A NEW CRANK-NICOLSON LEAP-FROG STABILIZATION: UNCONDITIONAL STABILITY AND TWO APPLICATIONS

NAN JIANG , MICHAELA KUBACKI , WILLIAM LAYTON , MARINA MORAITI , AND HOANG TRAN*

Abstract. We propose and analyze a linear stabilization of CNLF that removes all timestep conditions for stability, is parameter free and increases the SPD part of the linear system to be solved at each time step. We prove unconditional stability and give applications to uncoupling groundwater - surface water flows and Stokes flow plus a Coriolis term. The stabilization herein is not modular (unlike time filters) but it does remove all timestep conditions (also unlike time filters) and thus provides a complementary tool.

Key words. CNLF, stabilization, Stokes-Darcy

AMS subject classification. Primary 65M12; Secondary 65J08

1. Introduction. The implicit-explicit Crank-Nicolson and Leap-Frog (CNLF) method is widely used in atmosphere, ocean and climate codes, e.g., [3], [26], [28], [30] and has recently been used for uncoupling groundwater-surface water flows, [17]. Stability of CNLF by root conditions was proven in 1963 [15] and by energy methods for systems in [18]. Two related stability questions remain. First, the unstable mode (for which $u^{n+1} + u^{n-1} \equiv 0$) of LF is not damped by CN. Thus, modular time filters, like the Roberts-Asselin-Williams filter [3], [26], [30], have been developed. Second, the timestep restriction (1.6) below from the LF component can be too restrictive if the normal splitting into fast but low energy modes and slow but high energy modes is not perfectly done and if the parameters in the Stokes-Darcy problem are small.

This report presents a new CNLF stabilization, CNLFstab, addressing both issues. The method (1.4) below is unconditionally (no timestep condition) stable (Theorem 1, Section 2) and the unstable mode, while not eliminated, is controlled, Section 3. We test (1.4) in Section 3 for Stokes flow with strong rotation and coupled groundwater-surface water flows.

To present the method, consider an evolution equation

$$\frac{du}{dt} + N(u) + \Lambda u = 0. \tag{1.1}$$

(Both the algorithm and theory extend easily to nonzero right hand sides.) We assume that $X \hookrightarrow L \hookrightarrow X'$ are Hilbert spaces. Let $< \cdot, \cdot >, ||\cdot||$ denote the inner product and norm on L. Suppose

$$N : X \to X' \text{ satisfies } \quad \langle N(u), u\rangle \geq 0 \text{ for all } u \in X, \tag{1.2}$$

$$\Lambda : L \to L \text{ is a bounded, skew symmetric operator,} \tag{1.3}$$

where the X, X' duality pairing is an extension of the L inner product. These two assumptions ensure that

$$||u(t)||^2 \leq ||u_0||^2 \text{ and } ||u(t)||^2 = ||u_0||^2 \text{ if } N(u) \equiv 0.$$

*Dept. of Mathematics, Univ. of Pittsburgh, Pittsburgh, PA 15260, USA; emails: naj24, mjk63, wjl, mam328, and hat25@pitt.edu; www.math.pitt.edu/~wjl, www.pitt.edu/~mam328, www.pitt.edu/~hat25. All authors partially supported by NSF grant DMS 0810385.

These are the basic stability properties that must be preserved under discretization.

The method CNLFSTAB is: given $u^0, v^0, u^1, v^1 \in X$ find $u^n, v^n \in X$ for $n \geq 2$ satisfying

$$\frac{u^{n+1}-u^{n-1}}{2\triangle t} + \triangle \mathbf{t} \boldsymbol{\Lambda}^* \boldsymbol{\Lambda} \left(\mathbf{u}^{n+1} - \mathbf{u}^{n-1}\right) + \\ + N\left(\frac{u^{n+1}+u^{n-1}}{2}\right) + \Lambda u^n = 0. \tag{1.4}$$

The stabilization (in bold) $\triangle t \Lambda^* \Lambda \left(u^{n+1} - u^{n-1}\right)$ is linear and SPD in the unknown u^{n+1}, has no undetermined tuning parameters and the extra consistency error it contributes is formally $\triangle t^2 \Lambda^* \Lambda \left(u_t\right) = O(\triangle t^2)$, the same order as CNLF. CNLFSTAB, like CNLF, is a 3 level method and approximations are needed at the first two time steps to appropriate accuracy, [29]. The stabilization in CNLFSTAB is similar in spirit to [1], [DD], [8]. For a general theory of IMEX methods see [2], [7], [14].

1.1. The usual CNLF method. The usual CNLF method is

$$\frac{u^{n+1}-u^{n-1}}{2\triangle t} + N\left(\frac{u^{n+1}+u^{n-1}}{2}\right) + \Lambda u^n = 0. \tag{1.5}$$

With $||\cdot||$ the operator norm, CNLF is stable under the CFL- like condition

$$\triangle t||\Lambda|| < 1, \tag{1.6}$$

[15], [18]. When the nonlinear term is strictly positive

$$\langle N(u), u\rangle \geq \alpha ||u||^2 \text{ for some } \alpha > 0 \text{ and all } u \in X,$$

then $||u(t)|| \to 0$ as $t \to \infty$. In this common case, CNLF will damp only the mode $u^{n+1} + u^{n-1}$; there is no damping in the unstable mode where $u^{n+1} + u^{n-1} \equiv 0$. Roundoff error can lead to growth in the unstable mode, spurring development of corrective time filters, [3], [26], [30], [16].

2. Stability without a timestep restriction. We prove unconditional stability of (1.4). The proof shows that the coefficient of the stabilization term (here taken to be 1) may be reduced retaining unconditional stability. It also shows that if $N(u) \equiv 0$, then the following quantity is exactly conserved:

$$\frac{1}{4}\left(||u^{n+1}||^2 + ||u^n||^2\right) + \frac{\triangle t^2}{2}\left(||\Lambda u^{n+1}||^2 + ||\Lambda u^n||^2\right) + \frac{\triangle t}{2}\left\langle \Lambda u^n, u^{n+1}\right\rangle.$$

THEOREM 2.1. *Consider (1.1) under (1.2) and (1.3). The method (1.4) is unconditionally stable (with no timestep restriction): for every $n \geq 1$*

$$\frac{1}{2}||u^{n+1}||^2 + \frac{1}{4}||u^n||^2 + \triangle t^2 ||\Lambda u^n||^2 \leq \\ \leq \frac{1}{2}\left(||u^1||^2 + ||u^0||^2\right) + \triangle t^2\left(||\Lambda u^1||^2 + ||\Lambda u^0||^2\right) + \triangle t\left\langle \Lambda u^0, u^1\right\rangle.$$

Proof. Broadly, the proof follows the CNLF case in [18] with modified treatment of the critical term $\left\langle \Lambda u^n, u^{n+1} + u^{n-1}\right\rangle$ using the stabilizations. Multiply by $\triangle t$ and take the duality pairing of (1.4) with $\left(u^{n+1}+u^{n-1}\right)/2$. Add and subtract $||u^n||^2$; this gives

$$\frac{1}{4}(||u^{n+1}||^2+||u^n||^2)-\frac{1}{4}(||u^n||^2+||u^{n-1}||^2)+$$
$$+\triangle t^2\left\langle \Lambda^*\Lambda\left(u^{n+1}-u^{n-1}\right),\frac{u^{n+1}+u^{n-1}}{2}\right\rangle+$$
$$+\triangle t\left\langle N(\frac{u^{n+1}+u^{n-1}}{2}),\frac{u^{n+1}+u^{n-1}}{2}\right\rangle+\triangle t\left\langle \Lambda u^n,\frac{u^{n+1}+u^{n-1}}{2}\right\rangle=0.$$

From (1.2)

$$\triangle t\left\langle N(\frac{u^{n+1}+u^{n-1}}{2}),\frac{u^{n+1}+u^{n-1}}{2}\right\rangle+\triangle t^2\left\langle \Lambda^*\Lambda\left(u^{n+1}-u^{n-1}\right),\frac{u^{n+1}+u^{n-1}}{2}\right\rangle$$
$$\geq\frac{\triangle t^2}{2}<\Lambda\left(u^{n+1}-u^{n-1}\right),\Lambda(u^{n+1}+u^{n-1})>$$
$$=\frac{\triangle t^2}{2}\left(||\Lambda u^{n+1}||^2-||\Lambda u^{n-1}||^2\right)$$
$$=\frac{\triangle t^2}{2}\left[\left(||\Lambda u^{n+1}||^2+||\Lambda u^n||^2\right)-\left(||\Lambda u^n||^2+||\Lambda u^{n-1}||^2\right)\right].$$

Thus, define the stabilized system energy

$$E^{n+1/2}:=\frac{1}{4}\left(||u^{n+1}||^2+||u^n||^2\right)+\frac{\triangle t^2}{2}\left(||\Lambda u^{n+1}||^2+||\Lambda u^n||^2\right).$$

We then have

$$E^{n+1/2}-E^{n-1/2}\leq-\frac{\triangle t}{2}\left\langle \Lambda u^n,u^{n+1}+u^{n-1}\right\rangle.$$

Let $C^{n+1/2}:=\left\langle \Lambda u^n,u^{n+1}\right\rangle$; using skew symmetry of Λ we have

$$\left\langle \Lambda u^n,u^{n+1}+u^{n-1}\right\rangle=C^{n+1/2}-C^{n-1/2}.$$

Thus, the stability equation becomes

$$E^{n+1/2}+\frac{\triangle t}{2}C^{n+1/2}\leq E^{n-1/2}+\frac{\triangle t}{2}C^{n-1/2},\ \text{for } n\geq 2,$$

so stability follows provided $E^{n+1/2}+\frac{\triangle t}{2}C^{n+1/2}>0$ for $u^n,u^{n+1}\neq 0$. By repeated application of the Cauchy-Schwarz-Young inequality we have

$$\frac{\triangle t}{2}C^{n+1/2}\leq\frac{\triangle t^2}{2}||\Lambda u^{n+1}||^2+\frac{1}{8}||u^n||^2.$$

Thus, stability follows since:

$$E^{n+1/2}+\frac{\triangle t}{2}C^{n+1/2}\geq\frac{1}{4}||u^{n+1}||^2+\frac{1}{8}||u^n||^2+\frac{\triangle t^2}{2}||\Lambda u^n||^2>0.$$

□

3. Two Applications. We apply (1.4) to uncoupling groundwater-surface water flows and to Stokes flow plus a Coriolis force term, a great simplification of the geophysical flow, [16]. We give a stability analysis of an interpretation of (1.4) for both problems, incorporating the time and space discretizations.

3.1. The evolutionary Stokes-Darcy problem. See, e.g., [6], [9], [22], [20], [19], [10] for background on the numerical analysis of the Stokes-Darcy model. Let Ω_f, Ω_p lie across an interface I from each other. For specificity, we take $\Omega_f = (0,1) \times (0,1)$, $\Omega_p = (0,1) \times (-1,0)$ and $I = \{(x,0), 0 < x < 1\}$. The fluid velocity u and porous media's piezometric head ϕ satisfy

$$
\begin{gathered}
u_t - \nu \triangle u + \nabla p = f_f(x,t), \nabla \cdot u = 0, \text{ in } \Omega_f, \\
S_0 \phi_t - \nabla \cdot (\mathcal{K} \nabla \phi) = f_p(x,t), \text{ in } \Omega_p, \\
\phi(x,0) = \phi_0(x), \text{ in } \Omega_p \text{ and } u(x,0) = u_0(x), \text{ in } \Omega_f, \\
\phi(x,t) = 0, \text{ in } \partial\Omega_p \backslash I \text{ and } u(x,t) = 0, \text{ in } \partial\Omega_f \backslash I.
\end{gathered} \tag{3.1}
$$

Let $\widehat{n}_{f/p}, \widehat{\tau}_i$ denote the normal and tangent vectors on I. The coupling conditions across I are conservation of mass, balance of forces on I and the Beavers-Joseph-Saffman condition

$$
\begin{gathered}
u \cdot \widehat{n}_f - \mathcal{K} \nabla \phi \cdot \widehat{n}_p = 0 \text{ and } p - \nu\, \widehat{n}_f \cdot \nabla u \cdot \widehat{n}_f = g\phi \text{ on } I, \\
-\nu \widehat{\tau}_i \cdot\ \nabla u \cdot \widehat{n}_f = \alpha \sqrt{\frac{\nu g}{\widehat{\tau}_i \cdot \mathcal{K} \cdot \widehat{\tau}_i}} u \cdot \widehat{\tau}_i, \text{ on } I.
\end{gathered}
$$

see [5], [27]. Here g, $\mathcal{K}$, ν and S_0 are the gravitational acceleration constant, hydraulic conductivity tensor, kinematic viscosity and specific storage, all positive. Often $\lambda_{\min}(\mathcal{K})$ and S_0 are small, [4], [21], [17].

We denote the $L^2(I)$ norm by $||\cdot||_I$ and the $L^2(\Omega_{f/p})$ norm and inner product by $||\cdot||_{f/p}, (\cdot,\cdot)_{f/p}$, respectively; the $H_{DIV}(\Omega_f)$ norm is $||u||^2_{DIV} := ||u||^2_f + ||\nabla \cdot u||^2_f$. To discretize the Stokes-Darcy problem in space by the finite element method we choose conforming Velocity, Pressure, and Darcy pressure finite element spaces

$$
\begin{aligned}
\text{Velocity: } & X_f^h \subset\ X_f := \{v \in \left(H^1(\Omega_f)\right)^d : v = 0 \text{ on } \ \partial\Omega_f \backslash I\}, \\
\text{Darcy pressure: } & X_p^h \subset\ X_p := \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \ \partial\Omega_p \backslash I\}, \\
\text{Stokes pressure: } & Q_f^h \subset Q_f \ := L^2(\Omega_f).
\end{aligned}
$$

Continuity of any discrete variable across the interface I is not imposed strongly. The Stokes velocity-pressure finite element spaces (X_f^h, Q_f^h) are assumed to satisfy the usual discrete inf-sup condition, [12], [11]. Define

$$
\begin{aligned}
& a_f(u,v) = (\nu \nabla u, \nabla v)_f + \sum_i \int_I \alpha \sqrt{\frac{\nu g}{\widehat{\tau}_i \cdot \mathcal{K} \cdot \widehat{\tau}_i}} (u \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i) ds, \\
& a_p(\phi,\psi) = g(\mathcal{K} \nabla \phi, \nabla \psi)_p, \quad \text{and} \\
& c_I(u,\phi) = g \int_I \phi u \cdot \widehat{n}_f ds.
\end{aligned}
$$

CNLFstab **adapted to the Stokes-Darcy problem:** Find $(u_h^{n+1}, p_h^{n+1}, \phi_h^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ satisfying, for all $v_h \in X_f^h$, $q_h \in Q_f^h$, $\psi_h \in X_p^h$

$$\begin{gathered} gS_0(\frac{\phi_h^{n+1}-\phi_h^{n-1}}{2\triangle t}, \psi_h)_p + \\ +\triangle t g^2(\nabla(\phi_h^{n+1}-\phi_h^{n-1}), \nabla\psi_h)_p + \triangle t g^2(\phi_h^{n+1}-\phi_h^{n-1}, \psi_h)_p \\ +a_p(\frac{\phi_h^{n+1}+\phi_h^{n-1}}{2}, \psi_h) - c_I(u_h^n, \psi_h) = g(f_p^n, \psi_h)_p, \\ (\frac{u_h^{n+1}-u_h^{n-1}}{2\triangle t}, v_h)_f + (\nabla\cdot\frac{u_h^{n+1}-u_h^{n-1}}{2\triangle t}, \nabla\cdot v_h)_f + a_f(\frac{u_h^{n+1}+u_h^{n-1}}{2}, v_h) \\ -(p_h^{n+1/2}, \nabla\cdot v_h)_f + c_I(v_h, \phi_h^n) = (f_f^n, v_h)_f, \\ (q_h, \nabla\cdot\frac{u_h^{n+1}+u_h^{n-1}}{2})_f = 0. \end{gathered} \tag{3.2}$$

The stabilization terms in (3.2) are similar in spirit to [1] in the porous medium and grad-div stabilization of u_t, [23], in the fluid region. The following trace inequality from Moraiti [21], which holds for our Ω_f, Ω_p with constant 1, is essential:

$$\left|\int_I \phi u\cdot\widehat{n}ds\right| \le ||u||_{DIV}||\phi||_{H^1(\Omega_p)}, \text{ for all } u\in X_f, \phi\in X_p. \tag{3.3}$$

Remark 3.1 (On the form of the stabilization). *In* $\triangle \mathbf{t}\mathbf{\Lambda}^*\mathbf{\Lambda}\left(\mathbf{u}^{n+1}-\mathbf{u}^{n-1}\right)$ *one must define* $\Lambda = (\Lambda_f, \Lambda_p) : X_f^h \times X_p^h \to X_f^h \times X_p^h$ *through*

$$(\Lambda_f(u,\phi), v)_f + (\Lambda_p(u,\phi),\psi)_p = \int_I \psi u\cdot\widehat{n}ds - \int_I \phi v\cdot\widehat{n}ds.$$

Ignoring technical issues, the stabilization motivated by $\triangle \mathbf{t}\mathbf{\Lambda}^*\mathbf{\Lambda}\left(\mathbf{u}^{n+1}-\mathbf{u}^{n-1}\right)$ *most natural in appearance are boundary integral terms of the form*

$$\triangle t g^2\int_I(\phi_h^{n+1}-\phi_h^{n-1})\psi_h ds \text{ and } \triangle t\int_I\left(u_h^{n+1}-u_h^{n-1}\right)\cdot\widehat{n}v_h\cdot\widehat{n}ds.$$

It is an open problem to analyze if this stabilization suffices. The inequality (3.3) above suggests that the stabilization in (3.2) is connected to the one on I above.

Theorem 3.2 (Unconditional stability for Stokes-Darcy). *(3.2) is stable: for any $N > 0$, there holds*

$$\begin{gathered} \frac{1}{2}||u_h^{N+1}||_{DIV}^2 + \frac{1}{2}||u_h^N||_{DIV}^2 + gS_0\left(||\phi_h^{N+1}||_p^2 + ||\phi_h^N||_p^2\right) + \\ +\triangle t\sum_{n=1}^N \nu||\nabla\left(u_h^{n+1}+u_h^{n-1}\right)||_f^2 + gk_{\min}||\nabla\left(\phi_h^{n+1}+\phi_h^{n-1}\right)||_p^2 \le \\ \le ||u_h^1||_{DIV}^2 + ||u_h^0||_{DIV}^2 \\ +gS_0||\phi_h^1||_p^2 + \triangle t^2 g^2||\phi_h^1||_{H^1(\Omega_p)}^2 + gS_0||\phi_h^0||_p^2 + 2\triangle t^2 g^2||\phi_h^0||_{H^1(\Omega_p)}^2 \\ +2\triangle t[c_I(\phi_h^0, u_h^1) - c_I(\phi_h^1, u_h^0)] + \\ +2\triangle t\sum_{n=1}^N\left[(f_f^n, u_h^{n+1}+u_h^{n-1})_f + g(f_p^n, \phi_h^{n+1}+\phi_h^{n-1})_p\right]. \end{gathered}$$

Proof. We adapt the proof of Theorem 1 to the setting of (3.2). Set $v_h = (u_h^{n+1} + u_h^{n-1})/2$, $\psi_h = (\phi_h^{n+1} + \phi_h^{n-1})/2$, and add and subtract $||u_h^n||^2_{DIV}$. Similarly, in the porous media equation, set $\psi_h = \phi_h^{n+1} + \phi_h^n$ and add and subtract $||\phi_h^n||_p^2$. Adding the two energy estimates gives

$$E^{n+1/2} - E^{n-1/2} + Coupling$$
$$+\triangle t D^{n+1/2} = \frac{\Delta t}{2}\left((f_f^n, u_h^{n+1} + u_h^{n-1})_f + g(f_p^n, \phi_h^{n+1} + \phi_h^{n-1})\right).$$

Here

$$E^{n+1/2} = \frac{1}{4}\left(||u_h^{n+1}||^2_{DIV} + ||u_h^n||^2_{DIV}\right) +$$
$$+\frac{1}{4} g S_0 \left(||\phi_h^{n+1}||_p^2 + ||\phi_h^n||_p^2\right) + \frac{1}{2}\triangle t^2 g^2 \left(||\phi_h^{n+1}||^2_{H^1(\Omega_p)} + ||\phi_h^n||^2_{H^1(\Omega_p)}\right)$$
$$D^{n+1/2} = \frac{1}{4} a_f(u_h^{n+1} + u_h^{n-1}, u_h^{n+1} + u_h^{n-1}) + \frac{1}{4} a_p(\phi_h^{n+1} + \phi_h^{n-1}, \phi_h^{n+1} + \phi_h^{n-1}).$$

The coupling terms are

$$Coupling = \frac{\triangle t}{2}\left[c_I(\phi_h^n, u_h^{n+1} + u_h^{n-1}) - c_I(\phi_h^{n+1} + \phi_h^{n-1}, u_h^n)\right] =$$
$$= \frac{\triangle t}{2}\left[c_I(\phi_h^n, u_h^{n+1}) - c_I(\phi_h^{n+1}, u_h^n)\right] - \frac{\triangle t}{2}\left[c_I(\phi_h^{n-1}, u_h^n) - c_I(\phi_h^n, u_h^{n-1})\right].$$

Let us denote $C^{n+1/2} = c_I(\phi_h^n, u_h^{n+1}) - c_I(\phi_h^{n+1}, u_h^n)$ so we have

$$\left[E^{n+1/2} + \frac{\Delta t}{2} C^{n+1/2}\right] - \left[E^{n-1/2} + \frac{\Delta t}{2} C^{n-1/2}\right]$$
$$+\triangle t D^{n+1/2} = \frac{\Delta t}{2}\left((f_f^n, u_h^{n+1} + u_h^{n-1})_f + g(f_p^n, \phi_h^{n+1} + \phi_h^{n-1})\right).$$

Standard coercivity estimates show that

$$D^{n+1/2} \geq \frac{\nu}{4}||\nabla\left(u_h^{n+1} + u_h^{n-1}\right)||_f^2 + \frac{g k_{\min}}{4}||\nabla\left(\phi_h^{n+1} + \phi_h^{n-1}\right)||_p^2.$$

Summing, stability and the stated energy inequality thus follow provided

$$E^{n+1/2} + \frac{\triangle t}{2} C^{n+1/2} \geq \frac{1}{8}\left(||u_h^{N+1}||^2_{DIV} + ||u_h^N||^2_{DIV}\right) +$$
$$+ \frac{1}{4} g S_0 \left(||\phi_h^{N+1}||_p^2 + ||\phi_h^N||_p^2\right).$$

Consider the coupling terms. Using (3.3)

$$\frac{\Delta t}{2}\left|C^{n+1/2}\right| = \frac{\Delta t}{2} g \left|\int_I \phi_h^n u_h^{n+1}\cdot\widehat{n}_f - \phi_h^{n+1} u_h^n \cdot \widehat{n}_f ds\right|$$
$$\leq \frac{\Delta t}{2} g\left(||u_h^{n+1}||_{DIV}||\phi_h^n||_{H^1(\Omega_p)} + ||u_h^n||_{DIV}||\phi_h^{n+1}||_{H^1(\Omega_p)}\right)$$
$$\leq \frac{1}{8}\left[||u_h^{n+1}||^2_{DIV} + ||u_h^n||^2_{DIV}\right] + \frac{\Delta t^2 g^2}{2}\left[||\phi_h^{n+1}||^2_{H^1(\Omega_p)} + ||\phi_h^n||^2_{H^1(\Omega_p)}\right].$$

We subtract this from $E^{n+1/2}$ and cancel terms, completing the proof. □

3.2. Application to Stokes flow plus Coriolis force. The use of CNLF in geophysical flows is based on fast-slow wave decompositions and time filters, see [28], [30]. There are many complexities in geophysics we shall avoid in this first test by focusing on Stokes flow plus a Coriolis force $f_C \times u$:

$$u_t - \nu \triangle u + \nabla p + f_C \times u = f(x,t) \text{ and } \nabla \cdot u = 0 \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega, \text{ and } u(x,0) = u_0(x) \text{ in } \Omega.$$

Choose conforming velocity-pressure FEM spaces $X^h \subset H_0^1(\Omega)^d$, $Q^h \subset L_0^2(\Omega)$ satisfying the usual discrete inf-sup condition, [12], [11]. Define the bilinear form (with a grad-div term, [23])

$$a(u,v) = (\nu \nabla u, \nabla v) + (\nabla \cdot u, \nabla \cdot v)$$

Let $\Lambda u := f_C \times u$. The (1.4) realization is: find $(u_h^{n+1}, p_h^{n+1}) \in X^h \times Q^h$ satisfying, for all $v_h \in X^h, q_h \in Q^h$,

$$\begin{aligned}
&(\frac{u_h^{n+1} - u_h^{n-1}}{2\triangle t}, v_h) + 2\triangle t(\Lambda(u_h^{n+1} - u_h^{n-1}), \Lambda(v_h)) \\
&+(f_C \times u_h^n, v_h) + a(\frac{u_h^{n+1} + u_h^{n-1}}{2}, v_h) + \\
&-(\frac{p_h^{n+1} + p_h^{n-1}}{2}, \nabla \cdot v_h) + (q_h, \nabla \cdot \frac{u_h^{n+1} + u_h^{n-1}}{2}) = (f^n, v_h).
\end{aligned} \tag{3.4}$$

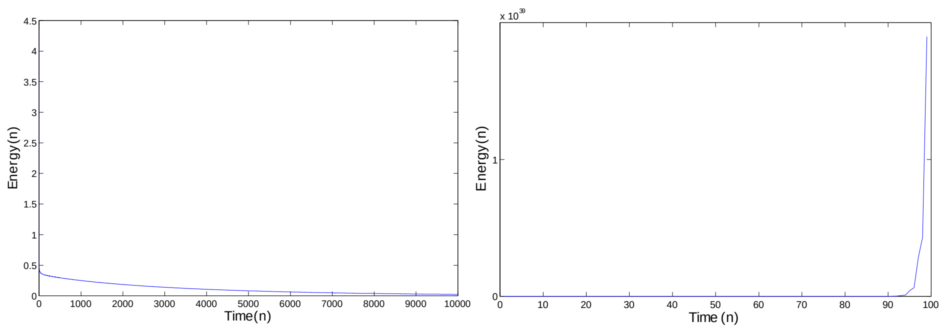
THEOREM 3.3 (Unconditional stability for flow + rotation). *(3.4) is unconditionally stable: for any $N > 0$,*

$$\begin{aligned}
&\frac{1}{2}[||u_h^{N+1}||^2 + \frac{3}{4}||u_h^N||^2 + 4\Delta t^2 ||\Lambda_h(u_h^N)||^2] \\
&+ \frac{\Delta t}{2} \sum_{n=1}^{N} \left[\nu ||\nabla(u_h^{n+1} + u_h^{n-1})||^2 + ||\nabla \cdot (u_h^{n+1} + u_h^{n-1})||^2\right] \\
&\leq \frac{1}{2}[||u_h^1||^2 + 4\Delta t^2 ||\Lambda_h(u_h^1)||^2 + ||u_h^0||^2 + 4\Delta t^2 ||\Lambda_h(u_h^0)||^2] \\
&+ \Delta t(\Lambda_h(u_h^0), u_h^1) + \Delta t \sum_{n=1}^{N} (f^n, u_n^{n+1} + u_h^n).
\end{aligned}$$

Proof. The proof in essential details follows that of Theorems 1 and 2. ◻

4. Numerical Illustrations. We present two tests of stability, performed using FreeFEM++ [13], one for Stokes-Darcy and one for Stokes flow plus strong rotation.

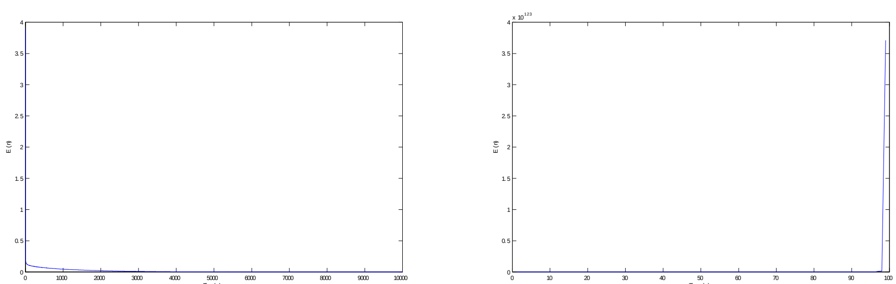
Example 1: Stokes-Darcy. We solve the Stokes-Darcy problem with and without stabilization, for small values of the parameters S_0 and k_{min} (all other parameters are 1) and $h = \triangle t = 0.1$. The true solution decays to zero as $t \to \infty$ so any growth in the approximate solution is an instability. The first test (and two figures) are for parameters $S_0 = 1, k_{min} = 10^{-4}$ and time steps that satisfy the timestep condition. CNLFSTAB is stable, as predicted, and after a long enough time, CNLF becomes weakly unstable, as often reported.



CNLFSTAB: stable CNLF: unstable

Figure 1: CNLFSTAB vs. CNLF, CFL condition holds

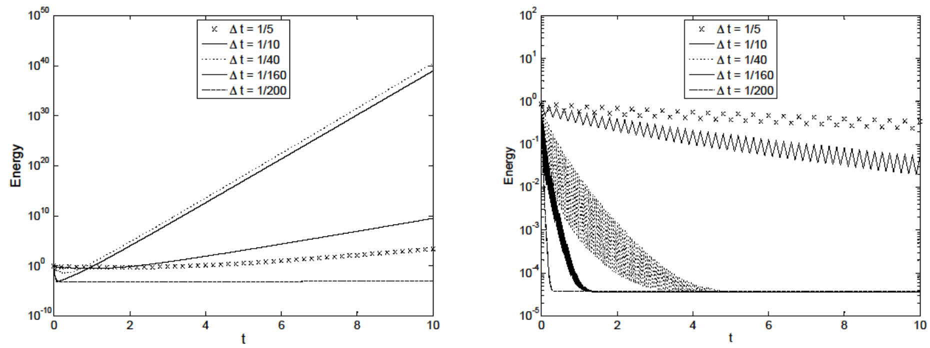
Thus CNLFSTAB controls CNLF's weak instability in this test. The second test (and two figures) are for parameters $S_0 = 0.1, k_{min} = 10^{-4}$ and the timestep condition violated. CNLFSTAB is stable, as predicted while CNLF is unstable, as expected. We have performed tests for other parameter values with the same result.



CNLFSTAB: stable. CNLF: unstable.

Figure 2: CNLFSTAB vs. CNLF, CFL condition violated

Example 2: Stokes flow + strong rotation. In this example we consider the $2d$ Stokes problem plus Coriolis forces with a speed of rotation $\omega = 100$. The computational domain is the square $[0,1] \times [0,1]$. Let $g_1(x) = x^2(1-x^2)\exp(7x)$, $g_2(y) = y^2(1-y)^2$ and the initial conditions be defined by $u_0 = g_1(x)g_2'(y)$, $v_0 = -g_1'(x)g_2(y)$. We solve the problem and plot the kinetic energy vs. time for CNLF first without and then with stabilization. As predicted by the theory, CNLF is unstable until $\triangle t||\Lambda|| < 1$ while CNLFstab is stable for all time steps.



CNLF: unstable until $\triangle t$ small   CNLFstab: always stable

Figure 3: Stability of CNLF vs. CNLFstab

5. Conclusions. The accepted view of CNLF without additional stabilizations or time filters is that it has two issues. First, a CFL type timestep condition is necessary for stability. Second, even under a CFL condition, non-damping of noise in the unstable mode over long time intervals can also lead to instabilities. Time filters are a wonderfully elegant and modular tool that addresses the second issue but not the first. We have presented a stabilization herein that, while not modular, addresses both issues. For the Stokes-Darcy problem CNLFstab is stable for all S_0 and k_{min} and is, to our knowledge, the first second order, parameter-uniform, long time stable partitioned method. Naturally, when a timestep condition is grossly violated (as the tests here did purposefully), the difficulty may be shifted from stability to accuracy. Thus, the next important step in studying CNLFstab must be precise error analysis and careful testing of accuracy for specific applications, like the two in Section 3.

REFERENCES

[1] M. Anitescu, W. Layton and F. Pahlevani, *Implicit for local effects, explicit for nonlocal is unconditionally stable*, ETNA 18 (2004), 174-187.

[2] U. Asher, S. Ruuth and B. Wetton, *Implicit-Explicit methods for time dependent partial differential equations*, SINUM 32(1995) 797-823.

[3] R.A. Asselin, *Frequency filter for time integration*, Mon. Weather Review 100(1972) 487-490.

[4] J. Bear, *Hydraulics of Groundwater*. McGraw-Hill, New York, 1979.

[5] G.S. Beavers and D. Joseph, *Boundary conditions at a naturally permeable wall*, JFM 30(1967) 197-207.

[6] Y. Cao, M. Gunzburger, X.-M. He, and X. Wang, *Parallel, non-iterative multi-physics domain decomposition methods for the time-dependent Stokes-Darcy problem*, tech report, 2011.

[7] M. Crouzeix, *Une méthode multipas implicite-explicite pour l'approximation des équationes d'évolution paraboliques*, Numer. Math. 35(1980) 257-276.

[8] L. Davis and F. Pahlevani, *Semi-Implicit Schemes for Transient Navier-Stokes Equations and Eddy Viscosity Models*, NMPDEs, 25 (2009), 212-231.

[9] M. Discacciati, E. Miglio and A. Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows*, ANM 43(2001) 57-74.

[DD] J. Douglas and T. Dupont, *Alternating-direction Galerkin methods on rectangles*, 133-214 in: Numerical Solution of PDEs, SYNSPADE II, Academic Press, New York, 1971.

[10] V.J. Ervin, E.W. Jenkins and S. Sun, *Coupling nonlinear Stokes and Darcy flow using mortar finite elements*, ANM, in Press, 2011.

[11] V. Girault and P.A. Raviart, *Finite Element Methods for Navier Stokes Equations*, Springer-Verlag, 1986.

[12] M.D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows - A Guide to Theory, Practices, and Algorithms*, Academic Press, 1989.
[13] F. Hecht and O. Pironneau, FreeFem++, URL: http://www.freefem.org.
[14] W. Hundsdorfer and J. Verwer, *Numerical solution of time dependent advection diffusion reaction equations*, Springer, Berlin, 2003.
[15] O. Johansson, H.-O. Kreiss, *Über das Verfahren der zentralen Differenzen zur Lösung des Cauchy problems für partielle Differentialgleichungen*, Nordisk Tidskr. Informations-Behandling 3 (1963) 97–107.
[16] E. Kalnay, *Atmospheric Modeling, data assimilation and predictability*, Cambridge Univ. Press, Cambridge, 2003.
[17] M. Kubacki, *Uncoupling Evolutionary Groundwater-Surface Water Flows Using the Crank-Nicolson Leapfrog Method*, TR12-07 at www.mathematics.pitt.edu/research/technical-reports.php, 2012.
[18] W. Layton and C. Trenchea, *Stability of two IMEX methods, CNLF and BDF2-AB2, for uncoupling systems of evolution equations*, ANM 62(2012), 112-120.
[19] W. Layton, H. Tran and C. Trenchea, *Analysis of Long Time Stability and Errors of Two Partitioned Methods for Uncoupling Evolutionary Groundwater - Surface Water Flows*, TR 11-05 at: www.mathematics.pitt.edu/research/technical-reports.php, 2011.
[20] Li Shan, Haibiao Zheng and W. Layton, *A Non-Iterative, Domain Decomposition Method with Different Time Step Sizes for the Evolutionary Stokes-Darcy Model*, to appear: NMPDEs 2012, DOI: 10.1002/num.21720.
[21] M. Moraiti, *On the quasistatic approximation in the Stokes-Darcy model of groundwater - surfacewater flows*, JMAA 394(2012) 796-808.
[22] M. Mu and X. Zhu, *Decoupled schemes for a non-stationary mixed Stokes -Darcy model*, Math Comp. 79(2010) 707-731.
[23] M.A. Olshanskii and A. Reusken, *Grad-Div stabilization for the Stokes equations*, Math. of Comp., 73 (2004), 1699–1718.
[24] L.E. Payne and B. Straughan, *Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modeling questions*, J Math. Pure Appl. 77(1998) 1959-1977.
[25] G.F. Pinder and M.A. Celia, *Subsurface hydrology*, Wiley, Hoboken, New Jersey, 2006.
[26] A. Robert, *The integration of a spectral model of the atmosphere by the implicit method*, Proc. WMO/IUGG Symposium on NWP, Japan Meteorological Soc., Tokyo, Japan, pp. 19-24, 1969.
[27] P.G. Saffman, *On the boundary condition at the interface of a porous medium*, Studies in Appl. Math. 1(1971) 93-101.
[28] S. J. Thomas, R. D. Loft, *The NCAR spectral element climate dynamical core: semi-implicit Eulerian formulation*, J. Sci. Comput. 25 (1-2) (2005) 307–322.
[29] J. Verwer, *Convergence and component splitting for the Crank-Nicolson Leap-Frog integration method*, CWI report MAS-E0902, 2009.
[30] P.D. Williams, *The RAW Filter: An Improvement to the Robert–Asselin Filter in Semi-Implicit Integrations*, Mon. Weather Rev., 139 (2011) 1996–2007.