Homework:

3-7 $M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$

$$\det|\lambda I - M| = (\lambda - \cos\mu)^2 + \underbrace{(\beta\gamma - \alpha^2)}_{1}\sin^2\mu$$

$$= \lambda^2 - 2\cos\mu\,\lambda + 1$$

$$\lambda_\pm = \cos\mu \pm \sqrt{\cos^2\mu - 1} = \cos\mu \pm i\sin\mu$$
$$= e^{\pm i\mu}$$

Eigenvectors: $(\lambda_\pm I - M)\begin{pmatrix} x \\ x' \end{pmatrix}_\pm = 0$

$$[\lambda_\pm - (\cos\mu + \alpha\sin\mu)]x - \beta\sin\mu\, x' = 0$$

$$\gamma\sin\mu\, x + [\lambda_\pm - (\cos\mu - \alpha\sin\mu)]x' = 0$$

$$\lambda_\pm = \cos\mu \pm i\sin\mu$$

$$(\pm i - \alpha)\sin\mu\, x - \beta\sin\mu\, x' = 0$$

$$\gamma\sin\mu\, x + (\pm i + \alpha)\sin\mu\, x' = 0$$

$x' = \dfrac{-\gamma}{\pm i + \alpha} x \Rightarrow \begin{pmatrix} 1 \\ \dfrac{-\gamma}{\pm i + \alpha} \end{pmatrix}$ Top row

$x' = \dfrac{\pm i - \alpha}{\beta} x \Rightarrow \begin{pmatrix} 1 \\ \dfrac{\pm i - \alpha}{\beta} \end{pmatrix}$ Bottom row

$$\frac{-\gamma}{\pm i + \alpha}\,\frac{\beta}{\pm i - \alpha} = \frac{-\gamma\beta}{-1 - \alpha^2} = \frac{\gamma\beta}{1 + \alpha^2} = 1$$

3-8 $M_T S M = S$ $\quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

3-9 $M = e^K = e^{\mu J} \equiv \sum_{n=0}^{\infty} \frac{\mu^n}{n!} J^n$

$$K = \mu J = \mu \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\mathrm{Tr}\,K = \mu\,\mathrm{Tr}\,J = 0$$

3-10 Forward $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Show Backward $= \begin{pmatrix} d & b \\ c & a \end{pmatrix}$

Suppose 1 simple element: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ = Forward

Backward $= \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\text{Same}} \Rightarrow a = d$

Induction: Assume for n elements

$$M_1 \cdots M_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$M_n \cdots M_1 = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

Let $M_{n+1} = \begin{pmatrix} a_+ & b_+ \\ c_+ & a_+ \end{pmatrix}$

$$M_1 \cdots M_n M_{n+1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_+ & b_+ \\ c_+ & a_+ \end{pmatrix}$$

$$M_{n+1} M_n \cdots M_1 = \begin{pmatrix} a_+ & b_+ \\ c_+ & a_+ \end{pmatrix}\begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$$\text{Forward}_{n+1} = \begin{pmatrix} aa_+ + bc_+ & ab_+ + ba_+ \\ ca_+ + dc_+ & cb_+ + da_+ \end{pmatrix}$$

$$\text{Backward}_{n+1} = \begin{pmatrix} cb_+ + da_+ & ab_+ + ba_+ \\ ca_+ + dc_+ & aa_+ + bc_+ \end{pmatrix}$$

3-11 $2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4$

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 8\beta\beta' K + 4\beta^2 K' = 0$$

$$\beta''' + 4\beta' K + 2\beta K' = 0$$

In element $\beta''' + 4K\beta' = 0$

$\beta' = -2\alpha \qquad \alpha'' + 4K\alpha = 0$

1. $K = 0 \Rightarrow \alpha'' = 0 \Rightarrow -2\alpha = a + bs$
2. $K < 0 \Rightarrow -2\alpha = a\cosh 2\sqrt{K}s + b\sinh 2\sqrt{K}s$
3. $K > 0 \Rightarrow -2\alpha = a\cos 2\sqrt{K}s + b\sin 2\sqrt{K}s$

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$$K=0 \Rightarrow \beta' = a + bs$$

$$K<0 \Rightarrow \beta' = a\cosh 2\sqrt{K}s + b\sinh 2\sqrt{K}s$$

$$K>0 \Rightarrow \beta' = a\cos 2\sqrt{K}s + b\sin 2\sqrt{K}s$$

$$K'=0 \Rightarrow \beta = c + as + \frac{1}{2}bs^2$$

$$K'<0 \Rightarrow \beta = c + \frac{a}{2\sqrt{K}}\sinh 2\sqrt{K}s + \frac{b}{2\sqrt{K}}\cosh 2\sqrt{K}s$$

$$K>0 \Rightarrow \beta = C + \frac{a}{2\sqrt{K}}\sin 2\sqrt{K}s - \frac{b}{2\sqrt{K}}\cos 2\sqrt{K}s$$

$$s=0:\quad K'=0 \Rightarrow \beta' = a = -2\alpha_0$$

$$\beta = c = \beta_0$$

$$\beta'' = b = -2\alpha_0'$$

$$\beta = \beta_0 - 2\alpha_0 s - \alpha_0' s^2$$

$$K>0:\quad \beta' = a = -2\alpha_0$$

$$\beta = c - \frac{b}{2\sqrt{K}}$$

$$\beta'' = -2\sqrt{K}\,a\,\cancelto{0}{\sin 2\sqrt{K}s} + 2\sqrt{K}\,b\,\underbrace{\cos 2\sqrt{K}s}_{1}$$

$$\|$$

$$-2\alpha' = 2\sqrt{K}\,b$$

$$b = -\frac{\alpha_0'}{\sqrt{K}}$$

$$c = \beta_0 - \frac{\alpha_0'}{2K}$$

$$\beta = \beta_0 - \frac{\alpha_0'}{2K} + \frac{\alpha_0'}{2K}\cos 2\sqrt{K}s - \frac{\alpha_0}{\sqrt{K}}\sin 2\sqrt{K}s$$