ANALYSIS OF A STABILIZED CNLF METHOD WITH FAST SLOW WAVE SPLITTINGS FOR FLOW PROBLEMS

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Abstract. In this work we study Crank-Nicolson Leap-Frog (CNLF) methods with fast slow wave splittings for Navier-Stokes equation plus a Coriolis force term, which is a simplification of geophysical flows. We present a new stabilized CNLF method where the added stabilization completely removes the method's CFL time step condition. We give a comprehensive stability and error analysis. We prove that for Oseen equation + rotation term, the unstable mode (for which $u^{n+1}+u^{n-1}\equiv 0$) of CNLF is asymptotically stable. Numerical results are provided to verify the stability and the convergence of the methods.

1. Introduction. One of the most common time discretizations in atmosphere and ocean codes is the implicit-explicit combination of the Crank-Nicolson and Leap-Frog methods, usually abbreviated as CNLF. The usual description is that CN is used to discretize physical effects corresponding to fast waves and low energy while LF for high energy slow waves. One issue about this method is that the time step restriction of CNLF can be very restrictive if the normal splitting is not perfectly done, i.e., if some fast waves are present in the LF terms. Our aim herein is to provide an analytic, nonlinear energy stability and convergence analysis of a stabilized CNLF introduced in [1], which is unconditionaly stable, based on a splitting of the NSE + rotation/Coriolis force motivated by the above description. This work is only one step to the very complex motivating application and many open questions (some surveyed in Section 1.1 below) remain.

Consider thus the NSE + a rotation/Coriolis force, a great simplification of the geophysical flow, [2], on a bounded domain subject to either no slip (the typical case herein) or periodic boundary conditions:

$$\begin{aligned} u_t + u\cdot\nabla u - \nu\Delta u + \nabla p + f_C \times u &= f(x,t) \text{ in } \Omega, \\ \nabla\cdot u &= 0 \text{ in } \Omega, \\ u(x,0) = u_0(x) \text{ in } \Omega &\text{ and } u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Suppressing the spatial discretization, we must identify the parts to be discretized by CN vs. LF. Let $U(x)$ denote a smooth, averaged fluid velocity. Rewrite the momentum equation as

$$u_t + N(u,p) + \Lambda(u) = f(x,t), \text{ and } \nabla\cdot u = 0, \text{ in } \Omega,$$

where

$$\begin{aligned} N(u,p) &:= (u-U)\cdot\nabla u - \nu\Delta u + \nabla p \quad \text{(fast wave component)}, \\ \Lambda(u) &:= U\cdot\nabla u + f_C\times u \quad \text{(slow wave component)}. \end{aligned}$$

We shall discretize in time by CNLF using the above splitting. The first time discretization we study is CNLF based on FAst-SLow splitting, which we call FASL: given u^0, u^1, p^1 (computed by some other method), find $u^2, u^3, \cdots, p^2, p^3, \cdots$ satisfying

$$\begin{aligned} \frac{u^{n+1}-u^{n-1}}{2\Delta t} + N\Big(\frac{u^{n+1}+u^{n-1}}{2}, \frac{p^{n+1}+p^{n-1}}{2}\Big) + \Lambda(u^n) &= f^n, \\ \nabla\cdot u^{n+1} &= 0, \text{ in } \Omega, \\ \text{and } u^{n+1} &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{1.2}$$

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Here selected mean flow $U(x)$ is assumed to be divergence free, satisfy the same boundary conditions as $u(x,t)$ and stationary. The stability region of LF is the interval $\{z : \text{Re}(z) = 0$ and $-1 \le \text{Im}(z) \le +1\}$. Thus, root condition analysis, [3], [4], suggests that skew symmetry of Λ is essential and for such Λ the time step restriction

$$\Delta t||\Lambda|| \le 1 \text{ or } \Delta t\left(\frac{|U|}{\Delta x} + |f_C|\right) \le \text{const} \tag{1.3}$$

is necessary.

This motivates us to develop for this application a stabilization of CNLF introduced in [1] (STAFASL), which eliminates all restrictions for stability while maintaining an accuracy comparable to that of FASL. Our scheme reads: given u^0, u^1, p^1 (computed by some other method), find $u^2, u^3, \cdots, p^2, p^3, \cdots$ satisfying

$$\begin{aligned}\frac{u^{n+1}-u^{n-1}}{2\Delta t} &+ N(\frac{u^{n+1}+u^{n-1}}{2}, \frac{p^{n+1}+p^{n-1}}{2}) \\ &+\Lambda(u^n) + 2\Delta t\Lambda^*\Lambda(u^{n+1}-u^{n-1}) = f^n, \\ &\nabla \cdot u^{n+1} = 0.\end{aligned} \tag{1.4}$$

The stabilization method was introduced in [1] and its stability was proven for the time dependent Stokes problem. The result is extended here to the fast-slow decomposition of NSE + rotation force, still without any CFL type time step restriction, with a new and comprehensive error analysis given.

Generally, it is believed that CNLF will damp only the mode $u^{n+1} + u^{n-1}$; there is no damping in the unstable mode where $u^{n+1} + u^{n-1} \equiv 0$. Roundoff error can lead to growth in the unstable mode, spurring development of corrective time filters, [2], [5], [6], [7]. For the linearized form of NSE (Oseen equation), we prove that this is not the case: CNLF discretization also damps the unstable mode. This extends a result from [8] to the flow problem (1.1). The damping of unstable mode for full NSE is an open question.

Our paper is organized as follows. In Section 2, we give necessary definitions and state well-known results which are used throughout the paper. In Section 3, for spatial discretization in FASL by a variational method (we specify FEMs but the same proof holds for Galerkin spectral methods and Thom's finite difference methods), we prove that FASL is stable in the absence of round off errors under CFL condition, see Theorem 3.1. The unconditional stability of STAFASL is established in Section 4. In Section 5, we prove that for Oseen equation + rotation term, the unstable mode of CNLF is asymptotically stable. Section 6 is devoted for a complete convergence analysis of STAFASL. Finally, numerical experiments are given in Section 7 for verifying our theoretical results.

1.1. Previous work. The implicit-explicit Crank-Nicolson and Leap-Frog method is widely used in atmosphere, ocean and climate codes, e.g., [5], [6], [7], [9] and has recently been used for uncoupling groundwater-surface water flows, [10]. Stability of CNLF by root conditions was proven in 1963 [3] and by energy methods for systems in [4]. Two related stability questions remain. First, the time step restriction (1.3) from the LF component can be too restrictive if the normal splitting into fast but low energy modes and slow but high energy modes is not perfectly done and if the problem parameters are too big. Second, the unstable mode (for which $u^{n+1} + u^{n-1} \equiv 0$) of LF is not damped by CN. Thus, modular time filters, like the Roberts-Asselin-Williams filter [5], [6], [7], have been developed to deal with this issue. The stabilized CNLF scheme presented in this paper addresses both of the issues: it is unconditionally (no time step condition) stable and the unstable mode, while not eliminated, is controlled for the linearized model. Stabilized CNLF, like CNLF, is a 3 level method and approximations are needed at the first two time steps to appropriate accuracy, [11]. The stabilization in stabilized CNLF herein is similar in spirit to [12], [13], [14]. For a general theory of IMEX methods see [15], [16], [17]. The instability of CNLF for nonautonomous systems was studied in [18].

2. Notation and preliminaries. Let Ω be an open, regular domain in $\mathbb{R}^d$ ($d =$ 2 or 3). The $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$ respectively. In particular, the $L^2(\Omega)$ norm and the inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. $H^k(\Omega)$ is used to represent the Sobolev space $W_2^k(\Omega)$, with norm $\|\cdot\|_k$. For functions $v(x,t)$ defined on $(0,T)$, we define $(1 \le m < \infty)$

$$\|v\|_{\infty,k} := EssSup_{[0,T]}\|v(t,\cdot)\|_k, \text{ and } \|v\|_{m,k} := \left(\int_0^T \|v(t,\cdot)\|_k^m dt\right)^{1/m} .$$

The space $H^{-k}(\Omega)$ is the dual space of bounded linear functions on $H_0^k(\Omega)$. A norm for $H^{-1}(\Omega)$ is given by

$$\|f\|_{-1} = \sup_{0\neq v\in H_0^1(\Omega)} \frac{(f,v)}{\|\nabla v\|} .$$

Let X be the velocity space and Q be the pressure space:

$$X := (H_0^1(\Omega))^d, \; Q := L_0^2(\Omega).$$

The space of divergence free functions is

$$V := \{v \in X : (\nabla\cdot v, q) = 0 , \forall q \in Q\}.$$

The norm on V^* (the dual of V) is defined as

$$\|f\|_* = \sup_{0\neq v\in V} \frac{(f,v)}{\|\nabla v\|} .$$

A weak formulation of (1.1) is: Find $u : [0,T] \to X$, $p : [0,T] \to Q$ for a.e. $t \in (0,T]$ satisfying:

$$(u_t, v) + (u\cdot\nabla u, v) + \nu(\nabla u, \nabla v) - (p, \nabla\cdot v) + (f_C \times u, v) = (f, v) , \forall v \in X$$
$$u(x,0) = u^0(x) \text{ in X and } (\nabla\cdot u, q) = 0, \forall q \in Q.$$

We denote conforming velocity, pressure finite element spaces based on an edge to edge triangulation of Ω (with maximum triangle diameter h) by

$$X_h \subset X , \; Q_h \subset Q.$$

We assume that X_h and Q_h satisfy the usual discrete inf-sup condition. Taylor-Hood elements, discussed in [19], [20], are one commonly used choice of velocity-pressure finite element spaces. The discretely divergence free subspace of X_h is

$$V_h := \{v_h \in X_h : (\nabla\cdot v_h, q_h) = 0 , \forall q_h \in Q_h\}.$$

We assume the mesh and finite element spaces satisfy the following inverse inequality (typical for locally quasi-uniform meshes and standard FEM spaces, see, e.g., [19]): for all $v_h \in X_h$

$$h\|\nabla v_h\| \le c_{inv}\|v_h\|$$

where c_{inv} depends only on the shape (h_e/ρ_e, $h_e = diam(e)$ and ρ_e is the diameter of the largest ball that can be inscribed in e) of the elements, [21].
Define the usual explicitly skew symmetric trilinear form

$$b^*(u,v,w) := \frac{1}{2}(u\cdot\nabla v, w) - \frac{1}{2}(u\cdot\nabla w, v).$$

Then for any $u_h, v_h, w_h \in X_h$,

$$b^*(u_h, v_h, w_h) = \int_\Omega u_h \cdot \nabla v_h \cdot w_h \, dx + \frac{1}{2}\int_\Omega (\nabla \cdot u_h)(v_h \cdot w_h) \, dx, \tag{2.1}$$

which can be easily proved by integrating by parts and using $u_h|_{\partial\Omega} = 0$.

$b^*(u, v, w)$ also satisfies the bound

$$b^*(u, v, w) \le C\|u\|_{\frac{1}{2}}\|\nabla v\|\|\nabla w\|, \text{ for all } u, v, w \in X.$$

3. Stability of FASL with the CFL condition. In this section we prove nonlinear, long time stability of FASL under a CFL type condition for Navier-Stokes equations with a Coriolis term.

The fully discrete approximation we study of (1.1) with FASL is: Given $u_h^{n-1}, u_h^n, p_h^{n-1}, p_h^n$, find $u_h^{n+1} \in X_h$, $p_h^{n+1} \in Q_h$ satisfying

$$\begin{aligned}
&(\frac{u_h^{n+1} - u_h^{n-1}}{2\Delta t}, v_h) + b^*(\frac{u_h^{n+1} + u_h^{n-1}}{2} - U, \frac{u_h^{n+1} + u_h^{n-1}}{2}, v_h) \\
&-(\frac{p_h^{n+1} + p_h^{n-1}}{2}, \nabla \cdot v_h) + \nu(\nabla\frac{u_h^{n+1} + u_h^{n-1}}{2}, \nabla v_h) \\
&+b^*(U, u_h^n, v_h) + (f_C \times u_h^n, v_h) = (f^n, v_h), \qquad \forall v_h \in X_h, \\
&(\nabla \cdot u_h^{n+1}, q_h) = 0, \qquad \forall q_h \in Q_h.
\end{aligned} \tag{3.1}$$

The selection of the averaged velocity is necessarily application dependent. Many choices of U are possible; the theorem below only needs $U = U(x)$ and $U \cdot \widehat{n}|_{\partial\Omega} = 0$. The computational aim is to choose so that the nonlinearity in the implicit CN discretized terms to be smaller.

THEOREM 3.1 (Stability of FASL). *Consider* FASL *method. Suppose the following timestep restriction holds*

$$\Delta t < (\|U\|_\infty c_{inv} h^{-1} + \|f_C\|_\infty)^{-1} \ , \tag{3.2}$$

then for any $N \ge 1$,

$$\begin{aligned}
&\frac{\beta}{2}\left(\|u_h^{N+1}\|^2 + \|u_h^N\|^2\right) + \sum_{n=1}^N \nu\Delta t \left\|\nabla\frac{u_h^{n+1} + u_h^{n-1}}{2}\right\|^2 \le \sum_{n=1}^N \frac{\Delta t}{\nu}\|f^n\|_*^2 \\
&+\frac{1}{2}\left(\|u_h^1\|^2 + \|u_h^0\|^2\right) + \Delta t b^*(U, u_h^0, u_h^1) + \Delta t(f_C \times u_h^0, u_h^1),
\end{aligned} \tag{3.3}$$

where $\beta = 1 - \Delta t(\|U\|_\infty c_{inv} h^{-1} + \|f_C\|_\infty)$.

Proof. Set $v_h = \frac{u_h^{n+1} + u_h^{n-1}}{2}$ in (3.1). Multiplying through by $2\Delta t$ and applying Young's inequality to the RHS, we get

$$\begin{aligned}
&\frac{1}{2}(\|u_h^{n+1}\|^2 - \|u_h^{n-1}\|^2) + \Delta t b^*(U, u_h^n, u_h^{n+1} + u_h^{n-1}) \\
&+\Delta t(f_c \times u_h^n, u_h^{n+1} + u_h^{n-1}) + \nu\Delta t\left\|\nabla\frac{u_h^{n+1} + u_h^{n-1}}{2}\right\|^2 \le \frac{\Delta t}{\nu}\|f^n\|_*^2.
\end{aligned} \tag{3.4}$$

We rearrange the terms on LHS of (3.4). Firstly,

$$\frac{1}{2}(\|u_h^{n+1}\|^2 - \|u_h^{n-1}\|^2) = \frac{1}{2}(\|u_h^{n+1}\|^2 + \|u_h^n\|^2) - \frac{1}{2}(\|u_h^n\|^2 + \|u_h^{n-1}\|^2).$$

The next two terms could be rewritten as follows, with notice that they are skew-symmetric

$$\Delta t b^*(U, u_h^n, u_h^{n+1} + u_h^{n-1}) = \Delta t b^*(U, u_h^n, u_h^{n+1}) - \Delta t b^*(U, u_h^{n-1}, u_h^n),$$
$$\Delta t(f_C \times u_h^n, u_h^{n+1} + u_h^{n-1}) = \Delta t(f_C \times u_h^n, u_h^{n+1}) - \Delta t(f_C \times u_h^{n-1}, u_h^n).$$

Summing up (3.4) from $n = 1$ to N, it gives

$$\begin{aligned} &\frac{1}{2}(\|u_h^{N+1}\|^2 + \|u_h^N\|^2) + \Delta t b^*(U, u_h^N, u_h^{N+1}) + \Delta t(f_C \times u_h^N, u_h^{N+1}) \\ &\quad + \sum_{n=1}^{N} \nu \Delta t \left\| \nabla \frac{u_h^{n+1} + u_h^{n-1}}{2} \right\|^2 \leq \sum_{n=1}^{N} \frac{\Delta t}{\nu} \|f^n\|_*^2 + \frac{1}{2}(\|u_h^1\|^2 + \|u_h^0\|^2) \\ &\qquad + \Delta t b^*(U, u_h^0, u_h^1) + \Delta t(f_C \times u_h^0, u_h^1). \end{aligned}$$

Denote

$$\epsilon^{n+\frac{1}{2}} = \frac{1}{2}(\|u_h^{n+1}\|^2 + \|u_h^n\|^2) + \Delta t b^*(U, u_h^n, u_h^{n+1}) + \Delta t(f_C \times u_h^n, u_h^{n+1}).$$

Applying the inverse inequality to the trilinear term gives

$$\Delta t b^*(U, u_h^n, u_h^{n+1}) \leq \Delta t \|U\|_\infty c_{inv} h^{-1} \|u_h^n\| \|u_h^{n+1}\|. \tag{3.5}$$

The other skew symmetric term can be estimated as follows

$$\Delta t(f_C \times u_h^n, u_h^{n+1}) \leq \Delta t \|f_C\|_\infty \|u_h^n\| \|u_h^{n+1}\|. \tag{3.6}$$

Hence,

$$\begin{aligned} \epsilon^{n+\frac{1}{2}} &\geq \frac{1}{2}(\|u_h^{n+1}\|^2 + \|u_h^n\|^2) \\ &\qquad - \Delta t \|U\|_\infty c_{inv} h^{-1} \|u_h^n\| \|u_h^{n+1}\| - \Delta t \|f_C\|_\infty \|u_h^n\| \|u_h^{n+1}\| \\ &\geq \left(\frac{1}{2} - \frac{1}{2} \Delta t (\|U\|_\infty c_{inv} h^{-1} + \|f_C\|_\infty) \right) \|u_h^n\|^2 \\ &\quad + \left(\frac{1}{2} - \frac{1}{2} \Delta t (\|U\|_\infty c_{inv} h^{-1} + \|f_C\|_\infty) \right) \|u_h^{n+1}\|^2 \\ &\geq (1 - \Delta t (\|U\|_\infty c_{inv} h^{-1} + \|f_C\|_\infty)) \left(\frac{1}{2} \|u_h^n\|^2 + \frac{1}{2} \|u_h^{n+1}\|^2 \right). \end{aligned} \tag{3.7}$$

Let $\beta = (1 - \Delta t(\|U\|_\infty c_{inv} h^{-1} + \|f_C\|_\infty))$. Then under the timestep condition, (3.3) follows. ∎

4. Stability of STAFASL. In this section, we prove the unconditional stability of STAFASL for Navier-Stokes equation with a Coriolis term. The added stabilization eliminates the restriction on the timestep from CNLF, which gives us the unconditional stability. Same as in Theorem 3.1, U only needs to satisfy $U = U(x)$ and $U \cdot \widehat{n}|_{\partial\Omega} = 0$.

With FEM discretizations, we apply the usual explicitly skew symmetric trilinear form defined in Section 2 for both nonlinear parts discretized by CN and LF respectively to maintain the skew symmetry of the nonlinear terms. For the part discretized by LF, by (2.1), we have

$$b^*(U, u_h^n, v_h) = \int_\Omega U \cdot \nabla u_h^n \cdot v_h \; dx + \frac{1}{2} \int_\Omega (\nabla \cdot U)(u_h^n \cdot v_h) \; dx, \tag{4.1}$$

Therefore for compactness, define the linear, skew symmetric operator

$$\Lambda_h(v) := U \cdot \nabla v + \frac{1}{2} (\nabla \cdot U) \, v + f_C \times v.$$

Then by (4.1),

$$b^*(U,u_h^n,v_h)+(f_C\times u_h^n,v_h)=(\Lambda_h(u_h^n),v_h)\,, \tag{4.2}$$

Note by (3.5) and (3.6) in the proof of Theorem 3.1, we derive an upper bound on the norm of Λ_h

$$\|\Lambda_h\|\le\|U\|_\infty c_{inv}h^{-1}+\|f_C\|_\infty, \tag{4.3}$$

and name $(\|U\|_\infty c_{inv}h^{-1}+\|f_C\|_\infty)^{-1}$ the **CFL limit**.

The stabilization we study is $2\Delta t\Lambda_h^*\Lambda_h(u_h^{n+1}-u_h^{n-1})$ and STAFASL scheme reads: find $(u_h^{n+1},p_h^{n+1})\in X_h\times Q_h$ satisfying, for all $v_h\in X_h, q_h\in Q_h$,

$$\begin{aligned}&\left(\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t},v_h\right)+b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}-U,\frac{u_h^{n+1}+u_h^{n-1}}{2},v_h)\\&+b^*(U,u_h^n,v_h)+(f_C\times u_h^n,v_h)+\nu(\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2},\nabla v_h)\\&-(\frac{p_h^{n+1}+p_h^{n-1}}{2},\nabla\cdot v_h)+2\Delta t\left(\Lambda_h(u_h^{n+1}-u_h^{n-1}),\Lambda_h(v_h)\right)=(f^n,v_h),\\&(\nabla\cdot u_h^{n+1},q_h)=0.\end{aligned} \tag{4.4}$$

THEOREM 4.1 (Unconditional Stability). STAFASL *is unconditionally stable. Specifically, for any $N\ge1$, there holds*

$$\begin{aligned}&\frac12\|u_h^{N+1}\|^2+\frac14\|u_h^N\|^2+\Delta t^2\|\Lambda_h(u_h^{N+1})\|^2+2\Delta t^2\|\Lambda_h(u_h^N)\|^2\\&+\Delta t\sum_{n=1}^N\nu\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2\le\frac12\|u_h^1\|^2+\frac12\|u_h^0\|^2+2\Delta t^2\|\Lambda_h(u_h^1)\|^2\\&+2\Delta t^2\|\Lambda_h(u_h^0)\|^2+\Delta t(\Lambda_h(u_h^0),u_h^1)+\frac{\Delta t}{\nu}\sum_{n=1}^N\|f^n\|_*^2.\end{aligned}$$

Proof. In (4.4) set $v_h=\frac{u_h^{n+1}+u_h^{n-1}}{2}$ and multiply through by $2\Delta t$. This gives, after adding and subtracting $\frac12||u_h^n||^2+2\Delta t^2||\Lambda_h(u_h^n)||^2$,

$$\begin{aligned}&\left(\frac12\|u_h^{n+1}\|^2+\frac12\|u_h^n\|^2+2\Delta t^2\|\Lambda_h(u_h^{n+1})\|^2+2\Delta t^2\|\Lambda_h(u_h^n)\|^2\right)\\&-\left(\frac12\|u_h^n\|^2+\frac12\|u_h^{n-1}\|^2+2\Delta t^2\|\Lambda_h(u_h^n)\|^2+2\Delta t^2\|\Lambda_h(u_h^{n-1})\|^2\right)\\&+\Delta tb^*(U,u_h^n,u_h^{n+1}+u_h^{n-1})+\Delta t(f_C\times u_h^n,u_h^{n+1}+u_h^{n-1})\\&+2\Delta t\nu\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2=2\Delta t(f^n,\frac{u_h^{n+1}+u_h^{n-1}}{2}).\end{aligned} \tag{4.5}$$

Applying Young's inequality to the RHS, (4.5) reduces to

$$\begin{aligned}&\left(\frac12\|u_h^{n+1}\|^2+\frac12\|u_h^n\|^2+2\Delta t^2\|\Lambda_h(u_h^{n+1})\|^2+2\Delta t^2\|\Lambda_h(u_h^n)\|^2\right)\\&-\left(\frac12\|u_h^n\|^2+\frac12\|u_h^{n-1}\|^2+2\Delta t^2\|\Lambda_h(u_h^n)\|^2+2\Delta t^2\|\Lambda_h(u_h^{n-1})\|^2\right)\\&+\Delta tb^*(U,u_h^n,u_h^{n+1}+u_h^{n-1})+\Delta t(f_C\times u_h^n,u_h^{n+1}+u_h^{n-1})\\&+\Delta t\nu\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2\le\frac{\Delta t}{\nu}\|f^n\|_*^2.\end{aligned} \tag{4.6}$$

By integration by parts we have

$$b^*(U,u_h^n,u_h^{n+1}+u_h^{n-1})=(U\cdot\nabla u_h^n+\frac{1}{2}(\nabla\cdot U)u_h^n,u_h^{n+1}+u_h^{n-1}).$$

Thus,

$$b^*(U,u_h^n,u_h^{n+1}+u_h^{n-1})+(f_C\times u_h^n,u_h^{n+1}+u_h^{n-1})=(\Lambda_h(u_h^n),u_h^{n+1}+u_h^{n-1}).$$

Define

$$\begin{aligned}C^{n+\frac{1}{2}}&:=(\Lambda_h(u_h^n),u_h^{n+1}),\ \text{and}\\ E^{n+\frac{1}{2}}&:=\frac{1}{2}\|u_h^{n+1}\|^2+\frac{1}{2}\|u_h^n\|^2+2\Delta t^2\|\Lambda_h(u_h^{n+1})\|^2+2\Delta t^2\|\Lambda_h(u_h^n)\|^2.\end{aligned}$$

Using skew symmetry of Λ_h, (4.6) can be rewritten as

$$\begin{aligned}E^{n+\frac{1}{2}}-E^{n-\frac{1}{2}}+\Delta t(C^{n+\frac{1}{2}}-C^{n-\frac{1}{2}})\\ +\Delta t\nu\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2\le\frac{\Delta t}{\nu}\|f^n\|_*^2.\end{aligned}\tag{4.7}$$

Summing up (4.7) from $n=1$ to N results in

$$\begin{aligned}E^{N+\frac{1}{2}}+\Delta tC^{N+\frac{1}{2}}+\Delta t\sum_{n=1}^{N}\nu\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2\\ \le E^{1-\frac{1}{2}}+\Delta tC^{1-\frac{1}{2}}+\frac{\Delta t}{\nu}\sum_{n=1}^{N}\|f^n\|_*^2.\end{aligned}\tag{4.8}$$

By Cauchy-Schwarz inequality

$$\begin{aligned}\Delta tC^{N+\frac{1}{2}}=\Delta t(\Lambda_h(u_h^N),u_h^{N+1})=-\Delta t(\Lambda_h(u_h^{N+1}),u_h^N)\\ \le\Delta t^2\|\Lambda_h(u_h^{N+1})\|^2+\frac{1}{4}\|u_h^N\|^2,\end{aligned}$$

and the stability follows. □

5. Asymptotic stability of the unstable mode for Oseen equation plus rotation term. In this section, we show that the unstable mode of CNLF is actually stable for the time dependent Oseen problem + rotation/Coriolis term:

$$\begin{aligned}u_t+U\cdot\nabla u-\nu\Delta u+\nabla p+f_C\times u=f(x,t)\text{ in }\Omega,\\ \nabla\cdot u=0\text{ in }\Omega,\\ u(x,0)=u_0(x)\text{ in }\Omega\text{ and }u=0\text{ on }\partial\Omega.\end{aligned}\tag{5.1}$$

Here, $U(x)$ is assumed to be a smooth, divergence free velocity field. The stability of unstable mode of CNLF for full NSE + rotation term is an open question.

5.1. The damping of unstable mode of FASL scheme. The corresponding FASL time stepping method for (5.1) is: Given $u_h^{n-1},u_h^n,p_h^{n-1},p_h^n$, find $u_h^{n+1}\in X_h$, $p_h^{n+1}\in Q_h$ satisfying

$$\begin{aligned}(\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t},v_h)-(\frac{p_h^{n+1}+p_h^{n-1}}{2},\nabla\cdot v_h)+\nu(\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2},\nabla v_h)\\ +b^*(U,u_h^n,v_h)+(f_C\times u_h^n,v_h)=(f^n,v_h),\qquad\forall v_h\in X_h,\\ (\nabla\cdot u_h^{n+1},q_h)=0,\qquad\forall q_h\in Q_h.\end{aligned}\tag{FASL-Oseen}$$

Theorem 5.1. *Consider* (FASL-Oseen). *Suppose the following conditions hold,*

$$\Delta t < \textbf{CFL limit}, \tag{5.2}$$

then both the stable mode and the unstable modes are stable. In particular, if the body force $f \equiv 0$*, all modes are asymptotically stable:*

$$u_h^{n+1} + u_h^{n-1} \longrightarrow 0 \quad \textit{and} \quad u_h^{n+1} - u_h^{n-1} \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty. \tag{5.3}$$

Proof. The only different term $(\frac{u_h^{n+1}+u_h^{n-1}}{2} - U) \cdot \nabla \frac{u_h^{n+1}+u_h^{n-1}}{2}$ of NSE from the time dependent Oseen problem vanishes after taking inner product with $\frac{u_h^{n+1}+u_h^{n-1}}{2}$. Therefore we use the stability results of Theorem 3.1 without proving. As in the proof of Theorem 3.1, we have for the time dependent Oseen problem with Coriolis term

$$\epsilon^{n+\frac{1}{2}} - \epsilon^{n-\frac{1}{2}} + \nu \Delta t \|\nabla \frac{u_h^{n+1} + u_h^{n-1}}{2}\|^2 \leq \frac{\Delta t}{\nu}\|f^n\|_*^2, \tag{5.4}$$

where

$$\epsilon^{n+\frac{1}{2}} = \frac{1}{2}(\|u_h^{n+1}\|^2 + \|u_h^n\|^2) + \Delta t b^*(U, u_h^n, u_h^{n+1}) + \Delta t(f_C \times u_h^n, u_h^{n+1}).$$

Under condition (5.2), which is exactly the same as (3.2), we have the following stability result

$$\beta\left(\frac{1}{2}\|u_h^{N+1}\|^2 + \frac{1}{2}\|u_h^N\|^2\right) + \sum_{n=1}^{N} \nu \Delta t \left\|\nabla \frac{u_h^{n+1} + u_h^{n-1}}{2}\right\|^2 \tag{5.5}$$

$$\leq \sum_{n=1}^{N} \frac{\Delta t}{\nu}\|f^n\|_*^2 + \frac{1}{2}\left(\|u_h^1\|^2 + \|u_h^0\|^2\right) + \Delta t b^*(U, u_h^0, u_h^1) + \Delta t(f_C \times u_h^0, u_h^1).$$

where $\beta = (1 - \Delta t(\|U\|_\infty c_{inv} h^{-1} + \|f_C\|_\infty))$. The stable mode is stable. In particular, if the body force $f = 0$, dropping nonnegative terms in the LHS, then (5.5) reduces to

$$\sum_{n=1}^{N} \left\|\nabla \frac{u_h^{n+1} + u_h^{n-1}}{2}\right\|^2 \leq C(\nu, u_h^0, u_h^1). \tag{5.6}$$

$C(\nu, u_h^0, u_h^1)$ is independent of N, so letting $N \longrightarrow \infty$ gives

$$\|\nabla(u_h^{n+1} + u_h^{n-1})\| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty. \tag{5.7}$$

By Poincaré inequality, we have

$$\|u_h^{n+1} + u_h^{n-1}\| \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty. \tag{5.8}$$

Next we prove, under the same condition, the unstable mode is also stable. Set $v_h = u_h^{n+1} - u_h^{n-1}$ in (FASL-Oseen), multiply though by $2\delta\Delta t$, where $\delta > 0$, and then add and subtract $\delta\Delta t\|u_h^n\|^2$, (FASL-Oseen) becomes

$$\begin{aligned} &\delta\nu\Delta t(\|\nabla u_h^{n+1}\|^2 + \|\nabla u_h^n\|^2) - \delta\nu\Delta t(\|\nabla u_h^n\|^2 + \|\nabla u_h^{n-1}\|^2) \\ &+2\delta\Delta t b^*(U, u_h^n, u_h^{n+1} - u_h^{n-1}) + 2\delta\Delta t(f_C \times u_h^n, u_h^{n+1} - u_h^{n-1}) \\ &+\delta\|u_h^{n+1} - u_h^{n-1}\|^2 = 2\delta\Delta t(f^n, u_h^{n+1} - u_h^{n-1}). \end{aligned} \tag{5.9}$$

Denote

$$D^{n+\frac{1}{2}} = \epsilon^{n+\frac{1}{2}} + \delta\nu\Delta t(\|\nabla u_h^{n+1}\|^2 + \|\nabla u_h^n\|^2).$$

Adding (5.4) and (5.9) and applying Cauchy-Schwarz and Young's inequalities give, for any α, $0 < \alpha < 1$,

$$D^{n+\frac{1}{2}} - D^{n-\frac{1}{2}} + \nu\Delta t\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2 + \alpha\delta\|u_h^{n+1} - u_h^{n-1}\|^2 \tag{5.10}$$
$$+2\delta\Delta t(\Lambda_h(u_h^n), u_h^{n+1} - u_h^{n-1}) \leq \frac{\Delta t}{\nu}\|f^n\|_*^2 + \frac{\delta\Delta t^2}{\alpha}\|f^n\|^2.$$

Recall that

$$\Lambda_h(v) := U\cdot\nabla v + \frac{1}{2}\left(\nabla\cdot U\right)v + f_C\times v.$$

Summing (5.10) from $n=1$ to N gives

$$D^{N+\frac{1}{2}} + \sum_{n=1}^{N}\left[\nu\Delta t\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2 + \alpha\delta\|u_h^{n+1} - u_h^{n-1}\|^2\right] \tag{5.11}$$
$$+B^N \leq D^{1+\frac{1}{2}} + \sum_{n=1}^{N}\left[\frac{\Delta t}{\nu}\|f^n\|_*^2 + \frac{\delta\Delta t^2}{\alpha}\|f^n\|^2\right],$$

where

$$B^N = \sum_{n=1}^{N}2\delta\Delta t\big(\Lambda_h(u_h^n), u_h^{n+1} - u_h^{n-1}\big).$$

Let $C_\infty = (\|U\|_\infty c_{inv}h^{-1} + \|f_C\|_\infty)$. Using (3.5) and (3.6) from the proof of Theorem 3.1, we have the following bound on B^N

$$B^N = \sum_{n=1}^{N}2\delta\Delta t\big(\Lambda_h(u_h^n), u_h^{n+1} - u_h^{n-1}\big) = 2\delta\Delta t\big(\Lambda_h(u_h^1), u_h^2 - u_h^0\big)$$
$$+2\delta\Delta t\sum_{n=2}^{N}\left[\frac{1}{2}\big(\Lambda_h(u_h^n - u_h^{n-2}), u_h^{n+1} - u_h^{n-1}\big) + \frac{1}{2}\big(\Lambda_h(u_h^n + u_h^{n-2}), u_h^{n+1} - u_h^{n-1}\big)\right]$$
$$\leq 2\delta\Delta t\big(\Lambda_h(u_h^1), u_h^2 - u_h^0\big) + 2\delta\Delta t\sum_{n=2}^{N}\left[\frac{1}{2}C_\infty\|u_h^n - u_h^{n-2}\|\|u_h^{n+1} - u_h^{n-1}\|\right.$$
$$\left.+\frac{1}{2}C_\infty\|u_h^n + u_h^{n-2}\|\|u_h^{n+1} - u_h^{n-1}\|\right] \tag{5.12}$$
$$\leq 2\delta\Delta t\big(\Lambda_h(u_h^1), u_h^2 - u_h^0\big) + 2\delta\Delta t\sum_{n=2}^{N}\left[\frac{1}{2}C_\infty\left(\frac{1}{2}\|u_h^n - u_h^{n-2}\|^2 + \frac{1}{2}\|u_h^{n+1} - u_h^{n-1}\|^2\right)\right.$$
$$\left.+\frac{1}{2}C_\infty\left(\frac{1}{2\epsilon}\|u_h^n + u_h^{n-2}\|^2 + \frac{\epsilon}{2}\|u_h^{n+1} - u_h^{n-1}\|^2\right)\right],$$
$$\leq 2\delta\Delta t\big(\Lambda_h(u_h^1), u_h^2 - u_h^0\big) + 2\delta\Delta t\sum_{n=1}^{N}\left[\frac{1}{2}C_\infty\|u_h^{n+1} - u_h^{n-1}\|^2\right.$$
$$\left.+C_\infty\left(\frac{C}{\epsilon}\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2 + \frac{\epsilon}{4}\|u_h^{n+1} - u_h^{n-1}\|^2\right)\right],$$

where $0 < \epsilon < 1$. Plugging (5.12) into (5.11) gives

$$D^{N+\frac{1}{2}} + \Delta t(\nu - \frac{2\delta C_\infty C}{\epsilon})\sum_{n=1}^{N}\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2$$

$$+\delta\Big(\alpha - (\Delta t C_\infty + \frac{\epsilon \Delta t C_\infty}{2})\Big) \sum_{n=1}^{N} \|u_h^{n+1} - u_h^{n-1}\|^2 \tag{5.13}$$

$$\leq 2\delta\Delta t\big(\Lambda_h(u_h^1), u_h^2 - u_h^0\big) + D^{1+\frac{1}{2}} + \sum_{n=1}^{N} \left[\frac{\Delta t}{\nu}\|f^n\|_*^2 + \frac{\delta \Delta t^2}{\alpha}\|f^n\|^2\right].$$

Under the conditions

$$\delta < \frac{\epsilon\nu}{CC_\infty} \quad \text{and} \quad \Delta t < \frac{\alpha}{1+\frac{\epsilon}{2}} C_\infty^{-1}, \tag{5.14}$$

both the stable and the unstable modes are stable. Since α and ϵ are arbitrary in $(0, 1)$, the second condition in (5.14) is equivalent to $\Delta t < C_\infty^{-1}$. This is the same condition, (3.2), to ensure stability of (FASL) in Theorem 3.1 and therefore also a sufficient condition for the stability of the time dependent Oseen problem. $\delta > 0$ is also arbitrary, so we can always pick δ to have the first condition in (5.14) satisfied. By Theorem 3.1, under the time step restriction (5.2), $\epsilon^{N+\frac{1}{2}}$ is positive and therefore $D^{N+\frac{1}{2}}$ is also positive:

$$D^{N+\frac{1}{2}} \geq \beta\left(\frac{1}{2}\|u_h^{N+1}\|^2 + \frac{1}{2}\|u_h^N\|^2\right) + \delta\nu\Delta t(\|\nabla u_h^{N+1}\|^2 + \|\nabla u_h^N\|^2). \tag{5.15}$$

Dropping nonnegative terms in LHS of (5.13) gives

$$\delta\Big(\alpha - (\Delta t C_\infty + \frac{\epsilon \Delta t C_\infty}{2})\Big) \sum_{n=1}^{N} \|u_h^{n+1} - u_h^{n-1}\|^2 \tag{5.16}$$

$$\leq 2\delta\Delta t\big(\Lambda_h(u_h^1), u_h^2 - u_h^0\big) + D^{1+\frac{1}{2}} + \sum_{n=1}^{N} \left[\frac{\Delta t}{\nu}\|f^n\|_*^2 + \frac{\delta \Delta t^2}{\alpha}\|f^n\|^2\right].$$

So the unstable mode is also stable. In particular, assume the body force $f = 0$, dropping some nonnegative terms, (5.16) reduces to

$$\sum_{n=1}^{N} \|u_h^{n+1} - u_h^{n-1}\|^2 < C(\delta, \epsilon, \nu, u_h^0, u_h^1) < \infty. \tag{5.17}$$

$C(\delta, \epsilon, \nu, u_h^0, u_h^1)$ is independent of N, so letting $N \longrightarrow \infty$ gives

$$\|u_h^{n+1} - u_h^{n-1}\|^2 \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty. \tag{5.18}$$

This completes the proof of Theorem 5.1.
□

5.2. The damping of unstable mode for STAFASL scheme. The stabilization we added to FASL eliminates the time step condition, controlling the stable mode. We next prove that this stabilization also eliminates the time step restriction for Oseen problem plus a rotation/Coriolis term, controlling both the stable and unstable modes.

The corresponding STAFASL time stepping method for (5.1) is: Given u_h^{n-1}, u_h^n, p_h^{n-1}, p_h^n, find $u_h^{n+1} \in X_h$, $p_h^{n+1} \in Q_h$ satisfying

$$\begin{aligned}
&(\frac{u_h^{n+1} - u_h^{n-1}}{2\Delta t}, v_h) - (\frac{p_h^{n+1} + p_h^{n-1}}{2}, \nabla \cdot v_h) && \text{(STAFASL-Oseen)}\\
&\quad +\nu(\nabla \frac{u_h^{n+1} + u_h^{n-1}}{2}, \nabla v_h) + 2\Delta t(\Lambda_h(u^{n+1} - u^{n-1}), \Lambda_h(v_h))\\
&\quad +b^*(U, u_h^n, v_h) + (f_C \times u_h^n, v_h) = (f^n, v_h), \qquad \forall v_h \in X_h,\\
&\quad (\nabla \cdot u_h^{n+1}, q_h) = 0, \qquad \forall q_h \in Q_h.
\end{aligned}$$

THEOREM 5.2. *Consider* (STAFASL-Oseen)*. Both the stable mode and the unstable modes are unconditionally stable. In particular, if the body force* $f \equiv 0$*, all modes are unconditionally, asymptotically stable:*

$$u_h^{n+1} + u_h^{n-1} \longrightarrow 0 \quad \textit{and} \quad u_h^{n+1} - u_h^{n-1} \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty. \tag{5.19}$$

Proof. The regular stability proof will follow exactly the proof of Theorem 4.1 and we will use its results without proving. As in the proof of Theorem 4.1, we have

$$\begin{aligned} E^{n+\frac{1}{2}} - E^{n-\frac{1}{2}} + \Delta t(C^{n+\frac{1}{2}} - C^{n-\frac{1}{2}}) \\ +\Delta t\nu\|\nabla \frac{u_h^{n+1} + u_h^{n-1}}{2}\|^2 \leq \frac{\Delta t}{\nu}\|f^n\|_*^2, \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} C^{n+\frac{1}{2}} &:= (\Lambda_h(u_h^n), u_h^{n+1}), \text{ and} \\ E^{n+\frac{1}{2}} &:= \frac{1}{2}\|u_h^{n+1}\|^2 + \frac{1}{2}\|u_h^n\|^2 + 2\Delta t^2\|\Lambda_h(u_h^{n+1})\|^2 + 2\Delta t^2\|\Lambda_h(u_h^n)\|^2. \end{aligned}$$

Now set $v_h = u_h^{n+1} - u_h^{n-1}$ in (STAFASL-Oseen), multiply through by $2\delta\Delta t$, where $\delta > 0$, and then add and subtract $\delta\Delta t\|u_h^n\|^2$, (STAFASL-Oseen) becomes

$$\begin{aligned} \delta\nu\Delta t(\|\nabla u_h^{n+1}\|^2 + \|\nabla u_h^n\|^2) - \delta\nu\Delta t(\|\nabla u_h^n\|^2 + \|\nabla u_h^{n-1}\|^2) \\ +2\delta\Delta t b^*(U, u_h^n, u_h^{n+1} - u_h^{n-1}) + 2\delta\Delta t(f_C \times u_h^n, u_h^{n+1} u_h^{n-1}) \\ +\delta\|u_h^{n+1} - u_h^{n-1}\|^2 + 4\delta\Delta t^2\|\Lambda_h(u_h^{n+1} - u_h^{n-1})\|^2 = 2\delta\Delta t(f^n, u_h^{n+1} - u_h^{n-2}). \end{aligned} \tag{5.21}$$

Denote

$$D^{n+\frac{1}{2}} = E^{n+\frac{1}{2}} + \Delta t C^{n+\frac{1}{2}} + \delta\nu\Delta t(\|\nabla u_h^{n+1}\|^2 + \|\nabla u_h^n\|^2).$$

Adding (5.20) and (5.21) and applying Cauchy-Schwarz inequality to the RHS gives

$$\begin{aligned} D^{n+\frac{1}{2}} - D^{n-\frac{1}{2}} + \nu\Delta t\|\nabla \frac{u_h^{n+1} + u_h^{n-1}}{2}\|^2 + \frac{\delta}{2}\|u_h^{n+1} - u_h^{n-1}\|^2 \\ +4\delta\Delta t^2\|\Lambda_h(u_h^{n+1} - u_h^{n-1})\|^2 + 2\delta\Delta t(\Lambda_h(u_h^n), u_h^{n+1} - u_h^{n-1}) \\ \leq \frac{\Delta t}{\nu}\|f^n\|_*^2 + 2\delta\Delta t^2\|f^n\|^2. \end{aligned} \tag{5.22}$$

Summing (5.22) from $n = 1$ to N gives

$$\begin{aligned} D^{N+\frac{1}{2}} + \sum_{n=1}^{N}\left[\nu\Delta t\|\nabla \frac{u_h^{n+1} + u_h^{n-1}}{2}\|^2 + \frac{\delta}{2}\|u_h^{n+1} - u_h^{n-1}\|^2\right. \\ \left.+4\delta\Delta t^2\|\Lambda_h(u_h^{n+1} - u_h^{n-1})\|^2\right] + B^N \leq D^{1+\frac{1}{2}} + \sum_{n=1}^{N}\left[\frac{\Delta t}{\nu}\|f^n\|_*^2 + 2\delta\Delta t^2\|f^n\|^2\right], \end{aligned} \tag{5.23}$$

where

$$B^N = \sum_{n=1}^{N} 2\delta\Delta t(\Lambda_h(u_h^n), u_h^{n+1} - u_h^{n-1}).$$

For B^N, applying Young's inequality gives, for any ϵ, $0 < \epsilon < \frac{1}{2}$

$$B^N \le \delta\epsilon \sum_{n=1}^{N} \|u_h^{n+1} - u_h^{n-1}\|^2 + \frac{\delta}{\epsilon} \sum_{n=1}^{N} \Delta t^2 \|\Lambda_h(u_h^n)\|^2. \tag{5.24}$$

The second term on the RHS can be rewritten as

$$\begin{aligned} \|\Lambda_h(u_h^n)\|^2 &= \|\Lambda_h(\frac{u_h^n + u_h^{n-2}}{2}) + \Lambda_h(\frac{u_h^n - u_h^{n-2}}{2})\|^2 \\ &= 2\|\Lambda_h(\frac{u_h^n + u_h^{n-2}}{2})\|^2 + 2\|\Lambda_h(\frac{u_h^n - u_h^{n-2}}{2})\|^2 - \|\Lambda_h(u_h^{n-2})\|^2. \end{aligned}$$

Then (5.24) becomes

$$\begin{aligned} B^N \le \delta\epsilon \sum_{n=1}^{N} \|u_h^{n+1} - u_h^{n-1}\|^2 + \frac{\delta}{\epsilon}\Delta t^2 \|\Lambda_h(u_h^1)\|^2 - \frac{\delta}{\epsilon} \sum_{n=2}^{N} \Delta t^2 \|\Lambda_h(u_h^{n-2})\|^2 \\ + \frac{\delta}{2\epsilon} \sum_{n=2}^{N} \Delta t^2 \left(\|\Lambda_h(u_h^n + u_h^{n-2})\|^2 + \|\Lambda_h(u_h^n - u_h^{n-2})\|^2 \right). \end{aligned} \tag{5.25}$$

After shifting the index of the second sum, we obtain

$$\begin{aligned} B^N \le \delta\epsilon \sum_{n=1}^{N} \|u_h^{n+1} - u_h^{n-1}\|^2 + \frac{\delta}{\epsilon}\Delta t^2 \|\Lambda_h(u_h^1)\|^2 - \frac{\delta}{\epsilon} \sum_{n=2}^{N} \Delta t^2 \|\Lambda_h(u_h^{n-2})\|^2 \\ + \frac{\delta}{2\epsilon} \sum_{n=1}^{N-1} \Delta t^2 \left(\|\Lambda_h(u_h^{n+1} + u_h^{n-1})\|^2 + \|\Lambda_h(u_h^{n+1} - u_h^{n-1})\|^2 \right). \end{aligned} \tag{5.26}$$

Plugging (5.26) into (5.23) gives

$$\begin{aligned} & D^{N+\frac{1}{2}} + \nu\Delta t \|\nabla \frac{u_h^{N+1} + u_h^{N-1}}{2}\|^2 + \frac{\delta}{2} \sum_{n=1}^{N} \|u_h^{n+1} - u_h^{n-1}\|^2 \\ & + \sum_{n=1}^{N-1} \left[\nu\Delta t \|\nabla \frac{u_h^{n+1} + u_h^{n-1}}{2}\|^2 - \frac{\delta}{2\epsilon}\Delta t^2 \|\Lambda_h(u_h^{n+1} + u_h^{n-1})\|^2 \right] \\ & + \sum_{n=1}^{N-1} \left[4\delta\Delta t^2 \|\Lambda_h(u_h^{n+1} - u_h^{n-1})\|^2 - \frac{\delta}{2\epsilon}\Delta t^2 \|\Lambda_h(u_h^{n+1} - u_h^{n-1})\|^2 \right] \\ & \le D^{1+\frac{1}{2}} + \frac{\delta}{\epsilon}\Delta t^2 \|\Lambda_h(u_h^1)\|^2 + \sum_{n=1}^{N} \left[\frac{\Delta t}{\nu} \|f^n\|_*^2 + 2\delta\Delta t^2 \|f^n\|^2 \right]. \end{aligned} \tag{5.27}$$

Under the following conditions

$$\Delta t \|\Lambda_h\|_*^2 < \frac{\epsilon\nu}{2\delta} \quad \text{and} \quad 4 > \frac{1}{2\epsilon}, \tag{5.28}$$

we have the stability of both the stable and unstable modes. Since ϵ in $(0, \frac{1}{2})$, we pick $\epsilon = \frac{1}{4}$ and the second condition is satisfied. Note $\|\Lambda_h\|_*$ is bounded

$$\|\Lambda_h\|_* \le C(\|U\| + \|\nabla \cdot U\| + \|f_C\|),$$

and $\delta > 0$ is arbitrary. So we can always pick δ, such that

$$\delta < \frac{\nu}{8\Delta t \|\Lambda_h\|_*^2}.$$

Let $\delta = \frac{\nu}{16\Delta t\|\Lambda_h\|_*^2}$. Then (5.27) reduces to

$$\begin{aligned}
&D^{N+\frac{1}{2}} + \nu\Delta t\|\nabla\frac{u_h^{N+1}+u_h^{N-1}}{2}\|^2 + \frac{\delta}{2}\sum_{n=1}^{N}\|u_h^{n+1}-u_h^{n-1}\|^2 \\
&+\frac{\nu\Delta t}{2}\sum_{n=1}^{N-1}\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2 + 2\delta\Delta t^2\sum_{n=1}^{N-1}\|\Lambda_h(u_h^{n+1}-u_h^{n-1})\|^2 \qquad (5.29)\\
&\leq D^{1+\frac{1}{2}} + 4\delta\Delta t^2\|\Lambda_h(u_h^1)\|^2 + \sum_{n=1}^{N}\left[\frac{\Delta t}{\nu}\|f^n\|_*^2 + 2\delta\Delta t^2\|f^n\|^2\right].
\end{aligned}$$

By Theorem 4.1, $E^{N+\frac{1}{2}} + \Delta t C^{N+\frac{1}{2}}$ is positive and therefore $D^{N+\frac{1}{2}}$ is also positive

$$\begin{aligned}
D^{N+\frac{1}{2}} \geq\ &\frac{1}{2}\|u_h^{N+1}\|^2 + \frac{1}{4}\|u_h^N\|^2 + \Delta t^2\|\Lambda_h(u_h^{N+1})\|^2 \qquad (5.30)\\
&+2\Delta t^2\|\Lambda_h(u_h^N)\|^2 + \delta\nu\Delta t(\|\nabla u_h^{N+1}\|^2 + \|\nabla u_h^N\|^2).
\end{aligned}$$

In particular, assume the body force $f = 0$, dropping some nonnegative term, (5.29) reduces to

$$\sum_{n=1}^{N}\left[\|u_h^{n+1}+u_h^{n-1}\|^2 + \|u_h^{n+1}-u_h^{n-1}\|^2\right] < C(\nu, u_h^0, u_h^1) < \infty. \qquad (5.31)$$

$C(\nu, u_h^0, u_h^1)$ is constant independent of N, so letting $N \longrightarrow \infty$ gives

$$\|u_h^{n+1}+u_h^{n-1}\|^2 \longrightarrow 0 \quad \text{and} \quad \|u_h^{n+1}-u_h^{n-1}\|^2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

This completes the proof of Theorem 5.2.
□

6. Error Analysis for STAFASL. We proceed to give an a priori error estimate for the approximations studied herein. Due to the intricateness of the proofs, for the compactness, we only analyze the error of STAFASL scheme. With minor modifications, we will get the analogous results of convergence rate for FASL.

Let $t^n := n\Delta t, n = 0, 1, 2, ..., N_T$, and $T := N_T\Delta t$. Denote $u^n = u(\cdot, t^n)$. We introduce the following discrete norms:

$$|||v|||_{m,k} := (\sum_{n=0}^{N_T}||v^n||_k^m\Delta t)^{1/m}, \qquad |||v|||_{\infty,k} := \max_{0\leq n\leq N_T}\|v^n\|_k.$$

Theorem 6.1 (Convergence of STAFASL). *Consider* STAFASL *scheme. Suppose* (u, p) *satisfies the following regularity conditions:*

$$\begin{aligned}
&u \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;H^{k+1}(\Omega)) \cap H^3(0,T;L^2(\Omega)),\\
&p \in L^2(0,T;H^{s+1}(\Omega)) \cap H^2(0,T;L^2(\Omega)), \text{ and } f \in L^2(0,T;L^2(\Omega)).
\end{aligned}$$

Then, for Δt *sufficiently small, there is a positive constant* C *independent of the mesh width and time step such that*

$$\begin{aligned}
&\frac{1}{2}\left(\|e^{N+1}\|^2 + \frac{3}{4}\|e^N\|^2 + 4\Delta t^2\|\Lambda_h(e^N)\|^2\right) + \Delta t\sum_{n=1}^{N}\frac{\nu}{4}||\nabla(e^{n+1}+e^{n-1})||^2\\
&\leq exp(\frac{C(T+\Delta t)(1+\nu^{-3}|||\nabla u|||_{\infty,0}^4)}{1-C\Delta t(1+\nu^{-3}|||\nabla u|||_{\infty,0}^4)})\Bigg[\frac{1}{2}\Big(\|e^1\|^2 + 8\Delta t^2\|\Lambda_h(e^1)\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{5}{4}\|e^0\|^2+4\Delta t^2\|\Lambda_h(e^0)\|^2\Big)+Ch^{2k}(\nu+\nu^{-3}+\nu^{-1}\|\nabla U\|^2)\||u\||^2_{2,k+1} \\
&+C(\Delta t)^4\nu^{-1}\|p_{tt}\|^2_{2,0}+Ch^{2s+2}\nu^{-1}\||p\||^2_{2,s+1}+Ch^{2k+1}\|\Lambda_h\|^2\||u\||^2_{2,k+1} \\
&+Ch^{2k+2}\|u_t\|^2_{2,k+1}+C(\Delta t)^4(\|u_{ttt}\|^2_{2,0}+\nu\|\nabla u_{tt}\|^2_{2,0}+\nu^{-1}\|\nabla u_{tt}\|^2_{2,0} \\
&+\nu^{-1}\|\nabla U\|^2\|\nabla u_{tt}\|^2_{2,0}+\|\Lambda_h^*\Lambda_h\|^2\|u_t\|^2_{2,0}+h^{2k+2}\|\Lambda_h^*\Lambda_h\|\|u_t\|^2_{2,k+1})\Big]\ .
\end{aligned} \tag{6.1}
$$

STAFASL is a 3 level method. To obtain (u_h^1, p_h^1) one must use another method, e.g., Crank-Nicholson. Note the errors in this first step will affect the overall convergence rate of our method.

Consequently, for Taylor-Hood elements, i.e. $k=2, s=1$, we have the following result.

COROLLARY 6.2. *Under the assumptions of Theorem 6.1, with (X_h, Q_h) given by the Taylor-Hood approximation elements ($k=2, s=1$), e^0 assumed to be 0, using a second order method in the first step, we have*

$$
\begin{aligned}
&\frac{1}{2}[||e^{N+1}||^2+\frac{3}{4}||e^N||^2+4\Delta t^2||\Lambda_h(e^N)||^2] \\
&\qquad +\Delta t\sum_{n=1}^{N}\frac{\nu}{4}||\nabla(e^{n+1}+e^{n-1})||^2\le C((\Delta t)^4+h^4).
\end{aligned}
$$

Proof. (Theorem 6.1)

The variational formulation of STAFASL is, for all $v_h\in V_h, q_h\in Q_h$,

$$
\begin{aligned}
&(\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t},v_h)+2\Delta t\left(\Lambda_h(u_h^{n+1}-u_h^{n-1}),\Lambda_h(v_h)\right) \\
&\qquad +b^*(U,u_h^n,v_h)+(f_C\times u_h^n,v_h) \\
&+\nu(\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2},\nabla v_h)+b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}-U,\frac{u_h^{n+1}+u_h^{n-1}}{2},v_h) \\
&-(\frac{p_h^{n+1}+p_h^{n-1}}{2},\nabla\cdot v_h)+(q_h,\nabla\cdot\frac{u_h^{n+1}+u_h^{n-1}}{2})=(f^n,v_h).
\end{aligned} \tag{6.2}
$$

At time t^n, the true solution (u,p) of the NSE + rotation force satisfies

$$
\begin{aligned}
&(\frac{u^{n+1}-u^{n-1}}{2\Delta t},v_h)+2\Delta t\left(\Lambda_h(u^{n+1}-u^{n-1}),\Lambda_h(v_h)\right) \\
&+b^*(U,u^n,v_h)+(f_C\times u^n,v_h)+\nu(\nabla\frac{u^{n+1}+u^{n-1}}{2},\nabla v_h) \\
&-(\frac{p^{n+1}+p^{n-1}}{2},\nabla\cdot v_h)=(f^n,v_h)+\tau(u^n;v_h)
\end{aligned} \tag{6.3}
$$

for all $v_h\in V_h$, where $\tau(u^n;v_h)$ represents the consistency error

$$
\begin{aligned}
\tau(u^n;v_h)=&\left(\frac{u^{n+1}-u^{n-1}}{2\Delta t}-u_t(\cdot,t^n),v_h\right)+2\Delta t\left(\Lambda_h(u^{n+1}-u^{n-1}),\Lambda_h(v_h)\right) \\
&-b^*(u^n,u^n,v_h)+b^*(U,u^n,v_h)+\nu\left(\nabla(\frac{u^{n+1}+u^{n-1}}{2}-u^n),\nabla v_h\right) \\
&-(\frac{p^{n+1}+p^{n-1}}{2}-p^n,\nabla\cdot v_h).
\end{aligned}
$$

Let

$$
e^n=u^n-u_h^n=(u^n-I_hu^n)+(I_hu^n-u_h^n)=\eta^n+\xi^n,
$$

where $I_h u^n \in V_h$ is an interpolation of u^n in V_h. Subtracting (6.2) from (6.3) gives

$$\begin{aligned}
&(\frac{\xi^{n+1}-\xi^{n-1}}{2\Delta t}, v_h) + 2\Delta t\left(\Lambda_h(\xi^{n+1}-\xi^{n-1}), \Lambda_h(v_h)\right) + (\Lambda_h(\xi^n), v_h) \\
&+\nu(\nabla\frac{\xi^{n+1}+\xi^{n-1}}{2}, \nabla v_h) + b^*(u^n, u^n, v_h) - b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}, \frac{u_h^{n+1}+u_h^{n-1}}{2}, v_h) \\
&-b^*(U, u^n - \frac{u_h^{n+1}+u_h^{n-1}}{2}, v_h) - 2\Delta t\left(\Lambda_h(u^{n+1}-u^{n-1}), \Lambda_h(v_h)\right) \qquad (6.4) \\
&= \left(\frac{u^{n+1}-u^{n-1}}{2\Delta t} - u_t(\cdot, t^n), v_h\right) - (\frac{\eta^{n+1}-\eta^{n-1}}{2\Delta t}, v_h) - 2\Delta t\left(\Lambda_h(\eta^{n+1}-\eta^{n-1}), \Lambda_h(v_h)\right) \\
&- (\Lambda_h(\eta^n), v_h) - \nu(\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}, \nabla v_h) + (\frac{p^{n+1}+p^{n-1}}{2} - q_h, \nabla\cdot v_h) \\
&+\nu\left(\nabla(\frac{u^{n+1}+u^{n-1}}{2} - u^n), \nabla v_h\right) - (\frac{p^{n+1}+p^{n-1}}{2} - p^n, \nabla\cdot v_h).
\end{aligned}$$

for all $q_h \in Q_h$. By the skew symmetry of Λ_h, we have

$$\left(\Lambda_h(\xi^n), \xi^{n+1}+\xi^{n-1}\right) = C^{n+\frac{1}{2}} - C^{n-\frac{1}{2}},$$

where

$$\begin{aligned}
C^{n+\frac{1}{2}} &= \left(\Lambda_h(\xi^n), \xi^{n+1}\right), \\
C^{n-\frac{1}{2}} &= -\left(\Lambda_h(\xi^n), \xi^{n-1}\right) = \left(\Lambda_h(\xi^{n-1}), \xi^n\right).
\end{aligned}$$

Set $v_h = \xi^{n+1}+\xi^{n-1} \in V_h$ and sum up (6.4) from $n=1$ to $n=N$

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\xi^{N+1}\|^2 + \|\xi^N\|^2 - \|\xi^1\|^2 - \|\xi^0\|^2) + 2\Delta t\Big(\|\Lambda_h(\xi^{N+1})\|^2 + \|\Lambda_h(\xi^N)\|^2 \\
&-\|\Lambda_h(\xi^1)\|^2 - \|\Lambda_h(\xi^0)\|^2\Big) + C^{N+\frac{1}{2}} - C^{1-\frac{1}{2}} + \sum_{n=1}^{N}\frac{\nu}{2}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 \\
&= \sum_{n=1}^{N}\Bigg\{\nu\left(\nabla(\frac{u^{n+1}+u^{n-1}}{2} - u^n), \nabla(\xi^{n+1}+\xi^{n-1})\right) \qquad (6.5) \\
&-b^*(u^n, u^n, \xi^{n+1}+\xi^{n-1}) + b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}, \frac{u_h^{n+1}+u_h^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \\
&+b^*(U, u^n - \frac{u_h^{n+1}+u_h^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) - (\frac{\eta^{n+1}-\eta^{n-1}}{2\Delta t}, \xi^{n+1}+\xi^{n-1}) \\
&+2\Delta t\Big(\Lambda_h(u^{n+1}-u^{n-1}), \Lambda_h(\xi^{n+1}+\xi^{n-1})\Big) \\
&-2\Delta t\left(\Lambda_h(\eta^{n+1}-\eta^{n-1}), \Lambda_h(\xi^{n+1}+\xi^{n-1})\right) - \left(\Lambda_h(\eta^n), \xi^{n+1}+\xi^{n-1}\right) \\
&-\nu\left(\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}, \nabla(\xi^{n+1}+\xi^{n-1})\right) + \left(\frac{p^{n+1}+p^{n-1}}{2} - q^h, \nabla\cdot(\xi^{n+1}+\xi^{n-1})\right) \\
&-\left(\frac{p^{n+1}+p^{n-1}}{2} - p^n, \nabla\cdot(\xi^{n+1}+\xi^{n-1})\right)\Bigg\}.
\end{aligned}$$

Define the ξ energy terms by

$$E_\xi^{n+\frac{1}{2}} := \frac{1}{2}\Big(\|\xi^{n+1}\|^2 + \|\xi^n\|^2 + 4\Delta t^2\|\Lambda_h(\xi^{n+1})\|^2 + 4\Delta t^2\|\Lambda_h(\xi^n)\|^2\Big).$$

Then (6.5) can be rewritten as

$$E_\xi^{N+\frac{1}{2}} + \Delta t C^{N+\frac{1}{2}} + \Delta t\sum_{n=1}^{N}\frac{\nu}{2}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2$$

$$
\begin{aligned}
&= E_{\xi}^{0+\frac{1}{2}} + \Delta t C^{1-\frac{1}{2}} + \Delta t \sum_{n=1}^{N} \Bigg\{ \nu \left(\nabla(\frac{u^{n+1}+u^{n-1}}{2} - u^n), \nabla(\xi^{n+1}+\xi^{n-1}) \right) \\
&\quad - b^*(u^n, u^n, \xi^{n+1}+\xi^{n-1}) + b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}, \frac{u_h^{n+1}+u_h^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \\
&\quad + b^*(U, u^n - \frac{u_h^{n+1}+u_h^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) - (\frac{\eta^{n+1}-\eta^{n-1}}{2\Delta t}, \xi^{n+1}+\xi^{n-1}) \qquad (6.6) \\
&\quad + 2\Delta t\Big(\Lambda_h(u^{n+1}-u^{n-1}), \Lambda_h(\xi^{n+1}+\xi^{n-1})\Big) \\
&\quad - 2\Delta t\Big(\Lambda_h(\eta^{n+1}-\eta^{n-1}), \Lambda_h(\xi^{n+1}+\xi^{n-1})\Big) - \Big(\Lambda_h(\eta^n), \xi^{n+1}+\xi^{n-1}\Big) \\
&\quad - \nu \left(\nabla \frac{\eta^{n+1}+\eta^{n-1}}{2}, \nabla(\xi^{n+1}+\xi^{n-1}) \right) + \left(\frac{p^{n+1}+p^{n-1}}{2} - q^h, \nabla\cdot(\xi^{n+1}+\xi^{n-1}) \right) \\
&\quad - \left(\frac{p^{n+1}+p^{n-1}}{2} - p^n, \nabla\cdot(\xi^{n+1}+\xi^{n-1}) \right) \Bigg\}.
\end{aligned}
$$

Now we bound the right hand side of the equation above. First,

$$
\begin{aligned}
&\nu \left(\nabla(\frac{u^{n+1}+u^{n-1}}{2} - u^n), \nabla(\xi^{n+1}+\xi^{n-1}) \right) \\
&\le \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 + 16\nu\|\nabla(\frac{u^{n+1}+u^{n-1}}{2} - u^n)\|^2 \qquad (6.7) \\
&\le \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 + 2\nu\Delta t^3 \int_{t^{n-1}}^{t^{n+1}} \|\nabla u_{tt}\|^2 dt \ .
\end{aligned}
$$

For the nonlinear term, adding and subtracting $b^*(\frac{u^{n+1}+u^{n-1}}{2}, \frac{u^{n+1}+u^{n-1}}{2}, \xi^{n+1}+\xi^{n-1})$, we have

$$
\begin{aligned}
&-b^*(u^n, u^n, \xi^{n+1}+\xi^{n-1}) + b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}, \frac{u_h^{n+1}+u_h^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \\
&\quad + b^*(U, u^n - \frac{u_h^{n+1}+u_h^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \qquad (6.8) \\
&= -b^*(u^n, u^n, \xi^{n+1}+\xi^{n-1}) + b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}, \frac{u_h^{n+1}+u_h^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \\
&\quad + b^*(U, u^n - \frac{u_h^{n+1}+u_h^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \\
&\quad + b^*(\frac{u^{n+1}+u^{n-1}}{2}, \frac{u^{n+1}+u^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \\
&\quad - b^*(\frac{u^{n+1}+u^{n-1}}{2}, \frac{u^{n+1}+u^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}).
\end{aligned}
$$

The RHS of (6.8) can be bounded as follows. First,

$$
\begin{aligned}
&b^*(\frac{u^{n+1}+u^{n-1}}{2}, \frac{u^{n+1}+u^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \\
&\quad - b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}, \frac{u_h^{n+1}+u_h^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \\
&= b^*(\frac{u^{n+1}+u^{n-1}}{2}, \frac{u^{n+1}+u^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \qquad (6.9) \\
&\quad - b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}, \frac{u^{n+1}+u^{n-1}}{2}, \xi^{n+1}+\xi^{n-1}) \\
&\quad + b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2}, \frac{u^{n+1}+u^{n-1}}{2}, \xi^{n+1}+\xi^{n-1})
\end{aligned}
$$

$$
\begin{aligned}
&-b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2},\frac{u_h^{n+1}+u_h^{n-1}}{2},\xi^{n+1}+\xi^{n-1})\\
&= b^*(\frac{\eta^{n+1}+\eta^{n-1}}{2}+\frac{\xi^{n+1}+\xi^{n-1}}{2},\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1})\\
&+b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2},\frac{\eta^{n+1}+\eta^{n-1}}{2}+\frac{\xi^{n+1}+\xi^{n-1}}{2},\xi^{n+1}+\xi^{n-1})\\
&= b^*(\frac{\eta^{n+1}+\eta^{n-1}}{2},\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1})\\
&+b^*(\frac{\xi^{n+1}+\xi^{n-1}}{2},\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1})\\
&+b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2},\frac{\eta^{n+1}+\eta^{n-1}}{2},\xi^{n+1}+\xi^{n-1})\ .
\end{aligned}
$$

Estimation of the right hand side of (6.9): First,

$$
\begin{aligned}
&b^*(\frac{\eta^{n+1}+\eta^{n-1}}{2},\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1}) \qquad (6.10)\\
&\le C\sqrt{\|\frac{\eta^{n+1}+\eta^{n-1}}{2}\|\|\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}\|}\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|\|\nabla(\xi^{n+1}+\xi^{n-1})\|\\
&\le \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2\\
&+C\nu^{-1}\|\frac{\eta^{n+1}+\eta^{n-1}}{2}\|\|\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}\|\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|^2\ .
\end{aligned}
$$

Next, applying Young's inequality

$$
\begin{aligned}
&b^*(\frac{\xi^{n+1}+\xi^{n-1}}{2},\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1}) \qquad (6.11)\\
&\le C\|\xi^{n+1}+\xi^{n-1}\|^{\frac{1}{2}}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^{\frac{3}{2}}\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|\\
&\le \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2+C\nu^{-3}\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|^4\|\xi^{n+1}+\xi^{n-1}\|^2\\
&\le \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2+C\nu^{-3}\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|^4(\|\xi^{n+1}\|^2+\|\xi^{n-1}\|^2)\ .
\end{aligned}
$$

The last term in (6.9) can be bounded as follows

$$
\begin{aligned}
&b^*(\frac{u_h^{n+1}+u_h^{n-1}}{2},\frac{\eta^{n+1}+\eta^{n-1}}{2},\xi^{n+1}+\xi^{n-1}) \qquad (6.12)\\
&\le C\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|\|\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}\|\|\nabla(\xi^{n+1}+\xi^{n-1})\|\\
&\le \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2+C\nu^{-1}\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2\|\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}\|^2\ .
\end{aligned}
$$

We next bound other terms in (6.8):

$$
\begin{aligned}
&b^*(\frac{u^{n+1}+u^{n-1}}{2},\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1})-b^*(u^n,u^n,\xi^{n+1}+\xi^{n-1})\\
&= b^*(\frac{u^{n+1}+u^{n-1}}{2}-u^n,\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1}) \qquad (6.13)\\
&+b^*(u^n,\frac{u^{n+1}+u^{n-1}}{2}-u^n,\xi^{n+1}+\xi^{n-1})\\
&\le C\|\nabla(\frac{u^{n+1}+u^{n-1}}{2}-u^n)\|\|\nabla(\xi^{n+1}+\xi^{n-1})\|(\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|+\|\nabla u^n\|)
\end{aligned}
$$

$$\leq \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2$$
$$+C\nu^{-1}\|\nabla(\frac{u^{n+1}+u^{n-1}}{2}-u^n)\|^2(\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|^2+\|\nabla u^n\|^2)$$
$$\leq \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2$$
$$+C\nu^{-1}\Delta t^3(\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|^2+\|\nabla u^n\|^2)\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{tt}\|^2dt.$$

Finally,

$$b^*(U,u^n-\frac{u_h^{n+1}+u_h^{n-1}}{2},\xi^{n+1}+\xi^{n-1})$$
$$=b^*(U,u^n-\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1})$$
$$-b^*(U,\frac{u^{n+1}+u^{n-1}}{2}-\frac{u_h^{n+1}+u_h^{n-1}}{2},\xi^{n+1}+\xi^{n-1})$$
$$=b^*(U,u^n-\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1})-b^*(U,\frac{\eta^{n+1}+\eta^{n-1}}{2},\xi^{n+1}+\xi^{n-1}),$$

where

$$b^*(U,u^n-\frac{u^{n+1}+u^{n-1}}{2},\xi^{n+1}+\xi^{n-1})$$
$$\leq C\|\nabla U\|\|\nabla(u^n-\frac{u^{n+1}+u^{n-1}}{2})\|\|\nabla(\xi^{n+1}+\xi^{n-1})\| \tag{6.14}$$
$$\leq \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2+C\nu^{-1}\|\nabla U\|^2\|\nabla(u^n-\frac{u^{n+1}+u^{n-1}}{2})\|^2$$
$$\leq \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2+C\nu^{-1}\Delta t^3\|\nabla U\|^2\int_{t^{n-1}}^{t^{n+1}}\|\nabla u_{tt}\|^2dt\ ,$$

and

$$b^*(U,\frac{\eta^{n+1}+\eta^{n-1}}{2},\xi^{n+1}+\xi^{n-1})$$
$$\leq C\|\nabla U\|\|\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}\|\|\nabla(\xi^{n+1}+\xi^{n-1})\| \tag{6.15}$$
$$\leq \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2+C\nu^{-1}\|\nabla U\|^2\|\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}\|^2\ .$$

(6.10)-(6.15) give a bound for the RHS of (6.9). Next, we estimate the pressure terms

$$\left(\frac{p^{n+1}+p^{n-1}}{2}-q_h,\nabla\cdot(\xi^{n+1}+\xi^{n-1})\right)$$
$$\leq \|\frac{p^{n+1}+p^{n-1}}{2}-q_h\|\|\nabla(\xi^{n+1}+\xi^{n-1})\| \tag{6.16}$$
$$\leq \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2+C\nu^{-1}\|\frac{p^{n+1}+p^{n-1}}{2}-q_h\|^2$$
$$\leq \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2+C\nu^{-1}(\|\frac{p^{n+1}+p^{n-1}}{2}-p^n\|^2+\|p^n-q_h\|^2)$$
$$\leq \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2+C\nu^{-1}(\Delta t^3\int_{t_{n-1}}^{t_{n+1}}\|p_{tt}\|^2dt+h^{2s+2}\|p^n\|_{s+1}^2)\ ,$$

and

$$\left(\frac{p^{n+1}+p^{n-1}}{2}-p^n,\nabla\cdot(\xi^{n+1}+\xi^{n-1})\right)$$

$$\le \|\frac{p^{n+1}+p^{n-1}}{2}-p^n\|\|\nabla(\xi^{n+1}+\xi^{n-1})\| \tag{6.17}$$
$$\le \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 + C\nu^{-1}\|\frac{p^{n+1}+p^{n-1}}{2}-p^n\|^2$$
$$\le \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 + C\nu^{-1}\Delta t^3\int_{t_n}^{t_{n+1}}\|p_{tt}\|^2 dt\ .$$

For the stabilization term,

$$\Delta t\left(\Lambda_h(u^{n+1}-u^{n-1}),\Lambda_h(\xi^{n+1}+\xi^{n-1})\right)$$
$$= \Delta t\left(\Lambda_h^*\Lambda_h(u^{n+1}-u^{n-1}),\xi^{n+1}+\xi^{n-1}\right)$$
$$\le C\Delta t\|\Lambda_h^*\Lambda_h(u^{n+1}-u^{n-1})\|\|\nabla(\xi^{n+1}+\xi^{n-1})\|$$
$$\le C\Delta t^2\nu^{-1}\|\Lambda_h^*\Lambda_h(u^{n+1}-u^{n-1})\|^2 + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 \tag{6.18}$$
$$\le C\Delta t^2\nu^{-1}\int_\Omega\left(\int_{t^{n-1}}^{t^{n+1}}\Lambda_h^*\Lambda_h(u_t)dt\right)^2 d\Omega + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2$$
$$\le C\Delta t^2\nu^{-1}\int_\Omega\left(2\Delta t\int_{t^{n-1}}^{t^{n+1}}|\Lambda_h^*\Lambda_h(u_t)|^2 dt\right) d\Omega + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2$$
$$\le C\Delta t^3\nu^{-1}\int_{t^{n-1}}^{t^{n+1}}\|\Lambda_h^*\Lambda_h(u_t)\|^2 dt + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2\ .$$

Also,

$$\Delta t\left(\Lambda_h(\eta^{n+1}-\eta^{n-1}),\Lambda_h(\xi^{n+1}+\xi^{n-1})\right)$$
$$= \Delta t\left(\Lambda_h^*\Lambda_h(\eta^{n+1}-\eta^{n-1}),\xi^{n+1}+\xi^{n-1}\right)$$
$$\le C\Delta t\|\Lambda_h^*\Lambda_h(\eta^{n+1}-\eta^{n-1})\|\|\nabla(\xi^{n+1}+\xi^{n-1})\|$$
$$\le C\Delta t^2\nu^{-1}\|\Lambda_h^*\Lambda_h(\eta^{n+1}-\eta^{n-1})\|^2 + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 \tag{6.19}$$
$$= C\Delta t^2\nu^{-1}\int_\Omega\left(\int_{t^{n-1}}^{t^{n+1}}\Lambda_h^*\Lambda_h(\eta_t)dt\right)^2 d\Omega + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2$$
$$\le C\Delta t^2\nu^{-1}\int_\Omega\left(\int_{t^{n-1}}^{t^{n+1}}|\Lambda_h^*\Lambda_h(\eta_t)|^2 dt\right) d\Omega + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2$$
$$\le C\Delta t^3\nu^{-1}\int_{t^{n-1}}^{t^{n+1}}\|\Lambda_h^*\Lambda_h(\eta_t)\|^2 dt + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2\ .$$

The rest of terms can be estimated as follow

$$(\frac{\eta^{n+1}-\eta^{n-1}}{2\Delta t},\xi^{n+1}+\xi^{n-1})$$
$$\le C\|\frac{\eta^{n+1}-\eta^{n-1}}{2\Delta t}\|\|\nabla(\xi^{n+1}+\xi^{n-1})\| \tag{6.20}$$
$$\le C\nu^{-1}\|\frac{\eta^{n+1}-\eta^{n-1}}{\Delta t}\|^2 + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2$$
$$\le C\nu^{-1}\int_\Omega\left(\frac{1}{\Delta t}\int_{t^{n-1}}^{t^{n+1}}(\eta_t)dt\right)^2 d\Omega + + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2$$
$$\le C\Delta t^{-1}\nu^{-1}\int_{t^{n-1}}^{t^{n+1}}\|\eta_t\|^2 dt + \frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2\ ,$$

and

$$
\begin{aligned}
&\big(\Lambda_h(\eta^n),\xi^{n+1}+\xi^{n-1}\big)\\
&\le C\|\Lambda_h(\eta^n)\|\|\nabla(\xi^{n+1}+\xi^{n-1})\| \\
&\le C\nu^{-1}\|\Lambda_h(\eta^n)\|^2+\frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 \ .
\end{aligned}
\tag{6.21}
$$

Also,

$$
\begin{aligned}
&\nu\left(\nabla(\frac{\eta^{n+1}+\eta^{n-1}}{2}),\nabla(\xi^{n+1}+\xi^{n-1})\right)\\
&\le \nu\|\nabla(\frac{\eta^{n+1}+\eta^{n-1}}{2})\|\|\nabla(\xi^{n+1}+\xi^{n-1})\| \\
&\le 2\nu\|\nabla(\eta^{n+1}+\eta^{n-1})\|^2+\frac{\nu}{32}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 \ .
\end{aligned}
\tag{6.22}
$$

Finally,

$$
\begin{aligned}
&\left(\frac{u^{n+1}-u^{n-1}}{2\Delta t}-u_t(t^n),\xi^{n+1}+\xi^{n-1}\right)\\
&\le C\nu^{-1}\|\frac{u^{n+1}-u^{n-1}}{2\Delta t}-u_t(t^n)\|^2+\frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 \\
&\le C\Delta t^3\nu^{-1}\int_{t^{n-1}}^{t^{n+1}}\|u_{ttt}\|^2dt+\frac{\nu}{64}\|\nabla(\xi^{n+1}+\xi^{n-1})\|^2 \ .
\end{aligned}
\tag{6.23}
$$

Having bounded each term on the right hand side from (6.8)-(6.23),we now have the following inequality:

$$
\begin{aligned}
&E_\xi^{N+\frac12}+\Delta t C^{N+\frac12}+\Delta t\sum_{n=1}^{N}\frac{\nu}{4}||\nabla(\xi^{n+1}+\xi^{n-1})||^2\\
&\le E_\xi^{0+\frac12}+\Delta t C^{1-\frac12}+\Delta t\sum_{n=1}^{N}\Bigg\{C\nu^{-3}\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|^4(\|\xi^{n+1}\|^2+\|\xi^{n-1}\|^2)\\
&\quad +C\Big(\nu+\nu^{-1}(1+\|\nabla U\|^2)\Big)\|\nabla(\eta^{n+1}+\eta^{n-1})\|^2\\
&\quad +C\nu^{-1}\|\nabla\frac{u_h^{n+1}+u_h^{n-1}}{2}\|^2\|\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}\|^2\\
&\quad +C\nu^{-1}\|\frac{\eta^{n+1}+\eta^{n-1}}{2}\|\|\nabla\frac{\eta^{n+1}+\eta^{n-1}}{2}\|\|\nabla\frac{u^{n+1}+u^{n-1}}{2}\|^2\\
&\quad +C\nu^{-1}(\Delta t^3\int_{t_{n-1}}^{t_{n+1}}\|p_{tt}\|^2dt+h^{2s+2}\|p^n\|^2_{s+1})+C\|\Lambda_h(\eta^n)\|^2\\
&\quad +\frac{C}{\Delta t}\int_{t^{n-1}}^{t^{n+1}}\|\eta_t\|^2dt+C\Delta t^3\int_{t^{n-1}}^{t^{n+1}}\Big(\|u_{ttt}\|^2+\nu\|\nabla u_{tt}\|^2+\nu^{-1}\|\nabla u_{tt}\|^2\\
&\quad +\nu^{-1}\|\nabla U\|^2\|\nabla u_{tt}\|^2+\|\Lambda_h^*\Lambda_h(u_t)\|^2+\|\Lambda_h^*\Lambda_h(\eta_t)\|^2\Big)dt\Bigg\}.
\end{aligned}
\tag{6.24}
$$

As in the proof of stability, we have

$$
\begin{aligned}
&E_\xi^{N+\frac12}+\Delta t C^{N+\frac12}\\
&\ge \frac12[\|\xi^{N+1}\|^2+4\Delta t^2\|\Lambda_h(\xi^{N+1})\|^2+\|\xi^N\|^2+4\Delta t^2\|\Lambda_h(\xi^N)\|^2]\\
&\quad -(2\Delta t^2\|\Lambda_h(\xi^{N+1})\|^2+\frac18\|\xi^N\|^2)
\end{aligned}
\tag{6.25}
$$

$$\geq \frac{1}{2}[\|\xi^{N+1}\|^2 + \frac{3}{4}\|\xi^N\|^2 + 4\Delta t^2\|\Lambda_h(\xi^N)\|^2] \ ,$$

and

$$\begin{aligned} & E_\xi^{0+\frac{1}{2}} + \Delta t C^{0+\frac{1}{2}} \\ \leq & \frac{1}{2}[\|\xi^1\|^2 + 4\Delta t^2\|\Lambda_h(\xi^1)\|^2 + \|\xi^0\|^2 + 4\Delta t^2\|\Lambda_h(\xi^0)\|^2] \\ & +(2\Delta t^2\|\Lambda_h(\xi^1)\|^2 + \frac{1}{8}\|\xi^0\|^2) \\ \leq & \frac{1}{2}[\|\xi^1\|^2 + 8\Delta t^2\|\Lambda_h(\xi^1)\|^2 + \frac{5}{4}\|\xi^0\|^2 + 4\Delta t^2\|\Lambda_h(\xi^0)\|^2]. \end{aligned} \tag{6.26}$$

Applying interpolation inequalities to (6.24) gives

$$\begin{aligned} & \frac{1}{2}\left(\|\xi^{N+1}\|^2 + \frac{3}{4}\|\xi^N\|^2 + 4\Delta t^2\|\Lambda_h(\xi^N)\|^2\right) + \Delta t\sum_{n=1}^{N}\frac{\nu}{4}||\nabla(\xi^{n+1}+\xi^{n-1})||^2 \\ & \leq \frac{1}{2}\left(\|\xi^1\|^2 + 8\Delta t^2\|\Lambda_h(\xi^1)\|^2 + \frac{5}{4}\|\xi^0\|^2 + 4\Delta t^2\|\Lambda_h(\xi^0)\|^2\right) \\ & +\Delta t\sum_{n=0}^{N+1} C(1+\nu^{-3}|||\nabla u|||^4_{\infty,0})\|\xi^n\|^2 + Ch^{2k}(\nu+\nu^{-3}+\nu^{-1}\|\nabla U\|^2)|||u|||^2_{2,k+1} \\ & +C(\Delta t)^4\nu^{-1}\|p_{tt}\|^2_{2,0} + Ch^{2s+2}\nu^{-1}|||p|||^2_{2,s+1} + Ch^{2k+1}\|\Lambda_h\|^2|||u|||^2_{2,k+1} \\ & +Ch^{2k+2}\|u_t\|^2_{2,k+1} + C(\Delta t)^4(\|u_{ttt}\|^2_{2,0} + \nu\|\nabla u_{tt}\|^2_{2,0} + \nu^{-1}\|\nabla u_{tt}\|^2_{2,0} \\ & +\nu^{-1}\|\nabla U\|^2\|\nabla u_{tt}\|^2_{2,0} + \|\Lambda_h^*\Lambda_h\|^2\|u_t\|^2_{2,0} + h^{2k+2}\|\Lambda_h^*\Lambda_h\|\|u_t\|^2_{2,k+1}) \ . \end{aligned} \tag{6.27}$$

We use the discrete Gronwall inequality, Lemma 6.3 below, without proving, for reference see [27].

Lemma 6.3. *Let $D \geq 0$ and $\kappa_n, A_n, B_n, C_n \geq 0$ for any integer $n \geq 0$ and satisfy*

$$A_N + \Delta t\sum_{n=0}^{N} B_n \leq \Delta t\sum_{n=0}^{N}\kappa_n A_n + \Delta t\sum_{n=0}^{N} C_n + D \ for \ N \geq 0.$$

Suppose that for all n, $\Delta t\kappa_n \leq 1$, and set $g_n = (1-\Delta t\kappa_n)^{-1}$. Then,

$$A_N + \Delta t\sum_{n=0}^{N} B_n \leq exp(\Delta t\sum_{n=0}^{N} g_n\kappa_n)[\Delta t\sum_{n=0}^{N} C_n + D] \ for \ N \geq 0.$$

Let Δt be sufficiently small, i.e., $C\Delta t < (1+\nu^{-3}|||\nabla u|||^4_{\infty,0})^{-1}$. We can apply the lemma and obtain

$$\begin{aligned} & \frac{1}{2}\left(\|\xi^{N+1}\|^2 + \frac{3}{4}\|\xi^N\|^2 + 4\Delta t^2\|\Lambda_h(\xi^N)\|^2\right) + \Delta t\sum_{n=1}^{N}\frac{\nu}{4}||\nabla(\xi^{n+1}+\xi^{n-1})||^2 \\ & \leq exp(\frac{C(T+\Delta t)(1+\nu^{-3}|||\nabla u|||^4_{\infty,0})}{1-C\Delta t(1+\nu^{-3}|||\nabla u|||^4_{\infty,0})})\Big[\frac{1}{2}\Big(\|\xi^1\|^2 + 8\Delta t^2\|\Lambda_h(\xi^1)\|^2 \\ & +\frac{5}{4}\|\xi^0\|^2 + 4\Delta t^2\|\Lambda_h(\xi^0)\|^2\Big) + Ch^{2k}(\nu+\nu^{-3}+\nu^{-1}\|\nabla U\|^2)|||u|||^2_{2,k+1} \\ & +C(\Delta t)^4\nu^{-1}\|p_{tt}\|^2_{2,0} + Ch^{2s+2}\nu^{-1}|||p|||^2_{2,s+1} + Ch^{2k+1}\|\Lambda_h\|^2|||u|||^2_{2,k+1} \\ & +Ch^{2k+2}\|u_t\|^2_{2,k+1} + C(\Delta t)^4(\|u_{ttt}\|^2_{2,0} + \nu\|\nabla u_{tt}\|^2_{2,0} + \nu^{-1}\|\nabla u_{tt}\|^2_{2,0} \end{aligned} \tag{6.28}$$

$$+\nu^{-1}\|\nabla U\|^2\|\nabla u_{tt}\|^2_{2,0} + \|\Lambda_h^*\Lambda_h\|^2\|u_t\|^2_{2,0} + h^{2k+2}\|\Lambda_h^*\Lambda_h\|\|u_t\|^2_{2,k+1})\Big] .$$

Recall that $e^n = \eta^n + \xi^n$. Use the triangle inequality on the error equation to split the error terms into terms of η and ξ:

$$\begin{aligned}
&\frac{1}{2}\left(||e^{N+1}||^2 + \frac{3}{4}||e^N||^2 + 4\Delta t^2||\Lambda_h(e^N)||^2\right) + \Delta t\sum_{n=1}^{N}\frac{\nu}{4}||\nabla(e^{n+1}+e^{n-1})||^2 \\
\leq &\frac{1}{2}\left(||\xi^{N+1}||^2 + \frac{3}{4}||\xi^N||^2 + 4\Delta t^2||\Lambda_h(\xi^N)||^2\right) + \Delta t\sum_{n=1}^{N}\frac{\nu}{4}||\nabla(\xi^{n+1}+\xi^{n-1})||^2 \\
&+\frac{1}{2}\left(||\eta^{N+1}||^2 + \frac{3}{4}||\eta^N||^2 + 4\Delta t^2||\Lambda_h(\eta^N)||^2\right) + \Delta t\sum_{n=1}^{N}\frac{\nu}{4}||\nabla(\eta^{n+1}+\eta^{n-1})||^2
\end{aligned} \tag{6.29}$$

Applying inequality (6.28), using the previous bounds for η terms, and absorbing constants into a new constant C, we have Theorem 6.1.
□

7. Numerical tests. We present three numerical experiments to test the algorithms proposed herein. First, given exact solutions, we verify the convergence rates of our methods. Second, we will test the stability and confirm that STAFASL successfully eliminates time step condition for stability which affects FASL. Finally, our methods is tested with the benchmark problem of flow around a cylinder. This experiment shows that STAFASL successfully produces the vortex street, thus not over-stabilizing the solutions, while flows generated by FASL are not stable and blow up shortly. The code was implemented using the software package *FreeFEM++*.

7.1. Test 1: Green-Taylor vortex. The first test is designed to verify the convergence rates of our methods. We select the velocity field given by the Green-Taylor vortex, [22], [23]. The exact velocity field is given by

$$\begin{aligned}
u_1(x,y,t) &= -\cos(\omega\pi x)\sin(\omega\pi y)e^{-2\omega^2\pi^2 t/\tau}\,, \\
u_2(x,y,t) &= \sin(\omega\pi x)\cos(\omega\pi y)e^{-2\omega^2\pi^2 t/\tau}\,, \\
u_3(x,y,t) &= 0\,, \\
p(x,y,t) &= -\frac{1}{4}(\cos(2\omega\pi x)+\cos(2\omega\pi y))e^{-4\omega^2\pi^2 t/\tau}\,.
\end{aligned} \tag{7.1}$$

defined on the domain $\Omega = (0,1)^2$. We take

$$\omega = 2,\ T = 1,\ \tau = Re = 500,\ h = 1/m,\ \Delta t = h,$$

where m is the number of subdivisions of the interval $(0,1)$. We choose the Coriolis term $f_C = (0,0,1)$ and utilize Taylor-Hood finite elements for the discretization. Newton iterations are applied to solve the nonlinear system with a $\|w_{(j+1)} - w_{(j)}\|_{H^1(\Omega)} < 10^{-10}$ as a stopping criterion. Convergence rates are calculated from the error at two successive values of h in the usual manner by postulating $e(h) = Ch^\beta$ and solving for β via $\beta = \ln(e(h_1)/e(h_2))/\ln(h_1/h_2)$. The boundary conditions could be taken to be periodic (the easier case). Instead we take the boundary condition on the problem to be inhomogeneous Dirichlet: $u_h = u_{exact}$, on $\partial\Omega$. The exact velocity field is not very sensitive in time, so we choose the mean velocity field to be:

$$\begin{aligned}
U_1(x,y,t) &= -\cos(\omega\pi x)\sin(\omega\pi y)\,, \\
U_2(x,y,t) &= \sin(\omega\pi x)\cos(\omega\pi y)\,, \\
U_3(x,y,t) &= 0\,.
\end{aligned} \tag{7.2}$$

The errors and rates of convergence are presented in Table 9.1 and 9.2. From the tables, we see that the rates of convergence of both algorithms confirm the predicted convergence rates from theory, and the errors of STAFASL are comparable with those of FASL.

7.2. Test 2: Stability test. In this experiment, we test and compare the stability conditions of our methods for fluid flows with Coriolis force. We confirm the time step restriction for stability of FASL as well as the unconditional stability of STAFASL.

Let $\Omega = (0,1)^2$ and $\nu = 1/Re = 100$. We take the source term f and the boundary condition to be 0 and the initial condition is given by

$$
\begin{aligned}
u_1(x,y,0) &= 2x^2(1-x)^2y(1-y)(1-2y)\exp(7x),\\
u_2(x,y,0) &= x(1-x)(7x^2-3x-2)y^2(1-y)^2\exp(7x),\\
u_3(x,y,0) &= 0.
\end{aligned}
$$

We compute kinetic energy $E^n = \frac{1}{2}\|u_h\|^2$ using two methods proposed herein: FASL and STAFASL. For a system lacking of external energy exchange and body forces, the true kinetic energy decays over time, thus, the variation of its numerical approximation could give us a clear conclusion on the schemes' stability: big growth means the scheme is unstable, decay means the scheme is stable and the unstable mode is damped. Since the velocity also quickly converges to 0, it is reasonable to choose mean velocity $U = 0$. Under this setting, the CFL condition (3.2) of FASL becomes $\Delta t < \|f_C\|_\infty^{-1}$.

Let $h = 1/10$ and final time $T = 10.0$, the variation of energy in time is plotted at different time step size to show the experimental stability restriction of our methods. We run the test with different values of f_C, i.e., $f_C = (0,0,200)$, $f_C = (0,0,20)$, $f_C = (0,0,0.02)$ to verify how time step restriction depends on rotation force.

Our result is shown in Figure 9.1. We observe that STAFASL is unconditionally stable in all cases. In the meantime, FASL is shown to be stable with $\Delta t \lesssim \frac{1}{200}$ and $\Delta t \lesssim \frac{1}{20}$ for $f_C = (0,0,200)$ and $(0,0,20)$ respectively. In case $f_C = (0,0,0.02)$, FASL is virtually stable with all time step size. These results are completely match with the CFL condition (3.2).

7.3. Test 3: Flow around a cylinder. Our final numerical experiment is for two dimensional Navier-Stokes flow around a cylinder. This is a well known benchmark problem taken from Shäfer and Turek [24]. The flow patterns are driven by interaction of a fluid with a wall, which is an important scenario for real, industrial type flows. Such flows are critical if stabilizations are considered to be useful. It is also interesting since success and failure are clear (vortex street or not) and thus comparison of higher order statistics is not necessary to reach a clear conclusion.

We consider domain Ω to be a 2.2×0.41 rectangular channel with a cylinder of radius 0.05 centered at $(0.2, 0.2)$. The cylinder, top and bottom of the channel are prescribed no-slip boundary conditions, and the inflow and outflow profiles are

$$
\begin{aligned}
u_1(0,y) = u_1(2.2,y) &= \frac{6}{0.41^2}\sin(\pi t/8)y(0.41-y),\\
u_2(0,y) = u_2(2.2,y) &= 0.
\end{aligned}
$$

We set the external force $f = 0$, the viscosity $\nu = 10^{-3}$ and the rotation term $f_C = 0$. The mean velocity is chosen to be the solution to the corresponding stationary problem with the inflow and outflow profiles

$$
\begin{aligned}
U_1(0,y) = U_1(2.2,y) &= \frac{1.2}{0.41^2}y(0.41-y),\\
U_2(0,y) = U_2(2.2,y) &= 0.
\end{aligned}
$$

For this setting, it is expected that, as flow increases, from $t = 2$ to $t = 4$, the eddy behind the cylinder becomes unstable. Between $t = 4$ and $t = 6$, the eddies are then shed on alternate sides of the cylinder and a vortex street develops. The vortices are still visible at $t = 8$.

The solutions to FASL and STAFASL are computed with Taylor-Hood elements on a triangular mesh providing 27803 total DOFs, refined near the cylinder (see Figure 9.2), and time step $\Delta t = 0.01$. Newton iterations are applied to solve the nonlinear system with a $\|w_{(j+1)} - w_{(j)}\|_{H^1(\Omega)} < 10^{-8}$ as a stopping criterion. These simulations are under-resolved; fully resolved computations of the NSE will require upwards of 100,000 DOFs and Δt less than 0.001.

First, we observe that for FASL, Newton iteration fails to converge at $t = 0.62$. Figure 9.3 shows that the approximated energy of FASL has already gone up and is 171% bigger than true value before Newton iteration diverges. Therefore, the instability of FASL at $\Delta t = 0.01$ must account for this. This instability illustrates the practical difficulty of assigning a precise numerical value to the **CFL limit** for the timestep. On the other hand, the simulation result of STAFASL is satisfactory: this method is stable and the flow pattern produced by STAFASL is matched with that of resolved solutions in [25] and [26]. This evolution can be seen in Figure 9.4, where the vorticity contours are plotted at t = 2, 4, 5, 6 and 8. Our test reaffirms that STAFASL produces acceptable simulations while unaffected by time step restriction like FASL.

8. Conclusions. CNLF is generally believed to be unstable in the unstable mode $(u^{n+1} - u^{n-1})$ and the timestep restriction imposed by LF can be very restrictive. This report gives a comprehensive stability analysis that proves, for Oseen+rotation term, the unstable mode of CNLF is actually stable under the same type timestep restriction imposed by LF for the stability of the stable mode $(u^{n+1} + u^{n-1})$. We also propose and analyze a stabilized CNLF (STAFASL) that eliminates the timestep restrictions of CNLF for NSE+rotation term, based on a fast-slow splitting method. Further, this stabilization method also removes the timestep restriction for stability of the unstable mode of the Oseen problem. Still, proof of controlling the unstable mode of the full NSE+rotation term is an open question. Numerical tests confirm the second order convergence rate of STAFASL and the superior of the STAFASL over FASL in stability.

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h	Δt	$\|\mathbf{u}-\mathbf{u}_h\|_\infty$	Rate	$\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|_2$	Rate
1/8	1/8	1.517e-1	–	5.813e+0	–
1/16	1/16	1.219e-2	3.637	9.897e-1	2.554
1/32	1/32	9.906e-4	3.621	1.366e-1	2.857
1/64	1/64	7.118e-5	3.799	1.848e-2	2.886
1/128	1/128	4.868e-6	3.870	2.615e-3	2.823

Table 9.1
The convergence performance for FASL.

h	Δt	$\|\mathbf{u}-\mathbf{u}_h\|_\infty$	Rate	$\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|_2$	Rate
1/8	1/8	1.468e-1	–	5.814e+0	–
1/16	1/16	1.211e-2	3.600	9.902e-1	2.554
1/32	1/32	9.898e-4	3.613	1.366e-1	2.858
1/64	1/64	7.865e-5	3.654	1.849e-2	2.885
1/128	1/128	1.740e-5	2.176	2.618e-3	2.820

Table 9.2
The convergence performance for STAFASL.

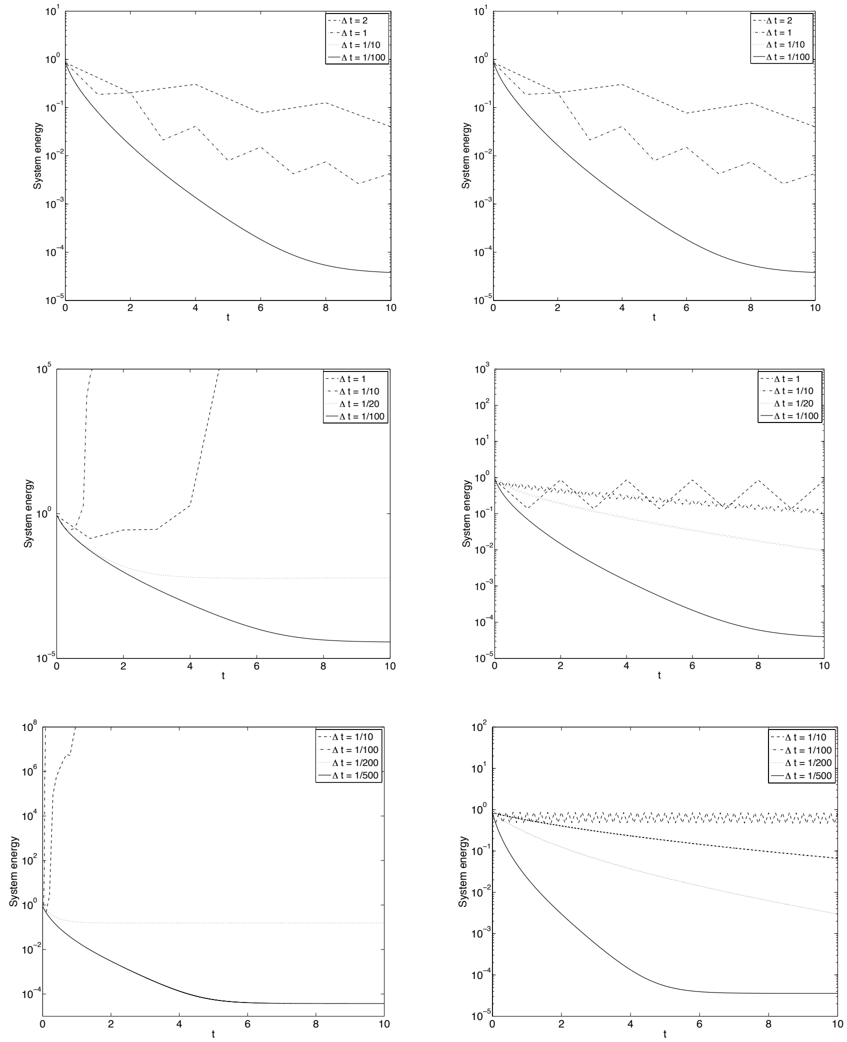

FIG. 9.1. *The decay of kinetic energy computed by* CNLF *(left) and* CNLFSTAB *(right) with several different time steps chosen. First row:* $f_C = 0.02$*, second row:* $f_C = 20$*, last row:* $f_C = 200$*.*

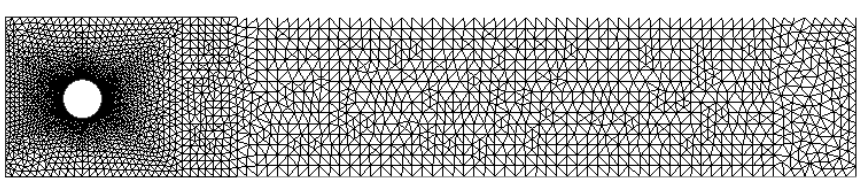

FIG. 9.2. *Mesh for the computation.*

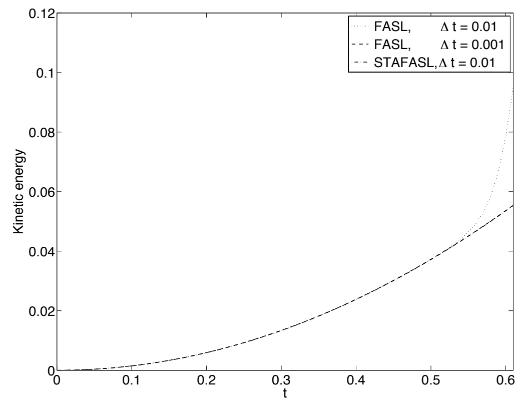

Fig. 9.3. *The kinetic energy of flow around a cylinder.*

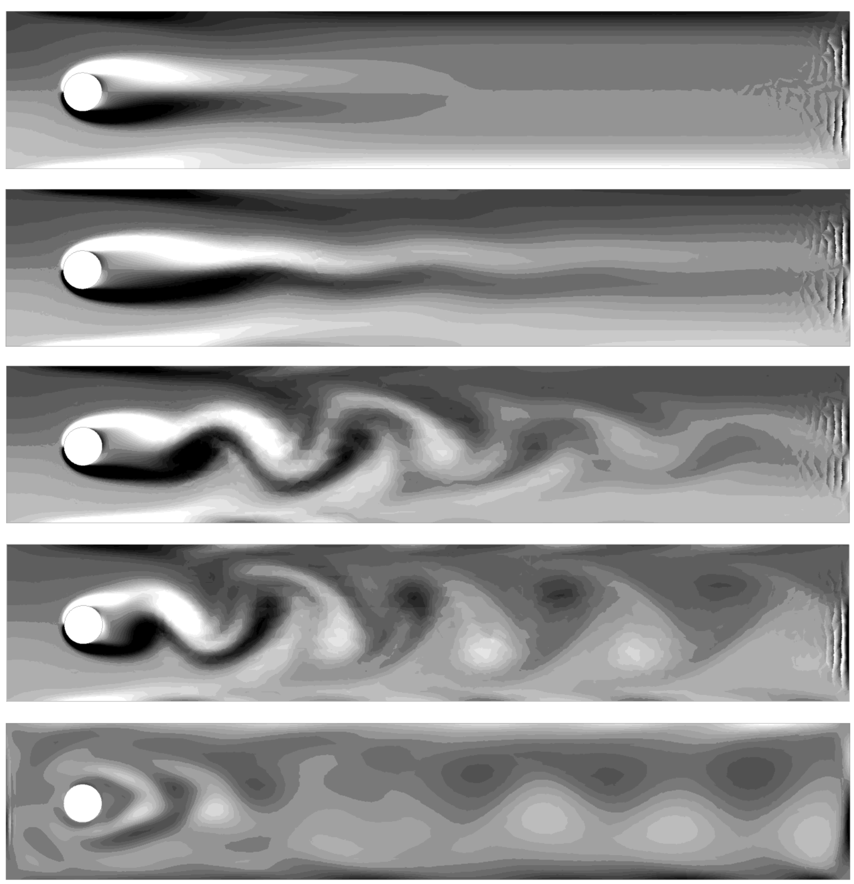

Fig. 9.4. STAFASL*: The vorticity contour at times 2, 4, 5, 6, 8.*