ORNL REPORT

ORNL/TM-2014/79
Unlimited Release
Printed January 2014

A convergence analysis of stochastic collocation method for Navier-Stokes equations with random input data

H. Tran, C. Trenchea, C. Webster

Prepared by
Department of Computational and Applied Mathematics
Computer Science and Mathematics Division
Oak Ridge National Laboratory
One Bethel Valley Road, Oak Ridge, Tennessee 37831

The Oak Ridge National Laboratory is operated by UT-Battelle, LLC,
for the United States Department of Energy under Contract DE-AC05-00OR22725.
Approved for public release; further dissemination unlimited.

OAK RIDGE NATIONAL LABORATORY
MANAGED BY UT-BATTELLE FOR THE DEPARTMENT OF ENERGY

ORNL/TM-2014/79
Date Published: January 2014

A CONVERGENCE ANALYSIS OF STOCHASTIC COLLOCATION METHOD FOR NAVIER-STOKES EQUATIONS WITH RANDOM INPUT DATA

Hoang Tran * Catalin Trenchea † Clayton Webster ‡

Abstract. Stochastic collocation method has proved to be an efficient method and been widely applied to solve various partial differential equations with random input data, including Navier-Stokes equations. However, up to now, rigorous convergence analyses are limited to linear elliptic and parabolic equations; its performance for Navier-Stokes equations was demonstrated mostly by numerical experiments. In this paper, we present an error analysis of the stochastic collocation method for a semi-implicit Backward Euler discretization for NSE and prove the exponential decay of the interpolation error in the probability space. Our analysis indicates that due to the nonlinearity, as final time T increases and NSE solvers pile up, the accuracy may be reduced significantly. Subsequently, an illustrative computational test of time dependent fluid flow around a bluff body is provided.

1. Introduction. Flow of liquids and gases is ubiquitous in nature and obtaining an accurate prediction of these flows is a central difficulty in diverse problems such as global change estimation, improving the energy efficiency of engines, controlling dispersal of contaminants, designing biomedical devices and many other venues. Most applications of fluid flows in engineering and science are affected by uncertainty in the input data and mathematical models, e.g., forcing terms, wall roughness, material properties, source and interaction terms, geometry, model coefficients, etc. In this case, it is necessary to introduce uncertainty in mathematical models to assess the reliability of predictions based on numerical simulations.

The literature on numerical methods for stochastic differential equations has grown extensively in the last decade. The Monte Carlo sampling method is the classical and most popular approach for approximating expected values and other statistical moments of quantities of interest (QoI) based on the solution of PDEs with random inputs. While being very flexible and easy to implement, Monte Carlo method requires a very large number of samples to achieve small errors. Recently, other approaches have been proposed that often feature fast convergence. These include stochastic Galerkin methods, stochastic collocation methods, and perturbation, Neumann and Taylor expansion methods.

Stochastic collocation methods (SCM) have emerged to be a modern, efficient technique for quantifying uncertainty in physical applications [22, 26, 39]. One advantage of SCMs concerns the much faster convergence rates, which can yield accurate predictions of the uncertainty at a small fraction of the cost of a Monte-Carlo simulation, while maintaining an ensemble-based, non-intrusive approach. The better convergence behavior of SCMs, however, requires analyticity of the solutions with respect to the random variables. In [3], such property was established and the error estimates of SCMs were given for elliptic PDEs. These results have been extended to linear parabolic equations in [40]. Often in nonlinear scientific and engineering problems, particularly in Navier-Stokes equations,

*Department of Computational and Applied Mathematics, Oak Ridge National Laboratory, One Bethel Valley Road, P.O. Box 2008, MS-6164, Oak Ridge, TN 37831-6164 (tranha@ornl.gov).
†Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, 15260 (trenchea@pitt.edu).
‡Department of Computational and Applied Mathematics, Oak Ridge National Laboratory, One Bethel Valley Road, P.O. Box 2008, MS-6164, Oak Ridge, TN 37831-6164 (webstercg@ornl.gov).

complex solutions arise and their dependence on the random input data varies rapidly. For these cases, the smoothness of solutions in probability space has been less studied. Consequently, the accuracy of SCMs (and their variants) has been demonstrated mostly by numerical experiments rather than by rigorous error analysis.

In this article, we establish, for the first time, some analyticity results and consequential priori error estimates for the solutions of fully discrete approximations for stochastic Navier-Stokes equations. Our primary goal is to give a convergence analysis in probability space for the time-dependent *backward Euler with constant extrapolation* (BECE) scheme, whose space-time error estimates were obtained in [33]:

$$\begin{aligned} \frac{u^{n+1}-u^n}{\Delta t} - \nu \Delta u^{n+1} + u^n \cdot \nabla u^{n+1} + \nabla p^{n+1} &= f^{n+1}, \\ \nabla \cdot u^{n+1} &= 0. \end{aligned} \tag{BECE}$$

This simple, first order scheme helps us focus on interpolation error of SCMs while avoiding the long computation associating with more complex time discretizations.

The result we present here is twofold. First, we show that at a fixed time T, under mild assumptions on the derivatives of random input data, the interpolation errors decay (sub-)exponentially fast with respect to the tensor product polynomial degree. This guarantees the efficiency of SCMs over Monte-Carlo method for Navier-Stokes modeling. In fact, SCMs has been successfully applied in numerous Navier-Stokes problems, where fast convergence was demonstrated, see, e.g., [6, 10, 29]. On the other side, the task of estimating and controlling approximation errors *over long time* is particularly difficult. For space-time discretizations, it is well-known that an error bound exists but this bound grows so rapidly in time thus rendering the results less useful, see [13,18]. Our analysis reflects this situation in stochastic context: At a fixed polynomial degree, we show that the upper bound of interpolation errors grows fast to $O(1)$ as T increases, leading to a possible accuracy decline in long time simulation. A related analysis of long-term behavior of polynomial chaos was derived by Wan and Karniadakis [35]. The authors indicated that for NSEs, polynomial chaos will lose convergence after a short time and increasing the polynomial order does little help. Their estimates, however, are limited to random frequency stochastic process and can be applied to a few stochastic flows, e.g., vortex shedding. Different from their work, this study conducts a convergence analysis for NSEs from a generic perspective.

The error estimation herein employs a standard approach [3] plus a special treatment of the nonlinear term. The main ingredient in the analysis is a study of analyticity of numerical solutions with respect to the probabilistic parameters. We prove the existence of analytical extensions of solutions in a subregion of the complex plane. Loosely speaking, the larger this extending region is, the better the convergence rate is. Unlike the parabolic equations where the radius of analyticity is unchanged over time, it could shrink exponentially in T for Navier-Stokes schemes, resulting in a reduction in accuracy.

The discrete scheme (BECE) we study here is just one example in the wide class of multi-step backward differentiation methods coupled with semi-implicit or explicit scheme for the nonlinear terms, see [2, 5]. These schemes are attractive because each time step requires only one discrete Stokes system and linear solve. In addition, in many cases, no time step restriction is needed for space-time stability and convergence. While the interpolation error estimations of higher order time discretization (such as Crank-Nicolson/Adams-Bashforth [14], Crank-Nicolson with linear extrapolation [15]) are not considered in this work, we expect they could be obtained in the same manner with longer computation.

Since a convection term with linearized schemes plays a pivotal role in our analysis, direct extensions for fully implicit time stepping methods, e.g., Backward Euler, Crank-Nicolson, are not likely. One possible alternative is to seek an error estimation of implicit method coupled with an iterative algorithm, where the nonlinear term is again "lagged", as they naturally interwind in solving NSEs. As we shall see, a convergence analysis of finite element approximation with fixed point iteration for stochastic steady-state NSEs can be established as an easy modification of our analysis for (BECE). An estimate for time-dependent schemes is nevertheless much trickier and beyond the scope of this paper.

The outline of the paper is as follows. In Section 2, we introduce the mathematical problem and the notation used throughout the paper, as well as a description of the stochastic collocation methods. In Section 3, we derive the analyticity of the fully discrete solution of (BECE) and the convergence rates of SCMs for this scheme. A brief analysis for fixed point methods for steady-state NSEs is followed in Section 4. We then give an illustrative numerical example of time dependent flow around a circular cylinder in Section 5. Finally, concluding remarks are given in Section 6.

1.1. Related works. There have been many sparse-grid techniques developed to improve the efficiency of stochastic numerical methods [27,28,32]. Numerous approaches have been introduced to deal with stochastic problems whose solutions exhibit sharp variation and even discontinuity with respect to random data. These include domain decomposition of parametric spaces [7, 36], wavelet-base methods [23, 25], adaptive refinement strategies for regions of singularity [11, 34, 41] and parameterization of output data [37, 38]. These approaches show significant improvements in efficiency and accuracy in several nonlinear (and particularly Navier-Stokes) applications.

Space-time convergence estimates for (BECE) are derived in [33]. For numerical analysis for higher order non-iterative extrapolating schemes, we refer to [9,14,15]. Their application in modeling engineering flows can be found in [30] (turbulent flows induced by wind turbine motion), [1] (reacting flows in complex geometries such as gas turbine combustors), [24] (turbulent flows transporting particles).

2. Problem setting.

2.1. Notation and preliminaries. Let D be a convex bounded polygonal domain in $\mathbb{R}^m$ ($m = 2, 3$). The $L^2(D)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^\infty(D)$ norm is denoted by $\|\cdot\|_{L^\infty(D)}$. $H^1(D)$ is used to represent the Sobolev space $W^{1,2}(D)$ and $H_0^1(D)$ is the closure in $H^1(D)$ of the space $C_c^\infty(D)$. The $L_0^2(D)$ space is given by

$$L_0^2(D) = \left\{ q : D \to \mathbb{R} : q \in L^2(D) \text{ and } \int_D q dx = 0 \right\}.$$

For functions $v(x,t)$ defined for almost every $t \in (0,T)$ on function space $\mathcal{L}(D)$, we define the norms $(1 \le p \le \infty)$

$$\|v\|_{L^p(0,T;\mathcal{L}(D))} = \left(\int_0^T \|v\|^p_{\mathcal{L}(D)} dt \right)^{1/p}.$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Here Ω is the set of outcomes, $\mathcal{F} \subset 2^\Omega$ is the σ-algebra of events, and $P : \mathcal{F} \to [0,1]$ is a probability measure. We define two random fields: the viscosity field $\nu(x,\omega) : \overline{D} \times \Omega \to \mathbb{R}$ and the forcing field $f(t,x,\omega) : [0,T] \times \overline{D} \times \Omega \to \mathbb{R}^m$. The stochastic time-dependent incompressible Navier-Stokes problem can be written as follows: find a random velocity, $u : [0,T] \times \overline{D} \times \Omega \to \mathbb{R}^m$ and random pressure $p : [0,T] \times \overline{D} \times \Omega \to \mathbb{R}$, such that P-almost surely (a.s.) the following equations hold in $(0,T] \times D \times \Omega$:

$$\begin{aligned} \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = f, \\ \nabla \cdot u = 0, \end{aligned} \tag{2.1}$$

subject to the initial condition

$$u(0,x,\omega) = u_0(x), \quad \text{on } D \times \Omega,$$

and the boundary condition

$$u(t,x,\omega) = 0, \quad \text{on } (0,T] \times \partial D \times \Omega.$$

Let $\mathbf{y} = (y_1, ..., y_d)$ denote a d-dimensional random variable in $(\Omega, \mathcal{F}, P)$ and define the space $L^2_P(\Omega)$ comprising all random variables $\mathbf{y}$ satisfying

$$\sum_{n=1}^d \int_\Omega |y_n(\omega)|^2 dP(\omega) < \infty.$$

Then the following Hilbert spaces can be defined

$$\begin{aligned} V = L^2(0,T; H^1_0(D)) \otimes L^2_P(\Omega) \text{ with norm } \|u\|^2_V = \int_0^T \int_D \mathbb{E}[|\nabla u|^2] dx dt, \\ W = L^2(0,T; L^2_0(D)) \otimes L^2_P(\Omega) \text{ with norm } \|p\|^2_W = \int_0^T \int_D \mathbb{E}[|p|^2] dx dt. \end{aligned}$$

In order to define the weak form of the Navier-Stokes equations, introduce two continuous bilinear forms

$$\begin{aligned} a(u,v) &= 2\nu \sum_{i,j=1}^m \int_D D_{ij}(u) D_{ij}(v) dx, \quad \forall u, v \in H^1(D), \\ b(v,q) &= -\int_D q \nabla \cdot v dx, \quad \forall q \in L^2(D), v \in H^1(D), \end{aligned}$$

where $D_{ij}(v) = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$, and the continuous trilinear form

$$c(w;u,v) = \frac{1}{2}\sum_{i,j=1}^{m}\left(\int_D w_j\left(\frac{\partial u_i}{\partial x_j}\right)v_i dx - \int_D w_j\left(\frac{\partial v_i}{\partial x_j}\right)u_i dx\right), \quad \forall w,u,v \in H^1(D).$$

We now define weak solution of the problem (2.1) a pair $(u,p) \in V \times W$ which satisfies the initial condition $u(0,x,\omega) = u_0(x,\omega)$ and for $T > 0$

$$\begin{aligned} &\mathbb{E}[(\partial_t u, v)] + \mathbb{E}[a(u,v)] + \mathbb{E}[c(u;u,v)] + \mathbb{E}[b(v,p)] \\ &\qquad\qquad = \mathbb{E}[(f,v)], \text{ for all } v \in H_0^1(D) \otimes L_P^2(\Omega), \\ &\mathbb{E}[b(u,q)] = 0, \text{ for all } q \in L_0^2(D) \otimes L_P^2(\Omega). \end{aligned} \tag{2.2}$$

Recall that m is the dimension of domain D. In the forthcoming analysis, we will make use of the following inequalities

Poincaré inequality:

$$\|v\| \le C_P\|\nabla v\|, \quad \forall v \in L^2(D),$$

Ladyzhenskaya inequalities (see [17]): For any $u,v,w \in H_0^1(D)$,

$$\begin{aligned} (u\cdot\nabla v, w) &\le \sqrt{2}\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla v\|\|w\|^{1/2}\|\nabla w\|^{1/2} \;\; (\text{if } m=2), \\ (u\cdot\nabla v, w) &\le \frac{16}{27}\|u\|^{1/4}\|\nabla u\|^{3/4}\|\nabla v\|\|w\|^{1/4}\|\nabla w\|^{3/4} \;\; (\text{if } m=3), \end{aligned}$$

and a consequence of *Cauchy-Bunyakovsky-Schwarz inequality*:

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} c_i^2, \quad \forall a_i, b_i, c_i \in \mathbb{R}. \tag{2.3}$$

2.2. Finite dimensional noise assumption. In many problems the source of randomness can be approximated using just a small number of uncorrelated or independent random variables; take, for example, the case of a truncated Karhunen-Loève expansion, [19]. This motivates us to make the following assumption.

Assumption 1. *The random input functions of the equation* (2.1) *have the form*

$$\begin{aligned} \nu(x,\omega) &= \nu(x,\mathbf{y}(\omega)), \;\; \textit{on } \overline{D}\times\Omega, \\ f(t,x,\omega) &= f(t,x,\mathbf{y}(\omega)), \;\; \textit{on } [0,T]\times\overline{D}\times\Omega, \end{aligned}$$

where $\mathbf{y}(\omega) = (y_1(\omega),...,y_d(\omega))$ is a vector of real-valued random variables with mean value zero and unit variance.

We will denote with $\Gamma_n \equiv y_n(\Omega)$ the image of y_n, $\Gamma = \prod_{n=1}^d \Gamma_n$ and assume that the random variables $[y_1,...,y_d]$ have a joint probability density function $\rho : \Gamma \to \mathbb{R}_+$, with $\rho \in L^\infty(\Gamma)$. Hence the probability space $(\Omega, \mathcal{F}, P)$ can be replaced by $(\Gamma, \mathcal{B}^d, \rho d\mathbf{y})$, where $\mathcal{B}^d$ is the d-dimensional Borel space.

Similar to V and W, we can define V_ρ and W_ρ as

$$V_\rho = L^2(0,T;H_0^1(D)) \otimes L_\rho^2(\Gamma) \text{ with norm } \|u\|_{V_\rho}^2 = \int_\Gamma \|u\|_{L^2(0,T;H_0^1(D))}^2 \rho d\mathbf{y},$$
$$W_\rho = L^2(0,T;L_0^2(D)) \otimes L_\rho^2(\Gamma) \text{ with norm } \|p\|_{W_\rho}^2 = \int_\Gamma \|p\|_{L^2(0,T;L_0^2(D))}^2 \rho d\mathbf{y}.$$

After making Assumption 1, the solution (u,p) of the stochastic NSE (2.2) can be described by just a finite number of random variables, i.e., $u(\omega,x) = u(y_1(\omega),...,y_d(\omega),x)$, $p(\omega,x) = p(y_1(\omega),...,y_d(\omega),x)$. Thus, the goal is to approximate the functions $u = u(\mathbf{y},x)$ and $p = p(\mathbf{y},x)$, where $\mathbf{y} \in \Gamma$ and $x \in \overline{D}$. Observe that the stochastic variational formulation (2.2) has a "deterministic" equivalent which is the following: find $u \in V_\rho, p \in W_\rho$ satisfying the initial condition and, for $T > 0$

$$\begin{aligned} &\int_\Gamma \rho(\partial_t u, v) d\mathbf{y} + \int_\Gamma \rho a(u,v) d\mathbf{y} + \int_\Gamma \rho c(u;u,v) d\mathbf{y} + \int_\Gamma \rho b(v,p) d\mathbf{y} \\ &\qquad = \int_\Gamma \rho(f,v) d\mathbf{y}, \text{ for all } v \in H_0^1(D) \otimes L_\rho^2(\Gamma), \\ &\int_\Gamma \rho b(u,q) d\mathbf{y} = 0, \text{ for all } q \in L_0^2(D) \otimes L_\rho^2(\Gamma). \end{aligned} \tag{2.4}$$

For a fixed T, the solution has the form $u(\omega,x) = u(y_1(\omega),...,y_d(\omega),x)$, $p(\omega,x) = p(y_1(\omega),...,y_d(\omega),x)$ and we use the notation $u(\mathbf{y})$, $p(\mathbf{y})$, $\nu(\mathbf{y})$, $f(\mathbf{y})$, and $u_0(\mathbf{y})$ in order to emphasize the dependence on the variable $\mathbf{y}$. Then, the weak formulation (2.4) for $T > 0$ is equivalent to

$$\begin{aligned} &(\partial_t u(\mathbf{y}), v) + a(u(\mathbf{y}), v) + c(u(\mathbf{y}); u(\mathbf{y}), v) + b(v, p(\mathbf{y})) \\ &\qquad = (f(\mathbf{y}), v), \text{ for all } v \in H_0^1(D), \ \rho\text{-a.e. in } \Gamma, \\ &b(u(\mathbf{y}), q) = 0, \text{ for all } q \in L_0^2(D), \ \rho\text{-a.e. in } \Gamma. \end{aligned} \tag{2.5}$$

2.3. Collocation method. Denote the conforming velocity, pressure finite element spaces based on an edge to edge triangulation of D (with maximum triangle diameter h) by

$$H_h \subset H_0^1(D) \ , \ L_h \subset L_0^2(D).$$

We assume that H_h and L_h satisfy the usual discrete inf-sup condition. Taylor-Hood elements, discussed in [4], [8], are one commonly used choice of velocity-pressure finite element spaces. The discretely divergence free subspace of H_h is

$$V_h \ := \{v_h \in H_h : (\nabla \cdot v_h, q_h) = 0 \ , \forall q_h \in L_h\}.$$

The spatial discrete approximation of (2.5) can be written as: find $u_h \in L^2(0,T;H_h)\otimes L^2_\rho(\Gamma)$ and $p_h \in L^2(0,T;L_h)\otimes L^2_\rho(\Gamma)$ satisfying initial condition and for $T>0$

$$\begin{aligned}(\partial_t u_h(\mathbf{y}), v_h) + a(u_h(\mathbf{y}), v_h) + c(u_h(\mathbf{y}); u_h(\mathbf{y}), v_h) + b(v_h, p_h(\mathbf{y}))\\ = (f(\mathbf{y}), v_h),\ \text{for all } v_h \in H_h,\ \rho\text{-a.e. in } \Gamma,\\ b(u_h(\mathbf{y}), q_h) = 0,\ \text{for all } q_h \in L_h,\ \rho\text{-a.e. in } \Gamma,\end{aligned} \tag{2.6}$$

or more simply: find $u_h \in L^2(0,T;V_h)\otimes L^2_\rho(\Gamma)$ satisfying initial condition and for $T>0$

$$\begin{aligned}(\partial_t u_h(\mathbf{y}), v_h) + a(u_h(\mathbf{y}), v_h) + c(u_h(\mathbf{y}); u_h(\mathbf{y}), v_h) = (f(\mathbf{y}), v_h),\\ \text{for all } v_h \in V_h,\ \rho\text{-a.e. in } \Gamma,\end{aligned} \tag{2.7}$$

We apply the stochastic collocation method to the weak form (2.6). Define $\mathcal{P}_r(\Gamma) \subset L^2_\rho(\Gamma)$ as the span of tensor product polynomials with degree at most $r = (r_1, ..., r_d)$. (We avoid using the popular notation p for the polynomial degree, since it is already reserved for the pressure). The dimension of $\mathcal{P}_r(\Gamma)$ is $N_r = \prod_{n=1}^d (r_n+1)$. We seek a numerical approximation to the solution of (2.6) in finite dimensional subspaces $V_{\rho,h} = L^2(0,T;H_h)\otimes \mathcal{P}_r(\Gamma)$ and $W_{\rho,h} = L^2(0,T;L_h)\otimes \mathcal{P}_r(\Gamma)$.

The procedure for solving (2.6) is divided into two parts. First, for a fixed $T>0$, at each collocation point (root of orthogonal polynomials) $\mathbf{y}\in\Gamma$, construct an approximation $u_h(T,\cdot,\mathbf{y}) \in H_h(D)$ and $p_h(T,\cdot,\mathbf{y}) \in L_h(D)$ satisfying

$$\begin{aligned}(\partial_t u_h(\mathbf{y}), v_h) + a(u_h(\mathbf{y}), v_h) + c(u_h(\mathbf{y}); u_h(\mathbf{y}), v_h)\\ + b(v_h, p_h(\mathbf{y})) = (f(\mathbf{y}), v_h),\ \text{for all } v_h \in H_h(D),\\ b(u_h(\mathbf{y}), q_h) = 0,\ \text{for all } q_h \in L_h(D).\end{aligned} \tag{2.8}$$

Next, we collocate (2.8) on those points and build the discrete solutions $u_{h,r} \in H_h(D)\otimes \mathcal{P}_r(\Gamma)$ and $p_{h,r} \in L_h(D)\otimes\mathcal{P}_r(\Gamma)$ by interpolating in $\mathbf{y}$ the collocated solutions, i.e.,

$$\begin{aligned}u_{h,r}(T,x,\mathbf{y}) &= \mathcal{I}_r u_h(T,x,\mathbf{y})\\ &= \sum_{j_1=1}^{r_1+1}\cdots\sum_{j_d=1}^{r_d+1} u_h(T,x,y_{j_1},\cdots,y_{j_d})(l_{j_1}\otimes\cdots\otimes l_{j_d}),\end{aligned}$$

where, for example, the functions $\{l_{j_k}\}_{k=1}^d$ can be taken as Lagrange polynomials. Obviously, the above product requires $\prod_{n=1}^d (r_n+1)$ function evaluations.

Because the random input data depend on a finite number of independent random variables and we collocate the weak formulation (2.8) at the zeros of orthogonal polynomials, the solution $u_{h,r}$ becomes a solution of uncoupled deterministic problems as in a Monte Carlo simulation but with much fewer collocation points.

3. Error analysis of the time stepping BECE scheme. In this section, we carry out an error analysis for the fully discrete scheme (BECE) for the approximation of the stochastic Navier-Stokes equation (2.1) in 3 spatial dimensions. The 2-dimensional case should follow similarly. Let $N \in \mathbb{N}_+$ and consider the uniform partition of the time interval $[0,T]$

$$0 = t_0 < t_1 < \cdots < t_N = T$$

with $t_j = t_0 + j\Delta t$, $j = 0,1,...,N$, and the time step $\Delta t = T/N$. For discretizing system (2.8), we apply and study the convergence of the first order backward Euler scheme with a semi-implicit treatment for the nonlinear term

Algorithm 1. *Given* $j \in \{0,\dots,N-1\}$ *and* $u_h^j \in H_h$, $p_h^j \in L_h$, *find* $u_h^{j+1} \in H_h$, $p_h^{j+1} \in L_h$ *satisfying*

$$\begin{aligned} \left(\frac{u_h^{j+1}-u_h^j}{\Delta t}, v_h\right) + a(u_h^{j+1}, v_h) + c(u_h^j; u_h^{j+1}, v_h) + b(v_h, p_h^{j+1}) \\ = (f^{j+1}, v_h), \ \textit{for all } v_h \in H_h, \\ b(u_h^{j+1}, q_h) = 0, \textit{ for all } q_h \in L_h. \end{aligned} \tag{BECE}$$

We will investigate the analyticity of the solution u_h with respect to $\mathbf{y}$, which is critical to establish the interpolation error estimates. Introducing the weight function $\sigma(\mathbf{y}) = \prod_{n=1}^d \sigma_n(y_n) \le 1$, where

$$\sigma_n(y_n) = 1 \ \text{ if } \Gamma_n \text{ is bounded,}$$
$$\sigma_n(y_n) = e^{-\lambda_n |y_n|} \quad \text{for some } \lambda_n > 0 \text{ if } \Gamma_n \text{ is unbounded,}$$

and the function space

$$C^0_\sigma(\Gamma; V) \equiv \left\{ v : \Gamma \to V, \ v \text{ continuous in } \mathbf{y}, \max_{\mathbf{y}\in\Gamma} \|\sigma(\mathbf{y}) v(\mathbf{y})\|_V < +\infty \right\}.$$

We denote

$$\Gamma_n^* = \prod_{\substack{j=1\\ j\neq n}}^d \Gamma_j \qquad \text{and} \qquad \sigma_n^* = \prod_{\substack{j=1\\ j\neq n}}^d \sigma_j$$

with y_n^* being an arbitrary element of Γ_n^* and make the following assumption on ν and f.

Assumption 2. *In what follows we assume that*

- $f \in C^0_\sigma(\Gamma; L^2(0,T;L^2(D)))$,
- ν *is uniformly bounded away from zero,*
- $f(\mathbf{y})$ *and* $\nu(\mathbf{y})$ *are infinitely differentiable with respect to each component of* $\mathbf{y}$,
- *There exists* $\gamma_n < +\infty$ *such that*

$$\left\| \frac{\partial^\ell_{y_n} \nu(\mathbf{y})}{\nu(\mathbf{y})} \right\|_{L^\infty(D)} \le \gamma_n^\ell \ell!, \quad \frac{\left[\Delta t \sum_{j=0}^{N-1} \|\partial^\ell_{y_n} f^{j+1}(\mathbf{y})\|^2\right]^{1/2}}{1 + \|f(\mathbf{y})\|_{L^2(0,T;L^2(D))}} \le \gamma_n^\ell \ell!,$$

for every $\mathbf{y} \in \Gamma$, $1 \le n \le d$.

Under the finite dimensional noise assumption made in Section 2.2, Assumption 2 is fulfilled by wide classes of random fields $\nu(x,\omega)$ and $f(t,x,\omega)$, such as linear truncated Karhunen-Loève or truncated exponential expansion. Details can be found in [3].

The following theorem makes the core of our paper. We prove there exists an analytical extension of solution in a subregion of the complex plane, and in the same time, indicate that that extending region can decay rapidly.

Theorem 1. *Under Assumption 2, if the solution $u_h^J(y_n,\mathbf{y}_n^*,x)$ to* (BECE)*, as a function of y_n, satisfies $u_h^J:\Gamma_n\to C^0_{\sigma_n^*}(\Gamma_n^*;H_h(D))$, then for all n, $1\le n\le d$, there exists $\alpha_n>0$ only depending on $\gamma_n,\Delta t$ and system parameters such that for every $J\in\{1,\dots,N\}$, $u_h^J(y_n,\mathbf{y}_n^*,x)$ admits an analytic extension $u_h^J(z,\mathbf{y}_n^*,x)$ in the region of the complex plane*

$$\Sigma(\Gamma_n;\tau_{n,J})\equiv\{z\in\mathbb{C}\mid dist(z,\Gamma_n)\le\tau_{n,J}\},\tag{3.1}$$

with $0<\tau_{n,J}<1/\alpha_n^J$.

Proof. At every point $y\in\Gamma$, the ℓ-th derivative of u_h w.r.t. y_n satisfies

$$\begin{aligned}&\left(\frac{1}{\Delta t}\partial^\ell_{y_n}u_h^{j+1}(\mathbf{y})-\frac{1}{\Delta t}\partial^\ell_{y_n}u_h^{j}(\mathbf{y}),v_h\right)+(\partial^\ell_{y_n}(\nu(\mathbf{y})\nabla u_h^{j+1}(\mathbf{y})),\nabla v_h)\\&+\left(\partial^\ell_{y_n}(u_h^j(\mathbf{y})\cdot\nabla u_h^{j+1}(\mathbf{y})),v_h\right)=\left(\partial^\ell_{y_n}f^{j+1}(\mathbf{y}),v_h\right),\quad\forall v_h\in V_h(D),\end{aligned}\tag{3.2}$$

or equivalently

$$\begin{aligned}\frac{1}{\Delta t}(\partial^\ell_{y_n}u_h^{j+1}(\mathbf{y}),v_h)-\frac{1}{\Delta t}(\partial^\ell_{y_n}u_h^{j}(\mathbf{y}),v_h)+\sum_{m=0}^{\ell}\binom{\ell}{m}\left(\partial^m_{y_n}\nu(\mathbf{y})\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y}),\nabla v_h\right)\\+\sum_{m=0}^{\ell}\binom{\ell}{m}\left(\partial^m_{y_n}u_h^j(\mathbf{y})\cdot\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y}),v_h\right)=\left(\partial^\ell_{y_n}f^{j+1}(\mathbf{y}),v_h\right).\end{aligned}$$

This implies that

$$\begin{aligned}&\left(\nu(\mathbf{y})\nabla\partial^\ell_{y_n}u_h^{j+1}(\mathbf{y}),\nabla v_h\right)+\left(u_h^j(\mathbf{y})\cdot\nabla\partial^\ell_{y_n}u_h^{j+1}(\mathbf{y}),v_h\right)\\&\qquad+\frac{1}{\Delta t}\left(\partial^\ell_{y_n}u_h^{j+1}(\mathbf{y}),v_h\right)-\frac{1}{\Delta t}\left(\partial^\ell_{y_n}u_h^{j}(\mathbf{y}),v_h\right)=\left(\partial^\ell_{y_n}f^{j+1}(\mathbf{y}),v_h\right)\\&-\sum_{m=1}^{\ell}\binom{\ell}{m}\left[(\partial^m_{y_n}\nu(\mathbf{y})\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y}),\nabla v_h)+(\partial^m_{y_n}u_h^j(\mathbf{y})\cdot\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y}),v_h)\right]\end{aligned}$$

for all $v_h\in V_h(D)$. Choosing $v_h:=\partial^\ell_{y_n}u_h^{j+1}(\mathbf{y})$ and utilizing the divergence free condition and the skew adjoint properties of the nonlinear term we obtain

$$\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\|^2+\frac{1}{2\Delta t}\|\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\|^2-\frac{1}{2\Delta t}\|\partial^{\ell}_{y_n}u_h^{j}(\mathbf{y})\|^2 \tag{3.3}$$
$$+\frac{1}{2\Delta t}\|\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})-\partial^{\ell}_{y_n}u_h^{j}(\mathbf{y})\|^2$$
$$=\Big(\partial^{\ell}_{y_n}f^{j+1}(\mathbf{y}),\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\Big)-\sum_{m=1}^{\ell}\binom{\ell}{m}\Big[\Big(\partial^{m}_{y_n}u_h^{j}(\mathbf{y})\cdot\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y}),\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\Big)$$
$$+\Big(\frac{\partial^{m}_{y_n}\nu(\mathbf{y})}{\nu(\mathbf{y})}\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y}),\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\Big)\Big].$$

Denoting the left hand side of (3.3) by LHS, applying Poincaré inequality and Ladyzhenskaya inequality, we get

$$LHS\le\frac{C_P}{\sqrt{\nu_{\min}}}\|\partial^{\ell}_{y_n}f^{j+1}(\mathbf{y})\|\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\| \tag{3.4}$$
$$+\sum_{m=1}^{\ell}\binom{\ell}{m}\left\|\frac{\partial^{m}_{y_n}\nu(\mathbf{y})}{\nu(\mathbf{y})}\right\|_{L^\infty(D)}\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y})\|\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\|$$
$$+\frac{16}{27\nu_{\min}^{7/8}}\sum_{m=1}^{\ell}\binom{\ell}{m}\left\|\partial^{m}_{y_n}u_h^{j}(\mathbf{y})\right\|^{1/4}\left\|\nabla\partial^{m}_{y_n}u_h^{j}(\mathbf{y})\right\|^{3/4}\left\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\right\|\times$$
$$\times\left\|\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y})\right\|^{1/4}\left\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y})\right\|^{3/4}$$
$$\le\frac{C_P}{\sqrt{\nu_{\min}}}\|\partial^{\ell}_{y_n}f^{j+1}(\mathbf{y})\|\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\|$$
$$+\sum_{m=1}^{\ell}\binom{\ell}{m}\left\|\frac{\partial^{m}_{y_n}\nu(\mathbf{y})}{\nu(\mathbf{y})}\right\|_{L^\infty(D)}\|\sqrt{\nu(y)}\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y})\|\|\sqrt{\nu(y)}\nabla\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\|$$
$$+\frac{16}{27}\frac{C_P^{1/2}}{\nu_{\min}^{3/2}}\sum_{m=1}^{\ell}\binom{\ell}{m}\left\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{m}_{y_n}u_h^{j}(\mathbf{y})\right\|\left\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell-m}_{y_n}u_h^{j+1}(\mathbf{y})\right\|\left\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{\ell}_{y_n}u_h^{j+1}(\mathbf{y})\right\|.$$

For $m\ge 0, J\ge 1$, we define

$$F_m^J=\left[\Delta t\sum_{j=0}^{J-1}\|\partial^{m}_{y_n}f^{j+1}(\mathbf{y})\|^2\right]^{1/2}\quad\text{and}\quad R_m^J=\left[\Delta t\sum_{j=0}^{J-1}\|\sqrt{\nu(\mathbf{y})}\nabla\partial^{m}_{y_n}u_h^{j+1}(\mathbf{y})\|^2\right]^{1/2}.$$

Multiplying (3.4) by $2\Delta t$, summing from $j=0$ to $J-1$ for any $J\ge 2$, $\ell\ge 1$ and applying

(2.3), it gives

$$\|\partial_{y_n}^{\ell} u_h^J\|^2 + 2(R_\ell^J)^2 \le \frac{2C_P}{\sqrt{\nu_{\min}}} F_\ell^J R_\ell^J + 2\sum_{m=1}^{\ell} \binom{\ell}{m} \left\| \frac{\partial_{y_n}^m \nu(\mathbf{y})}{\nu(\mathbf{y})} \right\|_{L^\infty(D)} R_{\ell-m}^J R_\ell^J + \frac{32}{27} \frac{C_P^{1/2}}{\Delta t^{1/2} \nu_{\min}^{3/2}} \sum_{m=1}^{\ell} \binom{\ell}{m} R_m^{J-1} R_{\ell-m}^J R_\ell^J. \tag{3.5}$$

For $J = 1$, $\ell \ge 1$, utilizing the fact that u_h^0 is independent of $\mathbf{y}$, there follows

$$\|\partial_{y_n}^{\ell} u_h^J\|^2 + 2(R_\ell^J)^2 \le \frac{2C_P}{\sqrt{\nu_{\min}}} F_\ell^J R_\ell^J + 2\sum_{m=1}^{\ell} \binom{\ell}{m} \left\| \frac{\partial_{y_n}^m \nu(\mathbf{y})}{\nu(\mathbf{y})} \right\|_{L^\infty(D)} R_{\ell-m}^J R_\ell^J.$$

Therefore, assigning $R_m^0 = 0$, $\forall m \ge 1$, (3.5) holds for every $J \ge 1$, $\ell \ge 1$. This leads us to

$$\frac{\|\partial_{y_n}^{\ell} u_h^J\|^2}{2R_\ell^J} + R_\ell^J \le \frac{C_P}{\sqrt{\nu_{\min}}} F_\ell^J + \sum_{m=1}^{\ell} \binom{\ell}{m} \left\| \frac{\partial_{y_n}^m \nu(\mathbf{y})}{\nu(\mathbf{y})} \right\|_{L^\infty(D)} R_{\ell-m}^J + \frac{16}{27} \frac{C_P^{1/2}}{\Delta t^{1/2} \nu_{\min}^{3/2}} \sum_{m=1}^{\ell} \binom{\ell}{m} R_m^{J-1} R_{\ell-m}^J, \quad \forall J \ge 1,\ \ell \ge 1.$$

Dividing both sides by $\ell!$ and denoting $S_m^J = \frac{R_m^J}{m!}$, using Assumption 2 we get

$$\frac{\|\partial_{y_n}^{\ell} u_h^J\|^2}{2R_\ell^J \ell!} + S_\ell^J \le \frac{C_P}{\sqrt{\nu_{\min}}} \gamma_n^\ell (1 + \|f\|) + \sum_{m=1}^{\ell} \gamma_n^m S_{\ell-m}^J + \frac{16}{27} \frac{C_P^{1/2}}{\Delta t^{1/2} \nu_{\min}^{3/2}} \sum_{m=1}^{\ell} S_m^{J-1} S_{\ell-m}^J, \quad \forall J \ge 1,\ \ell \ge 1. \tag{3.6}$$

Denoting two parameters independent of J, ℓ and u_h

$$A = \frac{C_P}{\sqrt{\nu_{\min}}}(1 + \|f\|), \quad \text{and} \quad B = \frac{16}{27} \frac{C_P^{1/2}}{\Delta t^{1/2} \nu_{\min}^{3/2}}.$$

We will prove that there exists $\alpha_n > 0$ independent of J, ℓ and u_h such that

$$S_\ell^J \le \alpha_n^{\ell J} \text{ for all } 1 \le \ell < \infty,\ 1 \le J \le N. \tag{3.7}$$

First, we consider three specific cases:

1. $\ell = 0$: Take $v = u_h^{j+1}(\mathbf{y})$ in (BECE), we have

$$\frac{1}{2\Delta t}\|u_h^{j+1}(\mathbf{y})\|^2 - \frac{1}{2\Delta t}\|u_h^j(\mathbf{y})\|^2 + \|\sqrt{\nu(\mathbf{y})}\nabla u_h^{j+1}(\mathbf{y})\|^2 \le \frac{C_P^2}{2\nu_{\min}}\|f^{j+1}(\mathbf{y})\|^2 + \frac{1}{2}\|\sqrt{\nu(\mathbf{y})}\nabla u_h^{j+1}(\mathbf{y})\|^2$$

Summing from $j=0$ to $J-1$ and multiplying by $2\Delta t$, there follows

$$\begin{aligned}\|u_h^J(\mathbf{y})\|^2 + \Delta t \sum_{j=0}^{J-1} \|\sqrt{\nu(\mathbf{y})}\nabla u_h^{j+1}(\mathbf{y})\|^2 \\ \le \frac{C_P^2}{\nu_{\min}} \Delta t \sum_{j=0}^{J-1} \|f^{j+1}(\mathbf{y})\|^2 + \|u_h^0\|^2,\end{aligned} \tag{3.8}$$

which gives $S_0^J \le \xi_0, \forall\, 1 \le J \le N$, where $\xi_0 = \frac{C_P}{\sqrt{\nu_{\min}}}(1+\|f\|) + \|u_h^0\|$.

2. $\ell = 1$: From (3.6) and Case 1, we have

$$S_1^J \le A\gamma_n + \gamma_n S_0^J + B S_1^{J-1} S_0^J \le A\gamma_n + \gamma_n \xi_0 + B\xi_0 S_1^{J-1}.$$

   There follows

$$\begin{aligned} S_1^J &\le (A\gamma_n + \gamma_n\xi_0) \sum_{j=0}^{J-1} (B\xi_0)^j \le (A\gamma_n + \gamma_n\xi_0) \frac{(B\xi_0)^J - 1}{B\xi_0 - 1} \\ &\le \left[\frac{B\xi_0}{B\xi_0 - 1}(A\gamma_n + \gamma_n\xi_0)\right] \cdot (B\xi_0)^{J-1} \le \alpha_n^J, \forall 1 \le J \le N, \end{aligned}$$

   with $\alpha_n = \max\left\{B\xi_0, \frac{B\xi_0}{B\xi_0 - 1}(A\gamma_n + \gamma_n\xi_0)\right\}$.

3. $J = 1$: From (3.6) and Case 1, recall that $S_m^0 = 0$, $\forall m \ge 1$, we have

$$S_\ell^1 \le A\gamma_n^\ell + \sum_{m=1}^{\ell} \gamma_n^m S_{\ell-m}^1 = A\gamma_n^\ell + \xi_0\gamma_n^\ell + \sum_{m=1}^{\ell-1} \gamma_n^m S_{\ell-m}^1.$$

   By induction, we will prove that $S_\ell^1 \le \alpha_n^\ell$, $\forall \ell \ge 1, \ell < \infty$. Assuming in addition $\alpha_n \ge \max\{4\gamma_n, 2(A+\xi_0)\gamma_n\}$, it gives

$$\begin{aligned} S_\ell^1 &\le \min\left\{\frac{A+\xi_0}{2^\ell(A+\xi_0)^\ell}, \frac{A+\xi_0}{4^\ell}\right\}\alpha_n^\ell + \sum_{m=1}^{\ell-1} \frac{\alpha_n^m}{4^m}\alpha_n^{\ell-m} \\ &\le \frac{1}{2}\alpha_n^\ell + \alpha_n^\ell \sum_{m=1}^{\ell-1} \frac{1}{4^m} < \alpha_n^\ell. \end{aligned}$$

Next, suppose that (3.7) occurs for all $J \le N$ with $\ell \le L-1$ and all $J \le M-1$ with $\ell = L$ ($L \ge 2$, $M \ge 2$), we will prove that (3.7) also occurs for $J = M$, $\ell = L$. Indeed, from (3.6),

$$S_L^M \le A\gamma_n^L + \sum_{m=1}^{L} \gamma_n^m S_{L-m}^M + B\sum_{m=1}^{L} S_m^{M-1} S_{L-m}^M.$$

By induction hypothesis, we have

$$
\begin{aligned}
S_L^M &\leq A\frac{\alpha_n^L}{4^L} + \xi_0\frac{\alpha_n^L}{4^L} + \sum_{m=1}^{L-1}\frac{\alpha_n^m}{4^m}\alpha_n^{ML-mM} + B\xi_0\alpha_n^{ML-L} + B\sum_{m=1}^{L-1}\alpha_n^{ML-m} \\
&\leq (A+\xi_0+B\xi_0)\alpha_n^{ML-1} + \alpha_n^{ML-1}\sum_{m=1}^{L-1}\frac{1}{4^m} + B\alpha_n^{ML-1}\frac{\alpha_n}{\alpha_n-1} \\
&\leq \left(A+\xi_0+B\xi_0+\frac{1}{3}+2B\right)\alpha_n^{ML-1} \text{ (assuming in addition that } \alpha_n \geq 2) \\
&\leq \alpha_n^{ML} \text{ (assuming in addition that } \alpha_n \geq (A+\xi_0+B\xi_0+\frac{1}{3}+2B))
\end{aligned}
$$

and (3.7) is proved completely.

Back to (3.6), it gives

$$
\frac{\|\partial_{y_n}^\ell u_h^J\|^2}{2R_\ell^J \ell!} + S_\ell^J \leq A\gamma_n^\ell + \sum_{m=1}^{\ell}\gamma_n^m S_{\ell-m}^J + B\sum_{m=1}^{\ell} S_m^{J-1}S_{\ell-m}^J
$$

for all ℓ. Employing the above estimation, we get

$$
\frac{\|\partial_{y_n}^\ell u_h^J\|^2}{2R_\ell^J \ell!} \leq \alpha_n^{\ell J}, \; \forall\, 1 \leq \ell < \infty, \; 1 \leq J \leq N,
$$

and there holds

$$
\|\partial_{y_n}^\ell u_h^J\| \leq \sqrt{2}(\ell!)\alpha_n^{\ell J}.
$$

We now define for every $y_n \in \Gamma_n$ the power series $u_h^J : \mathbb{C} \to C^0(\Gamma_n^*; H_h(D))$ as

$$
u_h^J(z, \mathbf{y}_n^*, x) = \sum_{\ell=0}^{\infty}\frac{(z-y_n)^\ell}{\ell!}\partial_{y_n}^\ell u_h^J(y_n, \mathbf{y}_n^*, x).
$$

Hence,

$$
\begin{aligned}
&\sigma_n(y_n)\|u_h^J(z)\|_{C^0_{\sigma_n^*}(\Gamma_n^*, H_h(D))} \leq \sum_{\ell=0}^{\infty}\frac{|z-y_n|^\ell}{\ell!}\sigma_n(y_n)\|\partial_{y_n}^\ell u_h^J(y_n)\|_{C^0_{\sigma_n^*}(\Gamma_n^*, H_h(D))} \\
&\leq \|u_h^J\|_{C^0_\sigma(\Gamma; H_h(D))}\sum_{\ell=0}^{\infty}(|z-y_n|\alpha_n^J)^\ell, \\
&\leq \left(\left\|\frac{C_P}{\sqrt{\nu_{\min}}} + \|u_h^0\|\right\|_{C^0_\sigma(\Gamma;\mathbb{R})} + \left\|\frac{C_P}{\sqrt{\nu_{\min}}}f\right\|_{C^0_\sigma(\Gamma; L^2(0,T;L^2(D)))}\right)\sum_{\ell=0}^{\infty}(|z-y_n|\alpha_n^J)^\ell \\
&\hspace{25em} \text{(from (3.8))}.
\end{aligned}
$$

The series converges for all $z \in \mathbb{C}$ satisfying $|z - y_n| \leq \tau_{n,J} < 1/\alpha_n^J$ and the function u_h^J

admits an analytical extension in the region $\Sigma(\Gamma_n, \tau_{n,J})$. □

With the analyticity result proved in Theorem 1, we proceed to estimate the interpolation error $\varepsilon = u_h - u_{h,r}$. The proof follows the same procedure as in [3], and thus, is omitted here.

Theorem 2. *Under the assumption of Theorem 1, suppose that the joint probability density ρ satisfies*

$$\rho(\mathbf{y}) \leq C_M e^{-\sum_{n=1}^{d}(\delta_n y_n)^2} \qquad \forall \mathbf{y} \in \Gamma$$

for some constant $C_M > 0$ and δ_n strictly positive if Γ_n is unbounded and zero otherwise. Then, for any integer $J \in \{1, \ldots, N\}$, there exists a positive constant C independent of h and r such that

$$\|u_h^J - u_{h,r}^J\|_{L^2(D)\otimes L^2_\rho(\Gamma)} \leq C \sum_{n=1}^{d} \beta_n(r_n) \exp(-R_{n,J} r_n^{\theta_n}), \tag{3.9}$$

where

$$\theta_n = \beta_n = 1 \text{ and } R_{n,J} = \log\left[\frac{2\tau_{n,J}}{|\Gamma_n|}\left(1 + \sqrt{1 + \frac{|\Gamma_n|^2}{4\tau_{n,J}^2}}\right)\right] \text{ if } \Gamma_n \text{ is bounded}$$

and

$$\theta_n = \frac{1}{2},\ \beta_n = O(\sqrt{r_n}) \text{ and } R_{n,J} = \tau_{n,J}\delta_n \text{ if } \Gamma_n \text{ is unbounded,}$$

where $\tau_{n,J}$ is the minimum distance between Γ_n and the nearest singularity in the complex plane, as defined in Theorem 1.

In short, Theorem 2 implies that:

- At a fixed time step J, the convergence rate of SCMs for bounded random variables is $O(\exp(-r))$ and for unbounded random variables is $O(\exp(-\sqrt{r}))$,
- $\tau_{n,J}$ converges to 0 as J increases, as indicated in Theorem 1, and so does $R_{n,J}$. Thus, at a fixed polynomial degree r, the interpolation error $u_h^J - u_{h,r}^J$ becomes $O(1)$ as $J \to \infty$.

4. Error analysis of the fixed-point iteration method for steady Navier-Stokes approximation. We proceed to give an error estimation of the fixed-point method for stationary NSEs. This can be obtained by slightly modifying the analysis in the last section. Assuming ν and f to be random fields, the finite element approximation of steady NSEs is given by: find $u_h \in H_h \otimes L^2_\rho(\Gamma)$ and $p_h \in L_h \otimes L^2_\rho(\Gamma)$ satisfying

$$\begin{aligned} a(u_h(\mathbf{y}), v_h) + c(u_h(\mathbf{y}); u_h(\mathbf{y}), v_h) + b(v_h, p_h(\mathbf{y})) \\ = (f(\mathbf{y}), v_h),\ \text{for all } v_h \in H_h,\ \rho\text{-a.e. in } \Gamma, \\ b(u_h(\mathbf{y}), q_h) = 0,\ \text{for all } q_h \in L_h,\ \rho\text{-a.e. in } \Gamma, \end{aligned} \tag{4.1}$$

Fixed point iteration is a commonly used method to solve nonlinear systems. Applying to (4.1), it reads:

Algorithm 2. *Let $N \geq 1$ be the maximum number of iterations and $0 \leq j \leq N-1$ be the current loop. Given $u_h^j \in H_h$, $p_h^j \in L_h$, find $u_h^{j+1} \in H_h$, $p_h^{j+1} \in L_h$ satisfying*

$$\begin{aligned} a(u_h^{j+1}, v_h) + c(u_h^j; u_h^{j+1}, v_h) + b(v_h, p_h^{j+1}) &= (f, v_h), \qquad \forall v_h \in H_h, \\ b(u_h^{j+1}, q_h) &= 0, \qquad \forall q_h \in L_h. \end{aligned} \tag{FP}$$

Here, at each collocation point, we start the process with some initial guess value u_h^0, which is deterministic. At each step, using the current intermediate solution (u_h^j, p_h^j) and formula (FP), the next (and hopefully better) approximation (u_h^{j+1}, p_h^{j+1}) is computed. The process will continue until a sufficiently accurate solution (u_h^J, p_h^J) (in physical error meaning) or the maximum number N is reached. If the method is convergent, in many cases, quite few iteration steps are needed.

Like the analysis for (BECE) above, we need some restrictive assumptions on the regularity of ν and f. Note that now f is a steady-state function.

Assumption 3. *In what follows we assume that*

- $f \in C^0_\sigma(\Gamma; L^2(D))$,
- *ν is uniformly bounded away from zero,*
- *$f(\mathbf{y})$ and $\nu(\mathbf{y})$ are infinitely differentiable with respect to each component of $\mathbf{y}$,*
- *There exists $\gamma_n < +\infty$ such that*

$$\left\| \frac{\partial^\ell_{y_n} \nu(\mathbf{y})}{\nu(\mathbf{y})} \right\|_{L^\infty(D)} \leq \gamma_n^\ell \ell!, \quad \frac{\|\partial^\ell_{y_n} f(\mathbf{y})\|}{1 + \|f(\mathbf{y})\|} \leq \gamma_n^\ell \ell!,$$

for every $\mathbf{y} \in \Gamma$, $1 \leq n \leq d$.

Let $J \geq 0$, recall that u_h^J is the solution to (FP) after J iteration. The analyticity result for u_h is presented in the following theorem. We prove there exists an analytical extension of solution in a subregion of the complex plane, and in the same time, indicate that that extending region can decay rapidly with the number of *iteration steps.*

Theorem 3. *Under Assumption 3, if $u_h^J(y_n, \mathbf{y}_n^*, x)$, as a function of y_n, satisfies $u_h^J : \Gamma_n \to C^0_{\sigma_n^*}(\Gamma_n^*; H_h(D))$, then for all n, $1 \leq n \leq d$, there exists $\alpha_n > 0$ only depending on γ_n and system parameters such that $u_h^J(y_n, \mathbf{y}_n^*, x)$ admits an analytic extension $u_h^J(z, \mathbf{y}_n^*, x)$ in the region of the complex plane*

$$\Sigma(\Gamma_n; \tau_{n,J}) \equiv \{z \in \mathbb{C} \mid dist(z, \Gamma_n) \leq \tau_{n,J}\}, \tag{4.2}$$

with $0 < \tau_{n,J} < 1/\alpha_n^J$.

Proof. The proof follows the argument in that of Theorem 1 closely, so we will only give a brief sketch here. At every point $y \in \Gamma$, take ℓ-th derivative of u_h^J w.r.t. y_n and choosing

$v_h = \partial_{y_n}^\ell u_h^J$, it gives

$$\begin{aligned} \|\sqrt{\nu(\mathbf{y})}\nabla\partial_{y_n}^\ell u_h^J(\mathbf{y})\| &\leq \frac{C_P}{\sqrt{\nu_{\min}}}\|\partial_{y_n}^\ell f^J(\mathbf{y})\| \\ &+ \sum_{m=1}^{\ell}\binom{\ell}{m}\left\|\frac{\partial_{y_n}^m\nu(\mathbf{y})}{\nu(\mathbf{y})}\right\|_{L^\infty(D)}\|\sqrt{\nu(y)}\nabla\partial_{y_n}^{\ell-m}u_h^J(\mathbf{y})\| \\ + \frac{16}{27}\frac{C_P^{1/2}}{\nu_{\min}^{3/2}}\sum_{m=1}^{\ell}\binom{\ell}{m}&\left\|\sqrt{\nu(\mathbf{y})}\nabla\partial_{y_n}^m u_h^{J-1}(\mathbf{y})\right\|\left\|\sqrt{\nu(\mathbf{y})}\nabla\partial_{y_n}^{\ell-m}u_h^J(\mathbf{y})\right\|, \quad \forall\ell\geq 1, J\geq 1. \end{aligned}$$

For $m \geq 0$, $J \geq 0$, defining $S_m^J = \|\sqrt{\nu(\mathbf{y})}\nabla\partial_{y_n}^m u_h^J(\mathbf{y})\|/m!$, using Assumption 3 we get

$$\begin{aligned} S_\ell^J \leq \frac{C_P}{\sqrt{\nu_{\min}}}\gamma_n^\ell(1+\|f\|) + \sum_{m=1}^{\ell}\gamma_n^m S_{\ell-m}^J \\ + \frac{16}{27}\frac{C_P^{1/2}}{\nu_{\min}^{3/2}}\sum_{m=1}^{\ell}S_m^{J-1}S_{\ell-m}^J, \quad \forall\ell\geq 1,\ J\geq 1. \end{aligned}$$

Similar to Theorem 1, we can construct $\alpha_n > 0$ depending on γ_n and system parameters such that $S_\ell^J \leq \alpha_n^{\ell J}$ for all $\ell \geq 1$, $J \geq 1$.

Applying Poincaré inequality, there holds

$$\|\partial_{y_n}^\ell u_h^J\| \leq \frac{C_P}{\sqrt{\nu_{\min}}}(\ell!)\alpha_n^{\ell J}, \quad \forall\ell\geq 1,\ J\geq 1.$$

The function u_h^J thus admits an analytical extension in the region $\Sigma(\Gamma_n, \tau_{n,J})$ with $\tau_{n,J} < 1/\alpha_n^J$. □

Theorem 3 leads to an estimation for interpolation error $u_h - u_{h,r}$, which is essentially identical to Theorem 2, and we do not restate here. It guarantees the exponential convergence of SCMs for the fixed point method for steady NSE approximations. The upper bound certainly grows with the number of iteration steps; however, in many applications, this is just a small number, making this bound realistic. Still, the fact that more iteration steps could lead to less accurate approximations in probability space seems to be counterintuitive. Whether a sharper estimation exists is an open question and worths further study.

5. Numerical examples. To illustrate the decrease of accuracy of SCMs in long time, in this section, we present a computational experiment of the two-dimensional flow around a circular cylinder, based on the well-known benchmark problem from Shäfer and Turek [31]. There is plenty of computations of stochastic flows in the literature, demonstrating both the fast convergence as well as the long-term growth of interpolation error of SCMs and polynomial chaos, see, e.g., [6, 10, 20, 21, 29, 37]. The problem we consider here was studied in [7, 35] with noisy boundary conditions. We revisit this problem with uncertain viscosity. Our test is programmed using the software package *FreeFem++* [12].

Let D be the channel with the cylinder presented in Figure 1. We consider the time-

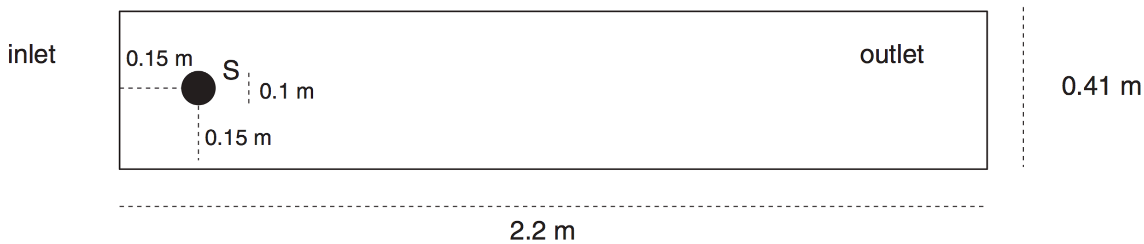

Figure 1: Domain D of the numerical test, [16].

dependent incompressible Navier-Stokes equation (2.1) subject to the following random viscosity

$$\nu = \nu_0(1 + Y/10),$$

where $\nu_0 = 0.8 \times 10^{-3}$ and Y is a uniform random variable of zero mean and unit variance. The cylinder, top and bottom of the channel are prescribed no-slip boundary conditions, and the inflow and outflow profiles are

$$\begin{aligned} u_1(0, y) &= u_1(2.2, y) = \frac{6}{0.41^2} y(0.41 - y), \\ u_2(0, y) &= u_2(2.2, y) = 0. \end{aligned}$$

Due to the randomness of the viscosity, the Reynolds number considered in this test is random. Based on the inflow velocity and the diameter of the cylinder $L = 0.1$, it satisfies $112.5 \leq Re \leq 137.5$. For this range of Reynolds number, the flow is in the laminar regime with a Kármán vortex street developing behind the cylinder. This results in a periodic response of the lift and drag coefficients. In what follows, we investigate the mean and error evolution of these two quantities simulated by SCM. By $C_{h,r}^{l|d}$, we denote the mean lift/drag coefficients corresponding to the fully discrete solutions in physical and probability spaces. The collocation points are Clenshaw-Curtis quadrature. In order to estimate the error, we compute a very high resolution approximate solution using SCM of 20^{th}-order and suppose it to be the "true" solution in probability space, the lift/drag coefficients corresponding to which are denoted by $C_h^{l|d}$.

The experiment is carried out up to $T = 50$, with zero forcing term and initial condition. The solutions are computed with Taylor-Hood elements on a triangular mesh providing 69174 total DOFs, refined near the cylinder, and time step $\Delta t = 0.005$. Since (BECE) cannot produce reliable space-time solutions for this flow, at each collocation point of Y, we employ the Crank-Nicolson time stepping scheme to solve the Navier-Stokes equations. The scheme reads:

Algorithm 3. *Given* $0 \le j \le N-1$ *and* $u_h^j \in H_h$, $p_h^j \in L_h$, *find* $u_h^{j+1} \in H_h$, $p_h^{j+1} \in L_h$ *satisfying*

$$\left(\frac{u_h^{j+1} - u_h^j}{\Delta t}, v_h\right) + \frac{1}{2}(a(u_h^{j+1}, v_h) + a(u_h^j, v_h)) + b(v_h, p_h^{j+1})$$
$$+\frac{1}{2}(c(u_h^{j+1}; u_h^{j+1}, v_h) + c(u_h^j; u_h^j, v_h)) = (f^{j+1/2}, v_h), \text{ for all } v_h \in H_h,$$
$$b(u_h^{j+1}, q_h) = 0, \text{ for all } q_h \in L_h.$$

Fixed point iterations are applied to solve the nonlinear system with a $\|u_{(i+1)} - u_{(i)}\|_{H_1(D)} < 10^{-10}$ as a stopping criterion. The numerical methods and space-time resolution we use herein were verified in deterministic problem to give an accurate approximation of lift and drag coefficients, [16].

Figure 2 show the instantaneous mean of lift and drag coefficients given by SCM. It can be seen that both the mean $C_{h,r}^l$ and $C_{h,r}^d$ oscillate periodically around a constant value. After a short time agreeing with the reference solution, SCM starts to lose accuracy. The higher order the method is, the later the divergence occurs. The evolution of errors of SCM at various polynomial orders r are shown in Figure 3. We observe that for all r plotted, the interpolation errors eventually increase to $O(1)$ with time. This confirms that the efficiency of SCM diminishes in long time flow simulations.

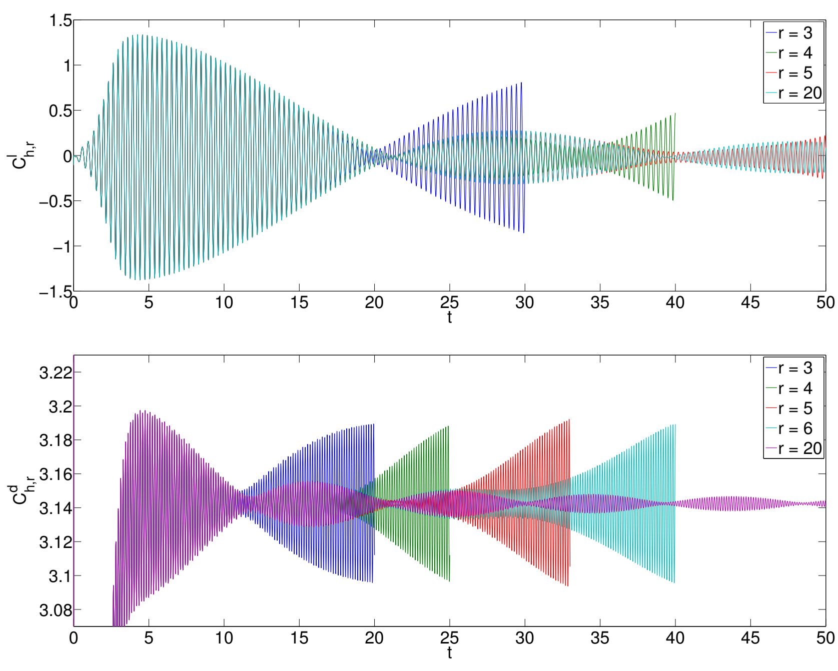

Figure 2: Evolution of mean of lift (upper) and drag (lower) coefficient.

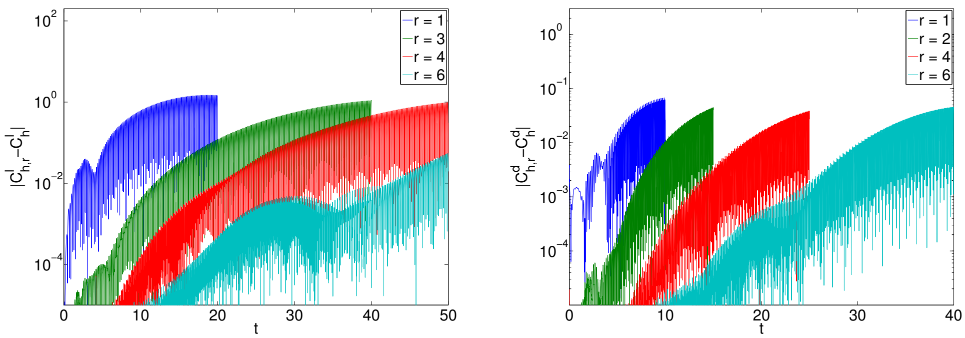

Figure 3: Evolution of error of lift (left) and drag (right) coefficient.

6. Conclusions. In this paper, error analysis of stochastic collocation methods for fully discrete Navier-Stokes approximations with random input data was carried out. Particularly, we considered the backward Euler with constant extrapolation scheme for time-dependent NSEs and fixed point iteration for steady NSEs. We proved the exponential convergence of the methods in the probability space. On the other side, at a fixed polynomial order, our analysis indicated that the interpolation error may grow to $O(1)$ in long term. A numerical example of 2D flow around a cylinder is given to illustrate our results.

The prospect of applying and improving SCMs to flow simulations essentially relies on understanding where SCMs produce reliable results and where they do not. On one hand, further analytical studies are desired to enhance this understanding. They include possible sharper estimations on the interpolation errors and the growth of the upper bounds, as well as convergence analysis for higher order semi-implicit and fully implicit discretization schemes. Simultaneously, as existing experiments are limited on flows at laminar or transitional regimes, it is unclear how fast the probability errors grow in turbulent flow simulations in practice. A computational demonstration of the performance of SCMs in such cases is an interesting question and extremely helpful.

REFERENCES

[1] G. Apte, S. Constantinescu, F. Ham, G. Iaccarino, P. Mahesh, and K. Moin, *Large-eddy simulation of reacting turbulent flows in complex geometries*, J. Appl. Mech., 73:364-371, 2006.

[2] G.A. Baker, V. Dougalis, and O. Karakashian, *On a higher order accurate, fully discrete galerkin approximation to the Navier-Stokes equations*, Math. Comp., 39:339-375, 1982.

[3] I. Babuska, F. Nobile, and R. Tempone, *A Stochastic Collocation Method for Elliptic Partial Differential Equations with Random Input Data*, SIAM Journal on Numerical Analysis, 45 (2007), pp. 1005-1034.

[4] S. Brenner and R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 3rd edition, 2008.

[5] T. Dupont, G. Fairweather and J. P. Johnson, *Three-level Galerkin methods for parabolic equations*, SIAM J. Numer. Anal., v.11, 1974, pp. 392-410.

[6] H. Elman, Q. Liao, *Reduced basis collocation methods for partial differential equations with random coefficients*, SIAM/ASA J. Uncertainty Quantification, v.1, pp. 192-217, 2013.

[7] J. Foo, X. Wan, and G. Karniadakis, *The multi-element probabilistic collocation method (ME-PCM): Error analysis and applications*, Journal of Computational Physics, 227 (2008), pp. 9572-9595.

[8] M.D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows - A Guide to Theory*, Practices, and Algorithms, Academic Press, 1989.

[9] V. Girault and P.A. Raviart, *Finite Element Approximations of the Navier-Stokes Equations*, Lecture Notes in Mathematics. Springer-Verlag, New York, 1979.

[10] B. Ganapathysubramanian, N. Zabaras, *Sparse grid collocation schemes for stochastic natural convection problems*, J. Comput. Phys. 225(1): 652-685, 2007.

[11] M. Gunzburger, C. Webster, G. Zhang, *An Adaptive Wavelet Stochastic Collocation Method for Irregular Solutions of Partial Differential Equations with Random Input Data*, Springer Lecture Notes on Computational Science, to appear.

[12] F. Hecht, *New development in freefem++*, J. Numer. Math. 20 (2012), no. 3-4, 251-265.

[13] J.G Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem, IV. Error analysis for second-order time discretization*, SIAM J. Numer. Anal., 27(2):353-384, 1990.

[14] Y. He and W. Sun, *Stability and convergence of the Crank-Nicolson/Adams-Bashforth scheme for the time-dependent Navier-Stokes equations*, SIAM J. Numer. Anal., 45:837-869, Feb. 2007.

[15] R. Ingram, *Unconditional convergence of high-order extrapolations of the Crank-Nicolson, finite element methods for the Navier-Stokes equations*, IJNAM, 10(2), pp. 257-297, 2013.

[16] V. John, *Reference values for drag and lift of a two-dimensional time-dependent flow around a cylinder*, Int. J. Numer. Meth. Fluids 44 (2004) 777-788.

[17] O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[18] W. Layton, *Introduction to the Numerical Analysis of Incompressible Viscous Flows*, Computational Science and Engineering Series, SIAM, 2008.

[19] M. Loève, *Probability Theory I*, Grad. Texts in Math. 45, Springer-Verlag, New York, 1987.

[20] G. Loeven, *Efficient uncertainty quantification in computational fluid dynamics*, Ph.D. thesis, TU Delft, 2010.

[21] D. Lucor, *Generalized Polynomial Chaos: Applications to Random Oscillators and Flow-structure Interactions*, Ph.D. thesis, Brown University, Division of Applied Mathematics, 2004.

[22] L. Mathelin, M. Hussaini, and T. Zang, *Stochastic approaches to uncertainty quantification in CFD simulations*, Numer. Algorithms, 38 (2005), pp. 209-236.

[23] O. P. Le Maître, O. M. Knio, H. N. Najm, and R. G. Ghanem, *Uncertainty propagation using Wiener-Haar expansions*, J. Comput. Phys., 197 (2004), pp. 28-57.

[24] K. Matsuzaki, M. Munekata, H. Ohba, and Ushijima, *Numerical study on particle motions in swirling flows in a cyclone separator*, J. Therm. Sci., 15(2):181-186, 2006.

[25] O. P. Le Maître, H. N. Najm, R. G. Ghanem, and O. M. Knio, *Multi-resolution analysis of Wiener-type uncertainty propagation schemes*, J. Comput. Phys., 197 (2004), pp. 502-531.

[26] F. Nobile and R. Tempone, *Analysis and implementation issues for the numerical approximation of parabolic equations with random coefficients*, Internat. J. Numer. Methods Engrg., 80 (2009), pp. 979-1006.

[27] F. Nobile, R. Tempone, and C. G. Webster, *A sparse grid stochastic collocation method for partial differential equations with random input data*, SIAM J. Numer. Anal., 46 (2008), pp. 2309-2345.

[28] F. Nobile, R. Tempone, and C. G. Webster, *An anisotropic sparse grid collocation method for elliptic partial differential equations with random input data*, SIAM J. Numer. Anal. 46 (5) (2008) 2411-2442.

[29] S. Sankaran, A. Marsden, *A stochastic collocation method for uncertainty quantification and propagation in cardiovascular simulations*, J. Biomech Eng 133(3):031001, Mar. 2011.

[30] J.N. Sorensen and W.Z Shen, *Numerical modeling of wind turbine wakes*, J. Fluids Eng., 124:393-399, Jun. 2002.

[31] M. Shäfer and S. Turek, *Benchmark computations of laminar flow around cylinder, Flow Simulation with High-Performance Computers II*, Vieweg, 1996.

[32] R.A. Todor, C. Schwab, *Convergence rates for sparse chaos approximations of elliptic problems with stochastic coefficients*, IMA J. Numer. Anal. 27 (2) (2007) 232-261.

[33] M. Tabata and D. Tagami, *Error estimates for finite element approximations of drag and lift in nonstationary Navier-Stokes flows*, Japan J. Indust. Appli. Math., 17:371-389, 2000.

[34] X.L. Wan and G.E. Karniadakis, *An adaptive multi-element generalized polynomial chaos method for stochastic differential equations*, J. Comput. Phys. 209 (2) (2005) 617-642.

[35] X.L. Wan and G.E. Karniadakis, *Long-term behavior of polynomial chaos in stochastic flow simulations*, Comput. Methods Appl. Mech. Engrg. 195 (2006) 5582-5596.

[36] X.L. Wan and G.E. Karniadakis, *Multi-element generalized polynomial chaos for arbitrary probability measures*, SIAM J. Sci. Comput. 28 (3) (2006) 901-928.

[37] J.A.S. Witteveen, G.J.A. Loeven, and H. Bijl, *Quantifying the effect of physical uncertainties in unsteady fluid-structure interaction problems*, in Proceedings of the 48th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, AIAA paper 2007-1942, Honolulu (HI), United States.

[38] J.A.S. Witteveen, G.J.A. Loeven, S. Sarkar, and H. Bijl, *Probabilistic collocation for period-1 limit cycle oscillations*, Journal of Sound and Vibration 311(1-2), 421-439.

[39] D. Xiu and J. S. Hesthaven, *High-order collocation methods for differential equations with random inputs*, SIAM J. Sci. Comput., 27 (2005), pp. 1118-1139.

[40] G. Zhang and M. Gunzburger, *Error analysis of a stochastic collocation method for parabolic partial differential equations with random input data*, SIAM Journal on Numerical Analysis, 50 (2012), pp. 1922-1940.

[41] G. Zhang, C. Webster, M. Gunzburger and J. Burkardt, *A Hyper-Spherical Adaptive Sparse-Grid Method for High-Dimensional Discontinuity Detection*, submitted, 2014.

Oak Ridge National Laboratory

MANAGED BY UT-BATTELLE FOR THE DEPARTMENT OF ENERGY