ON THE ESTIMATES OF DETERMINING MODES FOR NS-α AND NS-ω MODELS

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Abstract.

Key words. determining modes, Navier-Stokes equations

1. Introduction. Solutions of NSE can be represented adequately in a finite dimensional space whose basis is determining modes. Number of determining modes is an indicator of complexity of solutions to the NSE [2]. The problem of finding an optimal determining modes estimation to understand the complexity of the NSE continues to grow.

This report will give an estimate of the determining modes of NS-α and NS-ω models. We show that these models have fewer determining modes than in the case of NSE models [3], and when the radius filter goes to 0, these three models turn out to have the same number of determining modes.

We investigate the 3D equilibrium NS-α regularization

$$\begin{cases} -\nu\Delta u - \bar{u}\times(\nabla\times u) + \nabla p = f \text{ in } \Omega := (0,2\pi)^3 \\ \nabla\cdot\bar{u} = 0 \text{ in } \Omega \end{cases}$$

and NS-ω regularization

$$\begin{cases} -\nu\Delta u - u\times(\nabla\times \bar{u}) + \nabla p = f \text{ in } \Omega \\ \nabla\cdot u = 0 \text{ in } \Omega \end{cases}$$

with differential filter

$$\begin{cases} -\delta^2\Delta\bar{u} + \bar{u} = u \\ \nabla\cdot\bar{u} = 0 \end{cases}$$

and 2π-periodic boundary condition:

$$\int_\Omega u(x)dx = 0$$

Using Fourier transform, solution to both of the above models can be written

$$u(x) = \sum_{\mathbf{k}} \hat{u}(\mathbf{k})e^{-i\mathbf{k}x}$$

where $\mathbf{k}\in \mathbf{Z}^3$ and Fourier coefficient

$$\hat{u}(\mathbf{k}) = \frac{1}{(2\pi)^3}\int_\Omega u(x)e^{-i\mathbf{k}x}dx$$

Define $V_s = \{u(x) = \sum_{\mathbf{k}} \hat{u}(\mathbf{k})e^{-i\mathbf{k}x}$ such that $\sum_{\mathbf{k}} |\mathbf{k}|^{2s}|\hat{u}(\mathbf{k})|^2 < \infty\}$. In V_s define the norm

$$\|u\|_s = \|u\|_{V_s} = \left(\sum_{\mathbf{k}} |\mathbf{k}|^{2s}|u(\mathbf{k})|^2\right)^{1/2}$$

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We will use the following inequality in [3]

$$(u \cdot \nabla v, w) \le c\|u\|_{s_1}\|v\|_{s_2+1}\|w\|_{s_3} \tag{1.1}$$

where $s_1 + s_2 + s_3 = \frac{3}{2}, s_1, s_2, s_3 \ge 0$

A similar inequality can also be verified

$$(u \times (\nabla \times v), w) \le c\|u\|_{s_1}\|v\|_{s_2+1}\|w\|_{s_3} \tag{1.2}$$

The following identities will also be employed

$$(u \times (\nabla \times v), w) = -(w \times (\nabla \times v), u) \tag{1.3}$$
$$(u \times (\nabla \times v), u) = 0 \tag{1.4}$$
$$a \times (\nabla \times b) = (\nabla \times a) \times b - a \cdot \nabla b - b \cdot \nabla a + \nabla(a \cdot b) \tag{1.5}$$

2. Estimate of determining modes for NS-α models. We will state and prove the following theorem

THEOREM 2.1. *Let $k_\alpha(\delta) = \frac{1}{1+\delta^2}\frac{c^2\|f\|_*^2}{\nu^4}$ and X_α denote the finite dimensional space*

$$X_\alpha := span\{e^{i\mathbf{k}x} : |\mathbf{k}| \le k_\alpha(\delta)\} \cap V$$

If u_1 and u_2 are 2 solutions of the NS-α regularization with $P_{X_\alpha}(u_1 - u_2) = 0$ then $u_1 \equiv u_2$.

Here $V = \{v \in H^1(\Omega) | \int\limits_\Omega v dx = 0, \nabla \cdot v = 0\}$, $\|.\|_*$ is the norm in V^*, the dual space of V, P_{X_α} is the projection from V to X_α

Proof. Let u_1 and u_2 be 2 solutions to the equilibrium problem and $\phi = u_1 - u_2$

$$\begin{cases} -\nu\Delta u_1 - \bar{u}_1 \times (\nabla \times u_1) + \nabla p_1 = f \\ -\nu\Delta u_2 - \bar{u}_2 \times (\nabla \times u_2) + \nabla p_2 = f \end{cases}$$

It turns out

$$\begin{aligned} & -\nu\Delta(u_1 - u_2) - (\bar{u}_1 \times (\nabla \times u_1) - \bar{u}_2 \times (\nabla \times u_2)) + \nabla(p_1 - p_2) = 0 \\ \Rightarrow & -\nu\Delta\phi - (\bar{\phi} \times (\nabla \times u_1) + \bar{u}_2 \times (\nabla \times \phi)) + \nabla(p_1 - p_2) = 0 \\ \Rightarrow & -\nu\Delta\phi = \bar{\phi} \times (\nabla \times u_1) + \bar{u}_2 \times (\nabla \times \phi) - \nabla(p_1 - p_2) \end{aligned}$$

Multipying both sides by $\bar{\phi}$ and $\int_\Omega$ yields

LHS:

$$\begin{aligned} & -\nu(\Delta\phi, \bar{\phi}) = \nu(\nabla\phi, \nabla\bar{\phi}) = \nu(\nabla\bar{\phi}, \nabla\bar{\phi}) - \nu\delta^2(\nabla(\Delta\bar{\phi}), \nabla\bar{\phi}) \\ =& \nu\|\nabla\bar{\phi}\|^2 + \nu\delta^2\|\Delta\bar{\phi}\|^2 \end{aligned} \tag{2.1}$$

RHS:

$$(\bar{\phi} \times (\nabla \times u_1), \bar{\phi}) + (\bar{u}_2 \times (\nabla \times \phi), \bar{\phi}) - (\nabla(p_1 - p_2), \bar{\phi}) = (\bar{u}_2 \times (\nabla \times \phi), \bar{\phi})$$

Applying identity (1.5) we have

$$\begin{aligned} & (\bar{u}_2 \times (\nabla \times \phi), \bar{\phi}) \\ =& -(\phi \times (\nabla \times \bar{u}_2), \bar{\phi}) - (\bar{u}_2 \cdot \nabla\phi, \bar{\phi}) - (\phi \cdot \nabla\bar{u}_2, \bar{\phi}) + (\nabla(\bar{u}_2 \cdot \phi), \bar{\phi}) \\ =& -(\bar{\phi} \times (\nabla \times \bar{u}_2), \bar{\phi}) + \delta^2(\Delta\bar{\phi} \times (\nabla \times \bar{u}_2), \bar{\phi}) + (\bar{u}_2 \cdot \nabla\bar{\phi}, \bar{\phi}) - \delta^2(\bar{u}_2 \cdot \nabla\bar{\phi}, \Delta\bar{\phi}) - (\bar{\phi} \cdot \nabla\bar{u}_2, \bar{\phi}) + \delta^2(\Delta\bar{\phi} \cdot \nabla\bar{u}_2, \bar{\phi}) \\ =& \delta^2(\Delta\bar{\phi} \times (\nabla \times \bar{u}_2), \bar{\phi}) - \delta^2(\bar{u}_2 \cdot \nabla\bar{\phi}, \Delta\bar{\phi}) - (\bar{\phi} \cdot \nabla\bar{u}_2, \bar{\phi}) + \delta^2(\Delta\bar{\phi} \cdot \nabla\bar{u}_2, \bar{\phi}) \end{aligned}$$

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By (1.1) and (1.2) we get

$$\delta^2(\Delta\bar{\phi}\times(\nabla\times\bar{u_2}),\bar{\phi})\le c\delta^2\|\Delta\bar{\phi}\|\|\nabla\bar{u_2}\|\|\bar{\phi}\|_{3/2}$$
$$-\delta^2(\bar{u_2}\cdot\nabla\bar{\phi},\Delta\bar{\phi})\le c\delta^2\|\nabla\bar{u_2}\|\|\bar{\phi}\|_{3/2}\|\Delta\bar{\phi}\|$$
$$\delta^2(\Delta\bar{\phi}\cdot\nabla\bar{u_2},\bar{\phi})\le c\delta^2\|\Delta\bar{\phi}\|\|\nabla\bar{u_2}\|\|\bar{\phi}\|_{3/2}$$
$$-(\bar{\phi}\cdot\nabla\bar{u_2}),\bar{\phi})\le c\|\bar{\phi}\|_{1/2}\|\nabla\bar{u_2}\|\|\nabla\bar{\phi}\|$$

Therefore

$$\begin{aligned}&(\bar{u}_2\times(\nabla\times\phi),\bar{\phi})\\ \le& c\|\bar{\phi}\|_{1/2}\|\nabla\bar{u_2}\|\|\nabla\bar{\phi}\|+c\delta^2\|\Delta\bar{\phi}\|\|\nabla\bar{u_2}\|\|\bar{\phi}\|_{3/2}\\ \le&\frac{c^2}{2\nu}\|\bar{\phi}\|_{1/2}^2\|\nabla\bar{u}_2\|^2+\frac{\nu}{2}\|\nabla\bar{\phi}\|^2+\frac{c^2\delta^2}{2\nu}\|\bar{\phi}\|_{3/2}^2\|\nabla\bar{u}_2\|^2+\frac{\nu\delta^2}{2}\|\Delta\bar{\phi}\|^2\end{aligned}\tag{2.2}$$

From (2.1) and (2.2)

$$\frac{\nu}{2}\|\nabla\bar{\phi}\|^2+\frac{\nu\delta^2}{2}\|\Delta\bar{\phi}\|^2\le\frac{c^2}{2\nu}\|\bar{\phi}\|_{1/2}^2\|\nabla\bar{u}_2\|^2+\frac{c^2\delta^2}{2\nu}\|\bar{\phi}\|_{3/2}^2\|\nabla\bar{u}_2\|^2$$
$$\|\nabla\bar{\phi}\|^2+\delta^2\|\Delta\bar{\phi}\|^2\le\frac{c^2}{\nu^2}\|\nabla\bar{u}_2\|^2(\|\bar{\phi}\|_{1/2}^2+\delta^2\|\bar{\phi}\|_{3/2}^2)$$

Now by Fourier transform

$$\begin{aligned}\text{LHS}&=\sum_{\mathbf{k}}\frac{|\mathbf{k}|^2}{(1+\delta^2|\mathbf{k}|^2)^2}|\hat{\phi}(\mathbf{k})|^2+\sum_{\mathbf{k}}\frac{\delta^2|\mathbf{k}|^4}{(1+\delta^2|\mathbf{k}|^2)^2}|\hat{\phi}(\mathbf{k})|^2\\&=\sum_{\mathbf{k}}\frac{|\mathbf{k}|^2}{1+\delta^2|\mathbf{k}|^2}|\hat{\phi}(\mathbf{k})|^2\end{aligned}\tag{2.3}$$

RHS: Since

$$\begin{aligned}&-\nu\Delta u_2-\bar{u}_2\times(\nabla\times u_2)+\nabla p_2=f\\ \Rightarrow&-\nu(\Delta u_2,\bar{u}_2)=(f,\bar{u}_2)\\ \Rightarrow&\nu(\nabla u_2,\nabla\bar{u}_2)=(f,\bar{u}_2)\\ \Rightarrow&\nu(\nabla\bar{u}_2,\nabla\bar{u}_2)-\nu\delta^2(\nabla(\Delta\bar{u}_2),\nabla\bar{u}_2)=(f,\bar{u}_2)\\ \Rightarrow&\nu\|\nabla\bar{u}_2\|^2+\nu\delta^2\|\Delta\bar{u}_2\|^2=(f,\bar{u}_2)\le\|f\|_*\|\nabla\bar{u}_2\|\le\frac{\|f\|_*^2}{2\nu}+\frac{\nu}{2}\|\nabla\bar{u}_2\|^2\\ \Rightarrow&\|\nabla\bar{u}_2\|^2+\delta^2\|\Delta\bar{u}_2\|^2\le\frac{\|f\|_*^2}{\nu^2}\\ \Rightarrow&\sum_{\mathbf{k}}\left(\frac{|\mathbf{k}|^2}{(1+\delta^2|\mathbf{k}|^2)^2}+\frac{\delta^2|\mathbf{k}|^4}{(1+\delta^2|\mathbf{k}|^2)^2}\right)|\hat{u}(\mathbf{k})|^2=\sum_{\mathbf{k}}\frac{|\mathbf{k}|^2}{1+\delta^2|\mathbf{k}|^2}|\hat{u}(\mathbf{k})|^2\le\frac{\|f\|_*^2}{\nu^2}\\ \Rightarrow&\|\nabla\bar{u}_2\|^2=\sum_{\mathbf{k}}\frac{|\mathbf{k}|^2}{(1+\delta^2|\mathbf{k}|^2)^2}|\hat{u}(\mathbf{k})|^2\le\frac{1}{1+\delta^2}\sum_{\mathbf{k}}\frac{|\mathbf{k}|^2}{1+\delta^2|\mathbf{k}|^2}|\hat{u}(\mathbf{k})|^2\le\frac{1}{1+\delta^2}\frac{\|f\|_*^2}{\nu^2}\end{aligned}\tag{2.4}$$

Furthermore

$$\begin{aligned}\|\bar{\phi}\|_{1/2}^2+\delta^2\|\bar{\phi}\|_{3/2}^2&=\sum_{\mathbf{k}}\left(\frac{|\mathbf{k}|}{(1+\delta^2|\mathbf{k}|^2)^2}+\frac{\delta^2|\mathbf{k}|^3}{(1+\delta^2|\mathbf{k}|^2)^2}\right)|\hat{\phi}(\mathbf{k})|^2\\&=\sum_{\mathbf{k}}\frac{|\mathbf{k}|}{1+\delta^2|\mathbf{k}|^2}|\hat{\phi}(\mathbf{k})|^2\end{aligned}\tag{2.5}$$

From (2.3)-(2.5)

$$\sum_{\mathbf{k}} \frac{|\mathbf{k}|^2}{1+\delta^2|\mathbf{k}|^2}|\hat{\phi}(\mathbf{k})|^2 \le \frac{1}{1+\delta^2}\frac{c^2\|f\|_*^2}{\nu^4}\sum_{\mathbf{k}} \frac{|\mathbf{k}|}{1+\delta^2|\mathbf{k}|^2}|\hat{\phi}(\mathbf{k})|^2$$
$$\Rightarrow \sum_{\mathbf{k}} \left[\frac{|\mathbf{k}|^2}{1+\delta^2|\mathbf{k}|^2} - \frac{1}{1+\delta^2}\frac{c^2\|f\|_*^2}{\nu^4}\frac{|\mathbf{k}|}{1+\delta^2|\mathbf{k}|^2}\right]|\hat{\phi}(\mathbf{k})|^2 \le 0$$

For $|\mathbf{k}|$ large enough, the bracket term is positive. Then, the NS-α regularization will have a finite number of determining modes.

To estimate the number, let $k = |\mathbf{k}| > 0$. Then, the number of determining modes is the greatest integer in the positive roots of

$$\frac{k^2}{1+\delta^2k^2} - \frac{1}{1+\delta^2}\frac{c^2\|f\|_*^2}{\nu^4}\frac{k}{1+\delta^2k^2} = 0$$
$$\Rightarrow k = \frac{1}{1+\delta^2}\frac{c^2\|f\|_*^2}{\nu^4}$$

REMARK 2.1. $k_\alpha = \frac{1}{1+\delta^2}\frac{c^2\|f\|_*^2}{\nu^4} \le \frac{c^2\|f\|_*^2}{\nu^4} = k_{NSE}$.
When $\delta \to 0$, then $k_\alpha \to k_{NSE}$. Furthermore $k_\alpha \simeq (1-\delta^2)\frac{c^2\|f\|_*^2}{\nu^4}$ if δ small

3. Estimate of determining modes for NS-ω model. Below is the same theorem for NS-ω models

THEOREM 3.1. *Let $k_\omega(\delta)$ be the greatest integer in the positive root of*

$$k(1+\delta^2k^2)^2 = \frac{c^2\|f\|_*^2}{\nu^4}$$

Let X_ω denote the finite dimensional space

$$X_\omega := span\{e^{i\mathbf{k}x} : |\mathbf{k}| \le k_\omega(\delta)\} \cap V$$

If u_1 and u_2 are 2 solutions of the NS-ω regularization with $P_{X_\omega}(u_1 - u_2) = 0$ then $u_1 \equiv u_2$.

Proof. Let u_1 and u_2 be 2 solutions to the equilibrium problem and $\phi = u_1 - u_2$

$$\begin{cases} -\nu\Delta u_1 - u_1 \times (\nabla \times \bar{u}_1) + \nabla p_1 = f \\ -\nu\Delta u_2 - u_2 \times (\nabla \times \bar{u}_2) + \nabla p_2 = f \end{cases}$$

It turns out

$$\begin{aligned} &-\nu\Delta(u_1 - u_2) - (\phi \times (\nabla \times \bar{u}_1) + u_2 \times (\nabla \times \bar{\phi})) + \nabla(p_1 - p_2) = 0 \\ \Rightarrow &-\nu(\Delta\phi, \phi) = (\phi \times (\nabla \times \bar{u}_1), \phi) + (u_2 \times (\nabla \times \bar{\phi}), \phi) - (\nabla(p_1 - p_2), \phi) \\ \Rightarrow &\nu\|\nabla\phi\|^2 = (u_2 \times (\nabla \times \bar{\phi}), \phi) \end{aligned} \tag{3.1}$$

Again applying identity (1.5) we have

$$\begin{aligned} &(u_2 \times (\nabla \times \bar{\phi}), \phi) \\ = &-(\bar{\phi} \times (\nabla \times u_2), \phi) - (u_2 \cdot \nabla\bar{\phi}, \phi) - (\bar{\phi} \cdot \nabla u_2, \phi) + (\nabla(u_2 \cdot \bar{\phi}), \phi) \\ \le &c\|\bar{\phi}\|_{1/2}\|\nabla u_2\|\|\nabla\phi\| \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we have $\|\nabla\phi\|^2 \le \frac{c^2}{\nu^2}\|\bar{\phi}\|_{1/2}^2\|\nabla u_2\|^2$

But $-\nu\Delta u_2 - u_2 \times (\nabla \times \bar{u}_2) + \nabla p_2 = f$. Hence

$$\nu\|\nabla u_2\|^2 = (f, u_2) \le \|f\|_*\|\nabla u_2\| \text{ or } \|\nabla u_2\|^2 \le \frac{\|f\|_*^2}{\nu^2}$$

And we get $\|\nabla\phi\|^2 \le \frac{c^2\|f\|_*^2}{\nu^4}\|\bar{\phi}\|_{1/2}^2$
Using Fourier transform

$$\sum_{\mathbf{k}} |\mathbf{k}|^2|\hat{\phi}(\mathbf{k})|^2 \le \frac{c^2\|f\|_*^2}{\nu^4}\sum_{\mathbf{k}} \frac{|\mathbf{k}|}{(1+\delta^2|\mathbf{k}|^2)^2}|\hat{\phi}(\mathbf{k})|^2$$
$$\Rightarrow \sum_{\mathbf{k}} \left[|\mathbf{k}|^2 - \frac{c^2\|f\|_*^2}{\nu^4}\frac{|\mathbf{k}|}{(1+\delta^2|\mathbf{k}|^2)^2}\right]|\hat{\phi}(\mathbf{k})|^2 \le 0$$

For $|\mathbf{k}|$ large enough, the bracket term is positive. Then, the NS-ω regularization will have a finite number of determining modes.

To estimate the number, let $k = |\mathbf{k}| > 0$. Then, the number of determining modes is the greatest integer in the positive roots of

$$k^2 - \frac{c^2\|f\|_*^2}{\nu^4}\frac{k}{(1+\delta^2k^2)^2} = 0$$
$$\Rightarrow k(1+\delta^2k^2)^2 = \frac{c^2\|f\|_*^2}{\nu^4}$$

REMARK 3.1. $k_\omega \le \frac{c^2\|f\|_*^2}{\nu^4} = k_{NSE}$.
When $\delta \to 0$, then $k_\omega \to k_{NSE}$. Furthermore $k_\omega \simeq (1-2\delta^2)\frac{c^2\|f\|_*^2}{\nu^4}$ if δ small

REMARK 3.2. *By computation in this report, $k_\omega \le k_\alpha$. We could expect that NS-ω regularization need fewer modes than NS-α.*

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