag{3.13}$$
$$\leq 2\|u_h^1\|_f^2 + 2gS_0\|\phi_h^1\|_p^2 + 2\|u_h^0\|_f^2 + 2gS_0\|\phi_h^0\|_p^2 + 4\Delta tg\int_I(\phi_h^0 u_h^1\cdot\widehat{n}_f - \phi_h^1 u_h^0\cdot\widehat{n}_f)ds$$
$$+\Delta t\nu(2\|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2) + \Delta tgk_{\min}(2\|\nabla\phi_h^1\|_p^2 + \|\nabla\phi_h^0\|_p^2)$$
$$+4\Delta t\sum_{n=1}^{N-1}\frac{C_{P,f}^2}{\nu}\|\mathbf{f}_f^{n+1}\|_f^2 + 4\Delta t\sum_{n=1}^{N-1}\frac{gC_{P,p}^2}{k_{\min}}\|f_p^{n+1}\|_p^2.$$

As a consequence, if $\mathbf{f}_f \equiv 0, f_p \equiv 0$ *then*

$$u_h^N \to 0, \phi_h^N \to 0 \tag{3.14}$$

in $H^1(\Omega_f)$ *and* $H^1(\Omega_p)$ *respectively as* $N \to \infty$.

Proof. In (3.2) set $v_h = u_h^{n+1}, q_h = p_h^{n+1}$, $\psi_h = \phi_h^{n+1}$ respectively and add. This gives

$$\frac{1}{2\Delta t}\left(E^{n+1}-E^{n-1}\right) + \frac{1}{4\Delta t}\|u_h^{n+1}-u_h^{n-1}\|_f^2 + \frac{gS_0}{4\Delta t}\|\phi_h^{n+1}-\phi_h^{n-1}\|_p^2$$
$$+\nu\|\nabla u_h^{n+1}\|_f^2 + gk_{\min}\|\nabla\phi_h^{n+1}\|_p^2 + c_I(u_h^{n+1},\phi_h^n) - c_I(u_h^n,\phi_h^{n+1}) \tag{3.15}$$
$$\leq (\mathbf{f}_f^{n+1}, u_h^{n+1})_f + g(f_p^{n+1},\phi_h^{n+1})_p.$$

Write

$$c_I(u_h^{n+1},\phi_h^n) = \frac{1}{2}c_I(u_h^{n+1},\phi_h^n) + \frac{1}{2}c_I(u_h^{n-1},\phi_h^n) + \frac{1}{2}c_I(u_h^{n+1}-u_h^{n-1},\phi_h^n),$$
$$c_I(u_h^{n},\phi_h^{n+1}) = \frac{1}{2}c_I(u_h^{n},\phi_h^{n+1}) + \frac{1}{2}c_I(u_h^{n},\phi_h^{n-1}) + \frac{1}{2}c_I(u_h^{n},\phi_h^{n+1}-\phi_h^{n-1}).$$

Let us denote $C^{n+1/2} = c_I(u_h^{n+1},\phi_h^n) - c_I(u_h^n,\phi_h^{n+1})$. Multiplying (3.15) by 2, adding and subtracting E^n and rearranging the remainder gives

$$\begin{aligned}
&\frac{1}{\Delta t}\left[E^{n+1}+E^n+\Delta t C^{n+1/2}\right] - \frac{1}{\Delta t}\left[E^n+E^{n-1}+\Delta t C^{n-1/2}\right]\\
&+\frac{1}{2\Delta t}\|u_h^{n+1}-u_h^{n-1}\|_f^2 + \frac{gS_0}{2\Delta t}\|\phi_h^{n+1}-\phi_h^{n-1}\|_p^2 + 2\nu\|\nabla u_h^{n+1}\|_f^2 + 2gk_{\min}\|\nabla\phi_h^{n+1}\|_p^2\\
&+c_I(u_h^{n+1}-u_h^{n-1},\phi_h^n) - c_I(u_h^n,\phi_h^{n+1}-\phi_h^{n-1})\\
&\qquad\le\ 2\Big(\ (\mathbf{f}_f^{n+1},u_h^{n+1})_f + g(f_p^{n+1},\phi_h^{n+1})_p\Big).
\end{aligned}$$

Applying (2.3) and (2.4) with $\varepsilon_2 = \frac{\Delta t}{2gS_0}$ and $\varepsilon_3 = \frac{\Delta t}{2}$ we have

$$\begin{aligned}
&c_I(u_h^{n+1}-u_h^{n-1},\phi_h^n) - c_I(u_h^n,\phi_h^{n+1}-\phi_h^{n-1})\\
\ge&-\frac{gS_0}{2\Delta t}\|\phi_h^{n+1}-\phi_h^{n-1}\|_p^2 - \frac{\Delta t}{2gS_0}C_2\|\nabla(\phi_h^{n+1}-\phi_h^{n-1})\|_p^2 - \frac{\nu}{4}\|\nabla u_h^n\|_f^2\\
&-\frac{1}{2\Delta t}\|u_h^{n+1}-u_h^{n-1}\|_f^2 - \frac{\Delta t}{2}C_3\|\nabla(u_h^{n+1}-u_h^{n-1})\|_f^2 - \frac{gk_{\min}}{4}\|\nabla\phi_h^n\|_p^2\\
\ge&-\frac{gS_0}{2\Delta t}\|\phi_h^{n+1}-\phi_h^{n-1}\|_p^2 - \frac{\Delta t}{gS_0}C_2(\|\nabla\phi_h^{n+1}\|_p^2+\|\nabla\phi_h^{n-1}\|_p^2) - \frac{\nu}{4}\|\nabla u_h^n\|_f^2\\
&-\frac{1}{2\Delta t}\|u_h^{n+1}-u_h^{n-1}\|_f^2 - \Delta t C_3(\|\nabla u_h^{n+1}\|_f^2+\|\nabla u_h^{n-1}\|_f^2) - \frac{gk_{\min}}{4}\|\nabla\phi_h^n\|_p^2\\
\ge&-\frac{gS_0}{2\Delta t}\|\phi_h^{n+1}-\phi_h^{n-1}\|_p^2 - \frac{gk_{\min}}{4}(\|\nabla\phi_h^{n+1}\|_p^2+\|\nabla\phi_h^{n-1}\|_p^2) - \frac{\nu}{4}\|\nabla u_h^n\|_f^2\\
&-\frac{1}{2\Delta t}\|u_h^{n+1}-u_h^{n-1}\|_f^2 - \frac{\nu}{4}(\|\nabla u_h^{n+1}\|_f^2+\|\nabla u_h^{n-1}\|_f^2) - \frac{gk_{\min}}{4}\|\nabla\phi_h^n\|_p^2.
\end{aligned}$$

From Schwarz inequality and Poincaré inequality

$$2(\mathbf{f}_f^{n+1},u_h^{n+1})_f \le \frac{C_{P,f}^2}{\nu}\|\mathbf{f}_f^{n+1}\|_f^2 + \nu\|\nabla u_h^{n+1}\|_f^2,$$
$$2g(f_p^{n+1},\phi_h^{n+1})_p \le \frac{gC_{P,p}^2}{k_{\min}}\left\|f_p^{n+1}\right\|_p^2 + gk_{\min}\|\nabla\phi_h^{n+1}\|_p^2.$$

Thus

$$\begin{aligned}
&\frac{1}{\Delta t}\left[E^{n+1}+E^n+\Delta t C^{n+1/2}\right] - \frac{1}{\Delta t}\left[E^n+E^{n-1}+\Delta t C^{n-1/2}\right]\\
&+\frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 + \frac{\nu}{4}(\|\nabla u_h^{n+1}\|_f^2-\|\nabla u_h^n\|_f^2) + \frac{\nu}{4}(\|\nabla u_h^{n+1}\|_f^2-\|\nabla u_h^{n-1}\|_f^2)\\
&+\frac{gk_{\min}}{4}\|\nabla\phi_h^{n+1}\|_p^2 + \frac{gk_{\min}}{4}(\|\nabla\phi_h^{n+1}\|_p^2-\|\nabla\phi_h^n\|_p^2) + \frac{gk_{\min}}{4}(\|\nabla\phi_h^{n+1}\|_p^2-\|\nabla\phi_h^{n-1}\|_p^2)\\
&\qquad\le\ \frac{C_{P,f}^2}{\nu}\|\mathbf{f}_f^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{k_{\min}}\left\|f_p^{n+1}\right\|_p^2.
\end{aligned}$$

Summing this up from $n=1$ to $n=N-1$ results in

$$
\begin{aligned}
&E^N + E^{N-1} + \Delta t C^{N-1/2} + \Delta t \sum_{n=1}^{N-1} \frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 + \Delta t \sum_{n=1}^{N-1} \frac{gk_{\min}}{4}\|\nabla \phi_h^{n+1}\|_p^2 \\
&\quad + \frac{\Delta t\nu}{4}(2\|\nabla u_h^N\|_f^2 + \|\nabla u_h^{N-1}\|_f^2) + \frac{\Delta t g k_{\min}}{4}(2\|\nabla \phi_h^N\|_p^2 + \|\nabla \phi_h^{N-1}\|_p^2) \\
&\le E^1 + E^0 + \Delta t C^{1/2} + \frac{\Delta t\nu}{4}(2\|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2) + \frac{\Delta t g k_{\min}}{4}(2\|\nabla \phi_h^1\|_p^2 + \|\nabla \phi_h^0\|_p^2) \\
&\quad + \Delta t \sum_{n=1}^{N-1} \frac{C_{P,f}^2}{\nu}\|f_f^{n+1}\|_f^2 + \Delta t \sum_{n=1}^{N-1} \frac{gC_{P,p}^2}{k_{\min}}\|f_p^{n+1}\|_p^2.
\end{aligned}
$$

Applying (2.2) with $\varepsilon_1 = \frac{1}{2}$ and $\varepsilon_1 = 1$, under the time step condition (3.9) we have

$$
\begin{aligned}
C^{N-1/2} &\ge -\frac{\nu}{2}\|\nabla u_h^N\|_f^2 - \frac{\nu}{4}\|\nabla u_h^{N-1}\|_f^2 - \frac{gk_{\min}}{2}\|\nabla \phi_h^N\|_p^2 - \frac{gk_{\min}}{4}\|\nabla \phi_h^{N-1}\|_p^2 \\
&\quad - \frac{C_1}{2}\left(\|u_h^N\|_f^2 + \|\phi_h^N\|_p^2\right) - C_1(\|u_h^{N-1}\|_f^2 + \|\phi_h^{N-1}\|_p^2) \\
&\ge -\frac{\nu}{2}\|\nabla u_h^N\|_f^2 - \frac{\nu}{4}\|\nabla u_h^{N-1}\|_f^2 - \frac{gk_{\min}}{2}\|\nabla \phi_h^N\|_p^2 - \frac{gk_{\min}}{4}\|\nabla \phi_h^{N-1}\|_p^2 \\
&\quad - \frac{1}{4\Delta t}\|u_h^N\|_f^2 - \frac{gS_0}{4\Delta t}\|\phi_h^N\|_p^2 - \frac{1}{2\Delta t}\|u_h^{N-1}\|_f^2 - \frac{gS_0}{2\Delta t}\|\phi_h^{N-1}\|_p^2.
\end{aligned}
$$

Applying this inequality into the energy estimate, it gives

$$
\begin{aligned}
&\frac{1}{4}\|u_h^N\|_f^2 + \frac{gS_0}{4}\|\phi_h^N\|_p^2 + \Delta t \sum_{n=1}^{N-1} \frac{\nu}{4}\|\nabla u_h^{n+1}\|_f^2 + \Delta t \sum_{n=1}^{N-1} \frac{gk_{\min}}{4}\|\nabla \phi_h^{n+1}\|_p^2 \\
&\le \frac{1}{2}\|u_h^1\|_f^2 + \frac{gS_0}{2}\|\phi_h^1\|_p^2 + \frac{1}{2}\|u_h^0\|_f^2 + \frac{gS_0}{2}\|\phi_h^0\|_p^2 + \Delta t g \int_I (\phi_h^0 u_h^1 \cdot \widehat{n}_f - \phi_h^1 u_h^0 \cdot \widehat{n}_f) ds \\
&\quad + \frac{\Delta t\nu}{4}(2\|\nabla u_h^1\|_f^2 + \|\nabla u_h^0\|_f^2) + \frac{\Delta t g k_{\min}}{4}(2\|\nabla \phi_h^1\|_p^2 + \|\nabla \phi_h^0\|_p^2) \\
&\quad + \Delta t \sum_{n=1}^{N-1} \frac{C_{P,f}^2}{\nu}\|\mathbf{f}_f^{n+1}\|_f^2 + \Delta t \sum_{n=1}^{N-1} \frac{gC_{P,p}^2}{k_{\min}}\|f_p^{n+1}\|_p^2.
\end{aligned}
$$

Rearranging terms yields (3.13).

If $\mathbf{f}_f \equiv 0$ and $f_p \equiv 0$, the second assertion of Theorem 3.3 follows the convergence of series

$$
\sum_{n=1}^{\infty} \nu\|\nabla u_h^{n+1}\|_f^2 + \sum_{n=1}^{\infty} gk_{\min}\|\nabla \phi_h^{n+1}\|_p^2.
$$

□

4. Error analysis over $1 \le t_n < \infty$. We proceed to analyze the error over long time intervals. Recall that our FEM spaces are assumed to satisfy the usual approximation properties and the Stokes velocity-pressure spaces satisfy the discrete inf-sup condition. For compactness, we only analyze the error of the newer method BELF. With minor modifications in our proof, we will get the analogous results of convergence rates and long time behaviour for BEFE. Recall that the discretely divergence free velocities by

$$
V^h := X_f^h \cap \{v_h : (q_h, \nabla \cdot v_h)_f = 0, \text{ for all } q_h \in Q_f^h\}
$$

Let $t^n = n\Delta t$ and $T = N\Delta t$ (if $T = \infty$ then $N = \infty$). Also denote $u^n := u(t^n)$ (and similarly for other variables).

In order to establish the optimal error estimates for the approximation we need to assume the following regularity of the true solution:

$$\begin{aligned} &u \in L^\infty(0,T;H^{k+1}(\Omega_f)) \cap H^1(0,T;H^{k+1}(\Omega_f)) \cap H^2(0,T;L^2(\Omega_f)),\\ &\phi \in L^\infty(0,T;H^{k+1}(\Omega_p)) \cap H^1(0,T;H^{k+1}(\Omega_p)) \cap H^2(0,T;L^2(\Omega_p)),\\ &p \in L^2(0,T;H^{s+1}(\Omega_f)). \end{aligned} \tag{4.1}$$

We define the following abbreviations

$$\|v\|_{p,k,r} := \|v\|_{L^p(0,T;H^k(\Omega_r))} \qquad r \in \{f,p\}$$

and introduce discrete norms

$$\begin{aligned} |||v|||_{\infty,k,r} &:= \max_{0\le n\le N} \|v^n\|_{H^k(\Omega_r)},\\ |||v|||_{2,k,r} &:= \left(\sum_{n=0}^{N} \|v^n\|^2_{H^k(\Omega_r)}\Delta t\right)^{1/2}, \qquad r \in \{f,p\}. \end{aligned}$$

Denote the errors by $e_f^n := u^n - u_h^n$, $e_p^n := \phi^n - \phi_h^n$. The variational formulation of the continuous problem is first rewritten as the discrete problem driven by consistency errors as

$$\begin{aligned} &(\frac{u^{n+1}-u^{n-1}}{2\Delta t}, v_h)_f + a_f(u^{n+1}, v_h) - (p^{n+1} - \lambda_h^{n+1}, \nabla\cdot v_h)_f + c_I(v_h,\phi^n) =\\ &\qquad = (\mathbf{f}_f^{n+1}, v_h)_f + \varepsilon_f^{n+1}(v_h),\\ &gS_0(\frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t}, \psi_h)_p + a_p(\phi^{n+1},\psi_h) - c_I(u^n,\psi_h) =\\ &\qquad = g(f_p^{n+1},\psi_h)_p + \varepsilon_p^{n+1}(\psi_h),\\ &\qquad \text{for all } v_h \in V^h, \quad \psi_h \in X_p^h \text{ and any } \lambda_h^{n+1} \in Q_f^h. \end{aligned} \tag{4.2}$$

The consistency errors, $\varepsilon_f^{n+1}(v_h), \varepsilon_p^{n+1}(\psi_h)$, are defined, as usual, by

$$\begin{aligned} \varepsilon_f^{n+1}(v_h) &:= (\frac{u^{n+1}-u^{n-1}}{2\Delta t} - u_t^{n+1}, v_h)_f + c_I(v_h, \phi^n - \phi^{n+1}),\\ \varepsilon_p^{n+1}(\psi_h) &:= gS_0(\frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t} - \phi_t^{n+1}, \psi_h)_p - c_I(u^n - u^{n+1}, \psi_h). \end{aligned}$$

Subtraction gives the error equations:

$$\begin{aligned} &(\frac{e_f^{n+1}-e_f^{n-1}}{2\Delta t}, v_h)_f + a_f(e_f^{n+1}, v_h) + c_I(v_h, e_p^n) = \varepsilon_f^{n+1}(v_h) + (p^{n+1} - \lambda_h^{n+1}, \nabla\cdot v_h)_f,\\ &gS_0(\frac{e_p^{n+1}-e_p^{n-1}}{2\Delta t}, \psi_h)_p + a_p(e_p^{n+1},\psi_h) - c_I(e_f^n,\psi_h) = \varepsilon_p^{n+1}(\psi_h),\\ &\qquad \text{for all } v_h \in V^h, \quad \psi_h \in X_p^h \text{ and any } \lambda_h^{n+1} \in Q_f^h. \end{aligned} \tag{4.3}$$

THEOREM 4.1. *Consider BELF method* (3.2). *Suppose the time step condition* (3.9) *holds and* u *and* ϕ *satisfy regularity condition* (4.1). *Then, for any* $0 \le t_N < \infty$, *there is a positive constant* C_0 *such that*

$$\begin{aligned}
&\|e_f^N\|_f^2 + \|e_f^{N-1}\|_f^2 + S_0\|e_p^N\|_p^2 + S_0\|e_p^{N-1}\|_p^2 \\
&+ \Delta t \sum_{n=1}^{N-1} \left[\nu \left\| \nabla(e_f^{n+1} + e_f^{n-1}) \right\|_f^2 + k_{\min} \left\| \nabla(e_p^{n+1} + e_p^{n-1}) \right\|_p^2 \right] \\
&\leq C_0 \Bigg\{ \left\|u^1 - u_h^1\right\|_f^2 + \left\|u^0 - u_h^0\right\|_f^2 + S_0 \left\|\phi^1 - \phi_h^1\right\|_p^2 + S_0 \left\|\phi^0 - \phi_h^0\right\|_p^2 \\
&+ \Delta t \Big(\left\|\nabla\left(u^1 - u_h^1\right)\right\|_f^2 + \left\|\nabla\left(u^0 - u_h^0\right)\right\|_f^2 + \left\|\nabla\left(\phi^1 - \phi_h^1\right)\right\|_p^2 + \left\|\nabla\left(\phi^0 - \phi_h^0\right)\right\|_p^2 \Big) \\
&+ h^{2k} \left(|||u|||_{\infty,k+1,f}^2 + |||\phi|||_{\infty,k+1,p}^2 \right) + h^{2k+2} \left(\|u_t\|_{2,k+1,f}^2 + \|\phi_t\|_{2,k+1,p}^2 \right) \\
&+ \Delta t^2 \left(\|u_{tt}\|_{2,0,f}^2 + \|\phi_{tt}\|_{2,0,p}^2 \right) + \Delta t^2 \left(\|u_t\|_{2,1,f}^2 + \|\phi_t\|_{2,1,p}^2 \right) \\
&+ h^{2k} \left(|||u|||_{2,k+1,f}^2 + |||\phi|||_{2,k+1,p}^2 \right) + h^{2s+2} |||p|||_{2,s+1,f}^2 \Bigg\}.
\end{aligned} \tag{4.4}$$

Proof. Let U^{n+1}, Φ^{n+1} be the interpolation of u^{n+1} and ϕ^{n+1} in V^h and X_p^h correspondingly. Denote

$$\begin{aligned}
e_f^{n+1} &= (u^{n+1} - U^{n+1}) + (U^{n+1} - u_h^{n+1}) =: \eta_f^{n+1} + \xi_f^{n+1}, \\
e_p^{n+1} &= (\phi^{n+1} - \Phi^{n+1}) + (\Phi^{n+1} - \phi_h^{n+1}) =: \eta_p^{n+1} + \xi_p^{n+1}.
\end{aligned}$$

Rearranging terms of the error equations (4.3) gives

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\xi_f^{n+1} - \xi_f^{n-1}, v_h\right)_f + a_f(\xi_f^{n+1}, v_h) + c_I(v_h, \xi_p^n) = -\frac{1}{2\Delta t}\left(\eta_f^{n+1} - \eta_f^{n-1}, v_h\right)_f \\
&\qquad - a_f(\eta_f^{n+1}, v_h) - c_I(v_h, \eta_p^n) + \varepsilon_f^{n+1}(v_h) + (p^{n+1} - \lambda_h^{n+1}, \nabla \cdot v_h)_f, \\
&\frac{gS_0}{2\Delta t}\left(\xi_p^{n+1} - \xi_p^{n-1}, \psi_h\right)_p + a_p(\xi_p^{n+1}, \psi_h) - c_I(\xi_f^n, \psi_h) = -\frac{gS_0}{2\Delta t}\left(\eta_p^{n+1} - \eta_p^{n-1}, \psi_h\right)_p \\
&\qquad - a_p(\eta_p^{n+1}, \psi_h) + c_I(\eta_f^n, \psi_h) + \varepsilon_p^{n+1}(\psi_h),
\end{aligned} \tag{4.5}$$

for every $v_h \in V_h, \psi_h \in X_p^h$ and $\lambda_h^{n+1} \in Q_f^h$.

Choosing $v_h = \xi_f^{n+1} + \xi_f^{n-1}$ and $\psi_h = \xi_p^{n+1} + \xi_p^{n-1}$ and denoting

$$\begin{aligned}
\mathcal{A}^n &= \frac{1}{2} a_f(\xi_f^n, \xi_f^n) + \frac{1}{2} a_p(\xi_p^n, \xi_p^n), \\
\mathcal{B}^n &= \frac{1}{2} a_f(\xi_f^{n+1} + \xi_f^{n-1}, \xi_f^{n+1} + \xi_f^{n-1}) + \frac{1}{2} a_p(\xi_p^{n+1} + \xi_p^{n-1}, \xi_p^{n+1} + \xi_p^{n-1}), \\
\mathcal{C}^{n+1/2} &= c_I(\xi_f^{n+1}, \xi_p^n) - c_I(\xi_f^n, \xi_p^{n+1}), \\
\mathcal{E}^n &= \frac{1}{2}\|\xi_f^n\|_f^2 + \frac{gS_0}{2}\|\xi_p^n\|_p^2.
\end{aligned}$$

After adding (4.5) sides by sides

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\mathcal{E}^{n+1}-\mathcal{E}^{n-1}\right)+\mathcal{A}^{n+1}-\mathcal{A}^{n-1}+\mathcal{B}^{n}+\mathcal{C}^{n+1/2}-\mathcal{C}^{n-1/2}\\
&= -\frac{1}{2\Delta t}\left(\eta_f^{n+1}-\eta_f^{n-1},\xi_f^{n+1}+\xi_f^{n-1}\right)_f-\frac{gS_0}{2\Delta t}\left(\eta_p^{n+1}-\eta_p^{n-1},\xi_p^{n+1}+\xi_p^{n-1}\right)_p\\
&-a_f(\eta_f^{n+1},\xi_f^{n+1}+\xi_f^{n-1})-a_p(\eta_p^{n+1},\xi_p^{n+1}+\xi_p^{n-1})-c_I(\xi_f^{n+1}+\xi_f^{n-1},\eta_p^{n})\\
&+c_I(\eta_f^{n},\xi_p^{n+1}+\xi_p^{n-1})+\varepsilon_f^{n+1}(\xi_f^{n+1}+\xi_f^{n-1})+\varepsilon_p^{n+1}(\xi_p^{n+1}+\xi_p^{n-1})\\
&+(p^{n+1}-\lambda_h^{n+1},\nabla\cdot(\xi_f^{n+1}+\xi_f^{n-1}))_f.
\end{aligned} \tag{4.6}$$

Now we start to bound the right hand side of (4.6). First,

$$\begin{aligned}
&-\frac{1}{2\Delta t}\left(\eta_f^{n+1}-\eta_f^{n-1},\xi_f^{n+1}+\xi_f^{n-1}\right)_f-\frac{gS_0}{2\Delta t}\left(\eta_p^{n+1}-\eta_p^{n-1},\xi_p^{n+1}+\xi_p^{n-1}\right)_p\\
&\le \frac{4C_{P,f}^2}{\nu}\left\|\frac{\eta_f^{n+1}-\eta_f^{n-1}}{2\Delta t}\right\|_f^2+\frac{\nu}{20}\|\nabla(\xi_f^{n+1}+\xi_f^{n-1})\|_f^2\\
&+\frac{4C_{P,p}^2 gS_0^2}{k_{\min}}\left\|\frac{\eta_p^{n+1}-\eta_p^{n-1}}{2\Delta t}\right\|_p^2+\frac{gk_{\min}}{16}\left\|\nabla\left(\xi_p^{n+1}+\xi_p^{n-1}\right)\right\|_p^2.
\end{aligned} \tag{4.7}$$

The next terms can be controlled as follows

$$\begin{aligned}
&-a_f(\eta_f^{n+1},\xi_f^{n+1}+\xi_f^{n-1})-a_p(\eta_p^{n+1},\xi_p^{n+1}+\xi_p^{n-1})\\
&\le C_0\left(\|\nabla\eta_f^{n+1}\|_f^2+\|\nabla\eta_p^{n+1}\|_p^2\right)+\frac{\nu}{20}\|\nabla\left(\xi_f^{n+1}+\xi_f^{n-1}\right)\|_f^2\\
&+\frac{gk_{\min}}{16}\|\nabla\left(\xi_p^{n+1}+\xi_p^{n-1}\right)\|_p^2.
\end{aligned} \tag{4.8}$$

Using the trace inequality, Young's inequality and Poincare's inequality, we obtain

$$\begin{aligned}
&-c_I(\xi_f^{n+1}+\xi_f^{n-1},\eta_p^{n})+c_I(\eta_f^{n},\xi_p^{n+1}+\xi_p^{n-1})\\
&\le C_0\|\nabla\eta_p^{n}\|_p^2+C_0\|\nabla\eta_f^{n}\|_f^2+\frac{\nu}{20}\|\nabla\left(\xi_f^{n+1}+\xi_f^{n-1}\right)\|_f^2\\
&+\frac{gk_{\min}}{16}\|\nabla\left(\xi_p^{n+1}+\xi_p^{n-1}\right)\|_p^2.
\end{aligned} \tag{4.9}$$

The consistency errors are bounded as follows:

$$\begin{aligned}
|\varepsilon_f^{n+1}(\xi_f^{n+1}+\xi_f^{n-1})| &\le C_0\left\|u_t^{n+1}-\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right\|_f^2+C_0\|\nabla(\phi^{n+1}-\phi^{n})\|_p^2\\
&\quad+\frac{\nu}{20}\|\nabla(\xi_f^{n+1}+\xi_f^{n-1})\|_f^2,\\
|\varepsilon_p^{n+1}(\xi_p^{n+1}+\xi_p^{n-1})| &\le C_0\left\|\phi_t^{n+1}-\frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t}\right\|_p^2+C_0\|\nabla(u^{n+1}-u^{n})\|_f^2\\
&\quad+\frac{gk_{\min}}{16}\|\nabla(\xi_p^{n+1}+\xi_p^{n-1})\|_p^2.
\end{aligned} \tag{4.10}$$

Lastly, we bound the pressure term by

$$(p^{n+1}-\lambda_h^{n+1},\nabla\cdot(\xi_f^{n+1}+\xi_f^{n-1}))_f\le C_0\|p^{n+1}-\lambda_h^{n+1}\|_f^2+\frac{\nu}{20}\|\nabla(\xi_f^{n+1}+\xi_f^{n-1})\|_f^2. \tag{4.11}$$

A combination of estimates (4.6)–(4.11) gives

$$
\begin{aligned}
&\frac{1}{\Delta t}\left(\mathcal{E}^{n+1}-\mathcal{E}^{n-1}\right)+\mathcal{A}^{n+1}-\mathcal{A}^{n-1}+\mathcal{B}^{n}+\mathcal{C}^{n+1/2}-\mathcal{C}^{n-1/2}\\
&\leq \frac{4C_{P,f}^2}{\nu}\left\|\frac{\eta_f^{n+1}-\eta_f^{n-1}}{2\Delta t}\right\|_f^2+\frac{4C_{P,p}^2 gS_0^2}{k_{\min}}\left\|\frac{\eta_p^{n+1}-\eta_p^{n-1}}{2\Delta t}\right\|_p^2\\
&+C_0\left\|u_t^{n+1}-\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right\|_f^2+C_0\left\|\phi_t^{n+1}-\frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t}\right\|_p^2 \qquad (4.12)\\
&+C_0\left(\|\nabla\eta_f^{n+1}\|_f^2+\|\nabla\eta_f^{n}\|_f^2+\|\nabla\eta_p^{n+1}\|_p^2+\|\nabla\eta_p^{n}\|_p^2\right)\\
&+C_0\left(\|\nabla(u^{n+1}-u^n)\|_f^2+\|\nabla(\phi^{n+1}-\phi^n)\|_p^2\right)+C_0\|p^{n+1}-\lambda_h^{n+1}\|_f^2\\
&+\frac{\nu}{4}\|\nabla\left(\xi_f^{n+1}+\xi_f^{n-1}\right)\|_f^2+\frac{gk_{\min}}{4}\|\nabla\left(\xi_p^{n+1}+\xi_p^{n-1}\right)\|_p^2.
\end{aligned}
$$

Similar to Theorem 3.2, summing (4.12) from $n=1$ to $n=N-1$ and estimating

$$
\begin{aligned}
&\mathcal{E}^{N}+\mathcal{E}^{N-1}+\Delta t(\mathcal{A}^{N}+\mathcal{A}^{N-1})+\Delta t\sum_{n=1}^{N-1}\mathcal{B}^{n}+\Delta t\mathcal{C}^{N-1/2}\\
&\geq \frac{1}{4}\left(\|\xi_f^N\|_f^2+\|\xi_f^{N-1}\|_f^2+gS_0\|\xi_p^N\|_p^2+gS_0\|\xi_p^{N-1}\|_p^2\right)\\
&+\frac{\Delta t}{2}\sum_{n=1}^{N-1}\left(\nu\|\nabla\left(\xi_f^{n+1}+\xi_f^{n-1}\right)\|_f^2+gk_{\min}\|\nabla\left(\xi_p^{n+1}+\xi_p^{n-1}\right)\|_p^2\right),
\end{aligned}
$$

it gives

$$
\begin{aligned}
&\frac{1}{4}\left(\|\xi_f^N\|_f^2+\|\xi_f^{N-1}\|_f^2+gS_0\|\xi_p^N\|_p^2+gS_0\|\xi_p^{N-1}\|_p^2\right)\\
&+\frac{\Delta t}{4}\sum_{n=1}^{N-1}\left(\nu\|\nabla\left(\xi_f^{n+1}+\xi_f^{n-1}\right)\|_f^2+gk_{\min}\|\nabla\left(\xi_p^{n+1}+\xi_p^{n-1}\right)\|_p^2\right)\\
&\leq \mathcal{E}^1+\mathcal{E}^0+\Delta t(\mathcal{A}^1+\mathcal{A}^0)+\Delta t\mathcal{C}^{1/2}\\
&+\Delta t\sum_{n=1}^{N-1}\left\{\frac{4C_{P,f}^2}{\nu}\left\|\frac{\eta_f^{n+1}-\eta_f^{n-1}}{2\Delta t}\right\|_f^2+\frac{4C_{P,p}^2 gS_0^2}{k_{\min}}\left\|\frac{\eta_p^{n+1}-\eta_p^{n-1}}{2\Delta t}\right\|_p^2\right. \qquad (4.13)\\
&+C_0\left\|u_t^{n+1}-\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right\|_f^2+C_0\left\|\phi_t^{n+1}-\frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t}\right\|_p^2\\
&+C_0\left(\|\nabla\eta_f^{n+1}\|_f^2+\|\nabla\eta_f^{n}\|_f^2+\|\nabla\eta_p^{n+1}\|_p^2+\|\nabla\eta_p^{n}\|_p^2\right)\\
&\left.+C_0\left(\|\nabla(u^{n+1}-u^n)\|_f^2+\|\nabla(\phi^{n+1}-\phi^n)\|_p^2\right)+C_0\|p^{n+1}-\lambda_h^{n+1}\|_f^2\right\}.
\end{aligned}
$$

The terms on the right hand side of (4.13) can be bounded as below

$$
\begin{aligned}
&\mathcal{E}^1 + \mathcal{E}^0 + \Delta t(\mathcal{A}^1 + \mathcal{A}^0) + \Delta t \mathcal{C}^{1/2} \\
&\leq \left\|\eta_f^1\right\|_f^2 + \left\|\eta_f^0\right\|_f^2 + gS_0\left\|\eta_p^1\right\|_p^2 + gS_0\left\|\eta_p^0\right\|_p^2 + \left\|u^1 - u_h^1\right\|_f^2 \\
&\quad + \left\|u^0 - u_h^0\right\|_f^2 + gS_0\left\|\phi^1 - \phi_h^1\right\|_p^2 + gS_0\left\|\phi^0 - \phi_h^0\right\|_p^2 \\
&\quad + C_0\Delta t\Big(\left\|\nabla\eta_f^1\right\|_f^2 + \left\|\nabla\eta_f^0\right\|_f^2 + \left\|\nabla\eta_p^1\right\|_p^2 + \left\|\nabla\eta_p^0\right\|_p^2 + \left\|\nabla\left(u^1 - u_h^1\right)\right\|_f^2 \\
&\quad + \left\|\nabla\left(u^0 - u_h^0\right)\right\|_f^2 + \left\|\nabla\left(\phi^1 - \phi_h^1\right)\right\|_p^2 + \left\|\nabla\left(\phi^0 - \phi_h^0\right)\right\|_p^2\Big) \\
&\leq \left\|u^1 - u_h^1\right\|_f^2 + \left\|u^0 - u_h^0\right\|_f^2 + gS_0\left\|\phi^1 - \phi_h^1\right\|_p^2 + gS_0\left\|\phi^0 - \phi_h^0\right\|_p^2 \\
&\quad + C_0\Delta t\Big(\left\|\nabla\left(u^1 - u_h^1\right)\right\|_f^2 + \left\|\nabla\left(u^0 - u_h^0\right)\right\|_f^2 + \left\|\nabla\left(\phi^1 - \phi_h^1\right)\right\|_p^2 \\
&\quad + \left\|\nabla\left(\phi^0 - \phi_h^0\right)\right\|_p^2\Big) + C_0\left(\max_{n=0,\ldots,N}\left\|\nabla\eta_f^n\right\|_f^2 + \max_{n=0,\ldots,N}\left\|\nabla\eta_p^n\right\|_p^2\right) \\
&\leq \left\|u^1 - u_h^1\right\|_f^2 + \left\|u^0 - u_h^0\right\|_f^2 + gS_0\left\|\phi^1 - \phi_h^1\right\|_p^2 + gS_0\left\|\phi^0 - \phi_h^0\right\|_p^2 \\
&\quad + C_0\Delta t\Big(\left\|\nabla\left(u^1 - u_h^1\right)\right\|_f^2 + \left\|\nabla\left(u^0 - u_h^0\right)\right\|_f^2 + \left\|\nabla\left(\phi^1 - \phi_h^1\right)\right\|_p^2 \\
&\quad + \left\|\nabla\left(\phi^0 - \phi_h^0\right)\right\|_p^2\Big) + C_0 h^{2k}\Big(|||u|||_{\infty,k+1,f}^2 + |||\phi|||_{\infty,k+1,p}^2\Big),
\end{aligned}
\tag{4.14}
$$

also

$$
\begin{aligned}
&\Delta t\sum_{n=1}^{N-1}\left\{\frac{4C_P^2}{\nu}\left\|\frac{\eta_f^{n+1} - \eta_f^{n-1}}{2\Delta t}\right\|_f^2 + \frac{4C_P^2 gS_0^2}{k_{\min}}\left\|\frac{\eta_p^{n+1} - \eta_p^{n-1}}{2\Delta t}\right\|_p^2\right\} \\
&\leq C_0\left(\int_0^{t^N}\|\eta_{f,t}\|_f^2 dt + \int_0^{t^N}\|\eta_{p,t}\|_p^2 dt\right) \leq C_0\left(\|\eta_{f,t}\|_{2,0,f}^2 + \|\eta_{p,t}\|_{2,0,p}^2\right) \\
&\leq C_0 h^{2k+2}\left(\|u_t\|_{2,k+1,f}^2 + \|\phi_t\|_{2,k+1,p}^2\right),
\end{aligned}
\tag{4.15}
$$

and

$$
\begin{aligned}
&\Delta t\sum_{n=1}^{N-1}\left(C_0\left\|u_t^{n+1} - \frac{u^{n+1} - u^{n-1}}{2\Delta t}\right\|_f^2 + C_0\left\|\phi_t^{n+1} - \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t}\right\|_p^2\right) \\
&\leq C_0\Delta t^2\left(\int_0^{t^N}\|u_{tt}\|_f^2 dt + \int_0^{t^N}\|\phi_{tt}\|_p^2 dt\right) \leq C_0\Delta t^2\left(\|u_{tt}\|_{2,0,f}^2 + \|\phi_{tt}\|_{2,0,p}^2\right).
\end{aligned}
\tag{4.16}
$$

Furthermore

$$
\Delta t\sum_{n=1}^{N-1} C_0\left(\|\nabla(u^{n+1} - u^n)\|_f^2 + \|\nabla(\phi^{n+1} - \phi^n)\|_p^2\right) \tag{4.17}
$$

$$
\leq C_0\Delta t^2\left(\int_0^{t^N}\|u_t\|_{H^1(\Omega_f)}^2 dt + \int_0^{t^N}\|\phi_t\|_{H^1(\Omega_p)}^2 dt\right) \leq C_0\Delta t^2\left(\|u_t\|_{2,1,f}^2 + \|\phi_t\|_{2,1,p}^2\right),
$$

and

$$
\Delta t \sum_{n=1}^{N-1} C_0 \left(\|\nabla \eta_f^{n+1}\|_f^2 + \|\nabla \eta_f^n\|_f^2 + \|\nabla \eta_p^{n+1}\|_p^2 + \|\nabla \eta_p^n\|_p^2 \right) \tag{4.18}
$$

$$
\leq C_0 \Delta t \sum_{n=0}^{N} h^{2k} \left(\|u^n\|_{H^{k+1}(\Omega_f)}^2 + \|\phi^n\|_{H^{k+1}(\Omega_p)}^2 \right) \leq C_0 h^{2k} \left(\||u|\|_{2,k+1,f}^2 + \||\phi|\|_{2,k+1,p}^2 \right).
$$

Let λ_h^{n+1} be the interpolation of p^{n+1} in Q_f^h, we have

$$
\Delta t \sum_{n=1}^{N-1} C_0 \|p^{n+1} - \lambda_h^{n+1}\|_f^2 \leq C_0 \Delta t \sum_{n=0}^{N} h^{2s+2} \|p^n\|_{H^{s+1}(\Omega_f)}^2 \leq C_0 h^{2s+2} \||p|\|_{2,s+1,f}^2. \tag{4.19}
$$

Combining (4.13)–(4.19), there follows

$$
\begin{aligned}
&\frac{1}{4} \left(\|\xi_f^N\|_f^2 + \|\xi_f^{N-1}\|_f^2 + gS_0 \|\xi_p^N\|_p^2 + gS_0 \|\xi_p^{N-1}\|_p^2 \right) \\
&+ \frac{\Delta t}{4} \sum_{n=1}^{N-1} \left(\nu \|\nabla \left(\xi_f^{n+1} + \xi_f^{n-1} \right)\|_f^2 + g k_{\min} \|\nabla \left(\xi_p^{n+1} + \xi_p^{n-1} \right)\|_p^2 \right) \\
&\leq \left\| u^1 - u_h^1 \right\|_f^2 + \left\| u^0 - u_h^0 \right\|_f^2 + gS_0 \left\| \phi^1 - \phi_h^1 \right\|_p^2 + gS_0 \left\| \phi^0 - \phi_h^0 \right\|_p^2 \\
&+ C_0 \Delta t \left(\left\| \nabla \left(u^1 - u_h^1 \right) \right\|_f^2 + \left\| \nabla \left(u^0 - u_h^0 \right) \right\|_f^2 + \left\| \nabla \left(\phi^1 - \phi_h^1 \right) \right\|_p^2 + \left\| \nabla \left(\phi^0 - \phi_h^0 \right) \right\|_p^2 \right) \\
&+ C_0 h^{2k} \left(\||u|\|_{\infty,k+1,f}^2 + \||\phi|\|_{\infty,k+1,p}^2 \right) + C_0 h^{2k+2} \left(\|u_t\|_{2,k+1,f}^2 + \|\phi_t\|_{2,k+1,p}^2 \right) \\
&+ C_0 \Delta t^2 \left(\|u_{tt}\|_{2,0,f}^2 + \|\phi_{tt}\|_{2,0,p}^2 \right) + C_0 \Delta t^2 \left(\|u_t\|_{2,1,f}^2 + \|\phi_t\|_{2,1,p}^2 \right) \\
&+ C_0 h^{2k} \left(\||u|\|_{2,k+1,f}^2 + \||\phi|\|_{2,k+1,p}^2 \right) + C_0 h^{2s+2} \||p|\|_{2,s+1,f}^2.
\end{aligned} \tag{4.20}
$$

The estimate given in (4.4) follows from the triangle inequality and (4.20) with the notice that the upcoming new terms are already contained in the right hand side of the model. ◻

For Taylor-Hood elements, i.e. $k = 2$, $s = 1$, we have the following asymptotic estimate.

COROLLARY 4.2. *Consider BELF method* (3.2). *Under the assumptions of Theorem 4.1 with $T = \infty$, suppose that (X_f^h, Q_f^h) is given by P2-P1 Taylor-Hood approximation elements and X_p^h is P2 finite element. Then, there is a positive constant C_0 such that*

$$
\sup_{1 \leq N \leq \infty} \left\{ \|e_f^N\|_f^2 + \|e_f^{N-1}\|_f^2 + S_0 \|e_p^N\|_p^2 + S_0 \|e_p^{N-1}\|_p^2 \right\} \leq C_0((\Delta t)^2 + h^4).
$$

5. Numerical Experiments. We present numerical experiments to test the algorithms presented herein. First, using the exact solutions introduced in [MZ10], we confirm the predicted convergence rates from the theory. Second, we will look at errors over longer time intervals and small values of $k_{\min}$ to see the asymptotic stability of our proposed methods for $k_{\min}$ smaller than covered by the theory. The code was implemented using the software package *FreeFEM++* [HP].

5.1. Test 1. For the first test we select the velocity and pressure field given in [MZ10]. Let the domain Ω be composed of $\Omega_f = (0,1) \times (1,2)$ and $\Omega_p = (0,1) \times (0,1)$ with the interface $\Gamma = (0,1) \times \{1\}$. The exact velocity field is given by

$$
\begin{aligned}
u_1(x,y,t) &= (x^2(y-1)^2 + y)\cos t\,,\\
u_2(x,y,t) &= \left(-\frac{2}{3}x(y-1)^3 + 2 - \pi\sin(\pi x)\right)\cos t\,,\\
p(x,y,t) &= (2 - \pi\sin(\pi x))\sin\left(\frac{\pi}{2}y\right)\cos t\,,\\
\phi(x,y,t) &= (2 - \pi\sin(\pi x))(1 - y - \cos(\pi y))\cos t.
\end{aligned}
$$

We take the time interval $0 \le t \le 3$ and all the physical parameters $\eta, \rho, g, \nu, K, S_0$ and α are simply set to 1. We utilize Taylor-Hood P2-P1 finite elements for the Stokes equations and continuous piecewise quadratic finite element for the Darcy equation. The boundary condition on the problem is inhomogeneous Dirichlet: $u_h = u_{exact}$ on $\partial\Omega$. The initial data and source terms are chosen to correspond the exact solution.

For convenience, we denote $\|.\|_\infty = \|.\|_{\infty,0,f|p}$ and $\|.\|_2 = \|.\|_{2,0,f|p}$. The rates of convergence are computed using linear regression. Table 5.1–5.4 summarize the convergence rates with different order combinations of h and Δt. In particular, Table 5.3 and 5.4 confirm the convergence rates provided in Corollary 4.2.

h	Δt	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|p-p_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\nabla\phi-\nabla\phi_h\|_2$
1/5	1/5	3.565e-3	1.230e-1	9.863e-2	1.142e-2	2.050e-1
1/10	1/10	1.814e-3	3.563e-2	4.760e-2	5.760e-3	6.172e-2
1/20	1/20	9.113e-4	1.359e-2	2.354e-2	2.891e-3	2.42e-2
1/40	1/40	4.560e-4	6.166e-3	1.176e-2	1.448e-3	1.125e-2
1/80	1/80	2.280e-4	2.989e-3	5.882e-3	7.248e-4	5.506e-3
Rate of conv.		0.9926	1.3256	1.0152	0.9946	1.2891

Table 5.1
Convergence rate for BEFE with $\Delta t = h$.

h	Δt	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|p-p_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\nabla\phi-\nabla\phi_h\|_2$
1/5	1/5	4.484e-3	1.176e-1	1.334e-1	1.210e-2	1.937e-1
1/10	1/10	1.947e-3	3.680e-2	6.842e-2	6.038e-2	6.121e-2
1/20	1/20	9.805e-4	1.481e-2	3.473e-2	3.026e-3	2.451e-2
1/40	1/40	4.922e-4	6.883e-3	1.753e-2	1.515e-3	1.146e-2
1/80	1/80	2.467e-4	3.367e-3	8.812e-3	7.578e-4	5.615e-3
Rate of conv.		1.0352	1.2672	0.9805	0.9988	1.2634

Table 5.2
Convergence rate for BELF with $\Delta t = h$.

The performance of numerical methods we studied herein is also compared with the monolithic, coupled implicit method. Using the same test problem, the errors $\|u - u_h\|_\infty + \|\phi - \phi_h\|_\infty$ produced by three methods (Fully coupled Backward Euler, BEFE and BELF) are shown in Table 5.5.

h	Δt	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|p-p_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\nabla\phi-\nabla\phi_h\|_2$
1/5	1/5	3.565e-3	1.230e-1	1.024e-1	1.142e-2	2.050e-1
1/10	1/20	9.086e-4	3.143e-2	2.532e-2	2.894e-3	4.978e-2
1/20	1/80	2.279e-4	7.356e-3	6.282e-3	7.251e-4	1.202e-2
1/40	1/320	5.702e-5	1.822e-3	1.563e-3	1.814e-4	3.017e-3
1/80	1/1280	1.426e-5	4.673e-4	3.923e-4	4.532e-5	7.631e-4
Rate of conv.		1.9926	2.0189	2.0074	1.9950	2.0183

TABLE 5.3
Convergence rate for BEFE with $\Delta t = h^2/5$.

h	Δt	$\|u-u_h\|_\infty$	$\|\nabla u-\nabla u_h\|_2$	$\|p-p_h\|_2$	$\|\phi-\phi_h\|_\infty$	$\|\nabla\phi-\nabla\phi_h\|_2$
1/5	1/5	4.484e-3	1.176e-1	1.359e-1	1.210e-2	1.937e-1
1/10	1/20	1.004e-3	3.153e-2	3.597e-2	3.035e-3	4.926e-2
1/20	1/80	2.479e-4	7.492e-3	9.085e-3	7.584e-4	1.203e-2
1/40	1/320	6.188e-5	1.862e-3	2.273e-3	1.896e-4	3.027e-3
1/80	1/1280	1.547e-5	4.773e-4	5.699e-4	4.738e-5	7.661e-4
Rate of conv.		2.0378	1.9971	1.9779	1.9994	1.9989

TABLE 5.4
Convergence rate for BELF with $\Delta t = h^2/5$.

Next let the source terms $\mathbf{f}_f \equiv 0, f_p \equiv 0$ and the initial condition is given by

$$\begin{aligned}
&u_1(x,y,0) = (x^2(y-1)^2 + y)\,,\\
&u_2(x,y,0) = \left(-\frac{2}{3}x(y-1)^3 + 2 - \pi\sin(\pi x)\right)\,,\\
&p(x,y,0) = (2-\pi\sin(\pi x))\sin\left(\frac{\pi}{2}y\right)\,,\\
&\phi(x,y,0) = (2-\pi\sin(\pi x))(1-y-\cos(\pi y)).
\end{aligned}$$

We take $h = \frac{1}{20}$, $\Delta t = \frac{1}{20}$, $T = 2.0$ and compute $E^n = \|u_h^n\|_f^2 + \|\phi_h^n\|_p^2$ for each $n = 0, ..., N$ using three methods: fully coupled implicit method, BEFE and BELF. The variation of approximated kinetic energy E^n from 0.0 to 2.0 is shown in Figure 5.1. For the exact solution, we solve the problem with a small time step and mesh size ($h = \frac{1}{100}$, $\Delta t = \frac{1}{200}$) and use the solution obtained as reference. We note that all methods predict that $E^n \to 0$ as $t^n \to \infty$, which is completely consistent with our theoretical results when $\mathbf{f}_f, f_p \equiv 0$.

5.2. Test 2. Stokes-Darcy flows with very small hydraulic conductivity tensor $\mathcal{K}$ are of special interest in some applications, see [DQV07]. We test herein and compare the performance of our two proposed methods with that of the fully implicit method for such flows. In the following numerical experiment, we keep all initial condition, boundary condition, source data and parameters unchanged from the last test, except $k_{\min}$ is now set to be 10^{-6} and final time T is switched to 5.0, for a clearer representation of behavior of kinetic energy E^n over a longer time. Let $h = 1/10$, we plot E^n with four different time steps $\Delta t = 1/5, 1/8, 1/10, 1/20$ (Figure 5.2).

We observe that the fully implicit method is stable with no restriction on Δt. However, BELF and BEFE are already as stable as the implicit method for $\Delta t = 1/10$, which is a very mild constraint and far better than predicted by the theory.

Finally, we repeat the above experiment with ν and $k_{\min}$ set to be 10^{-1} and 10^{-6} correspondingly. We present the results for various values of Δt, purposely chosen to

h	Δt	Fully coupled implicit method	BEFE	BELF
1/5	1/5	9.305e-3	1.499e-2	1.658e-2
1/10	1/10	2.083e-3	7.574e-3	7.985e-3
1/20	1/20	9.604-4	3.802e-3	4.007e-3
1/40	1/40	4.797e-4	1.904e-3	2.007e-3
1/80	1/80	2.463e-4	9.528e-4	1.005e-3

TABLE 5.5

Errors $\|u - u_h\|_\infty + \|\phi - \phi_h\|_\infty$ of the fully coupled implicit method, BEFE and BELF.

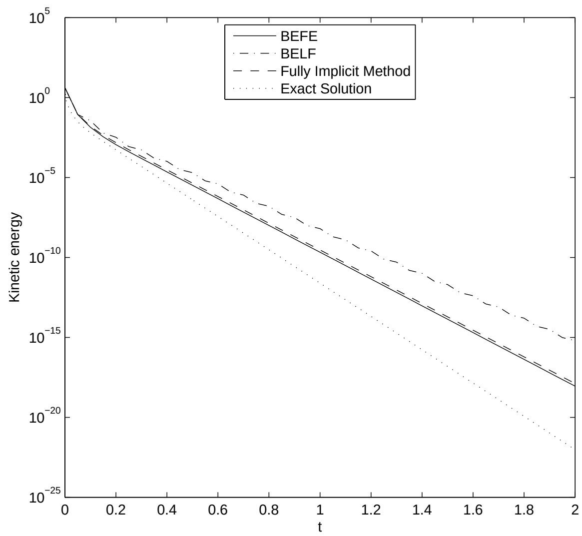

FIG. 5.1. *The decay of kinetic energy for different numerical methods.*

show the difference of the studied methods.

We see that the fully coupled Backward Euler method is the most stable, followed by BELF and then BEFE, as expected. We also note that BELF is already stable for $\Delta t = 1/30$ and so is BEFE for $\Delta t = 1/50$, again far better than conditions proposed in the theory. The problem of finding optimal conditions for the asymptotic stability of BEFE and BELF is thus an open question.

6. Conclusions. The evolutionary coupled Stokes-Darcy problem is a complex and high impact problem for which detailed numerical analysis can have a direct impact on algorithm development and solution strategies. Partitioned methods, which require one (per sub domain) solve of SPD system per time step are very attractive in computational complexity compared to monolithic methods (requiring one coupled, non symmetric system of roughly double in size). However, because the coupling is exactly skew symmetric, care must be taken in devising an appropriate uncoupling strategy. We have analyzed two first order partitioned methods which are also comparable in stability and accuracy to fully coupled, monolithic methods. Many open questions remain such as higher order partitioned methods that are long time stable and the precise behaviour of $\Delta t_{BEFE}, \Delta t_{BELF}$ with respect to the physical parameters (which we have attempted to indicate but not optimize). The question

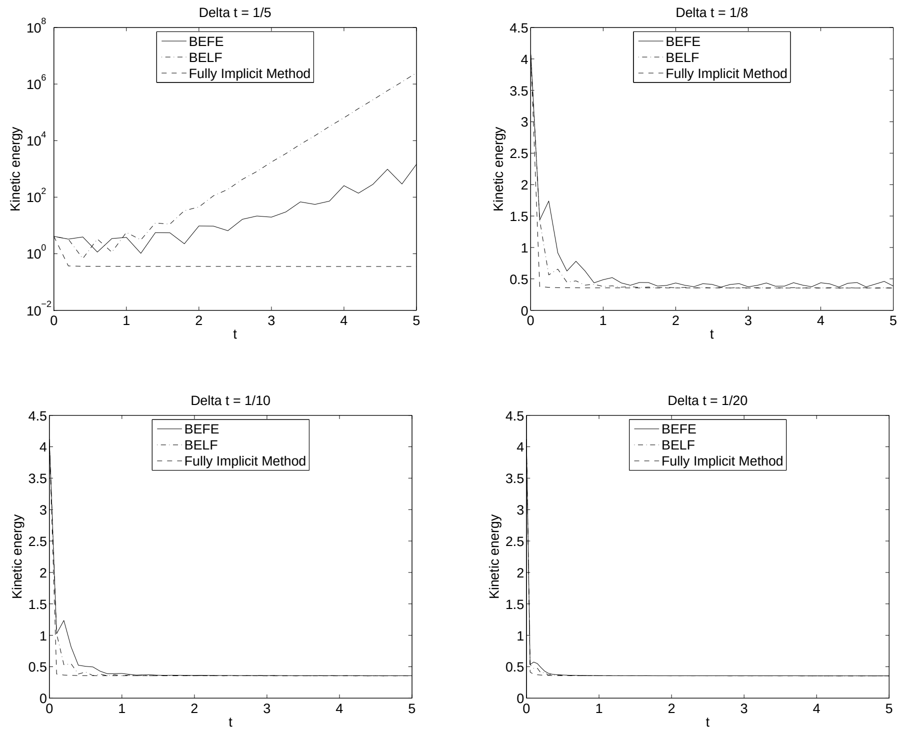

FIG. 5.2. *Variation of kinetic energy with* $\nu = 1$ *and* $k_{\min} = 10^{-6}$.

of dependence on physical parameters is important since many problems from motivating applications arise with $k_{\min}$ small, S_0 small, η small or in large but thin domains.

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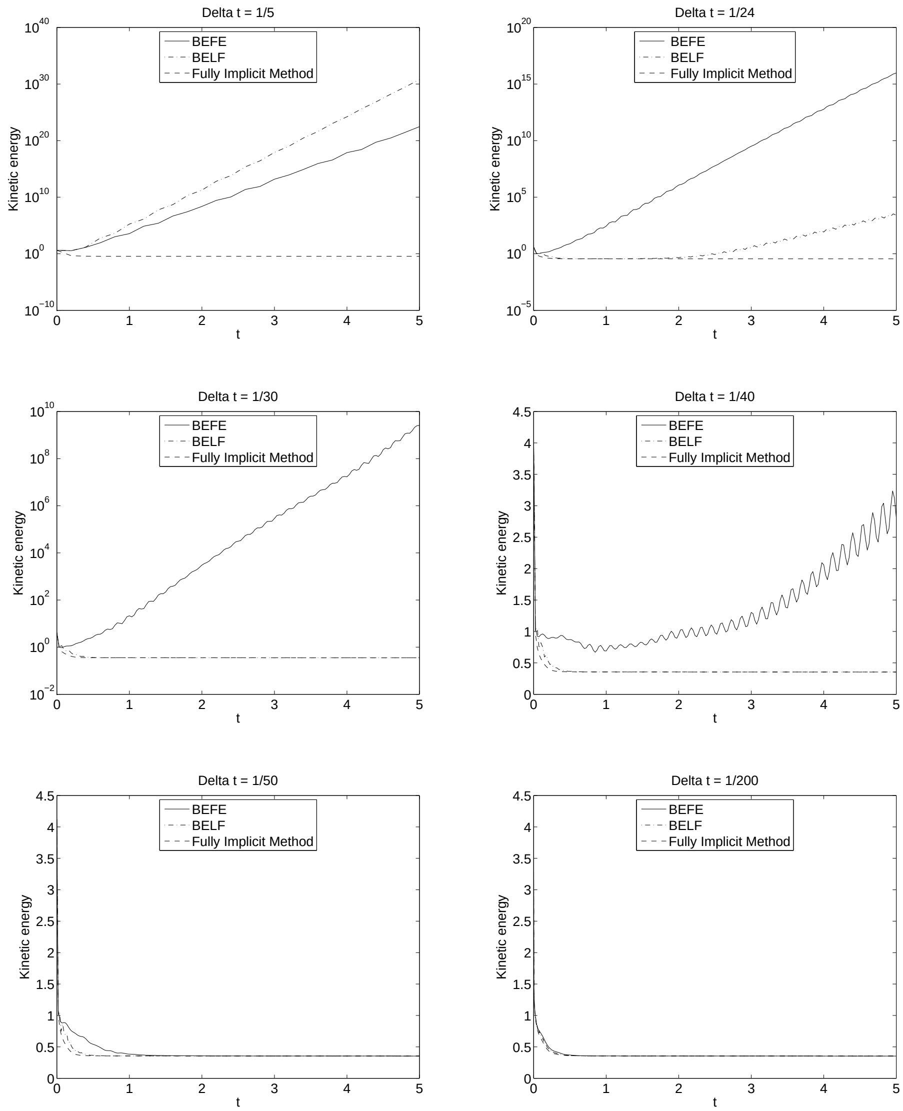

FIG. 5.3. *Variation of kinetic energy with* $\nu = 10^{-1}$ *and* $k_{\min} = 10^{-6}$.

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