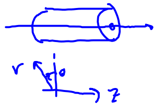
Cavities:

Want E_z strong to accelerate particles.



$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B}/\mu_0 = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$\mu_0 \varepsilon_0 = 1/c^2$

$\sqrt{\frac{\mu_0}{\varepsilon_0}} = Z_0 = 377\,\Omega$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B}$$

2 situations: Inside cavity $\vec{J} = 0$

In cavity walls $\vec{J} \neq 0$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

Curl of vector:

$$\vec{\nabla} \times \vec{V} = \hat{r}\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right) + \hat{\theta}\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \hat{z}\left(\frac{1}{r}\frac{\partial}{\partial r}(r V_\theta) - \frac{1}{r}\frac{\partial}{\partial \theta} V_r\right)$$

Take $\vec{\nabla} \times \vec{E}$ as $\vec{V}$ and plug in to $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{V}$ and equate to $-\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$

Look at $\hat{z}$ component and get (using $\vec{\nabla} \cdot \vec{E} = 0$)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} E_z + \frac{\partial^2 E_z}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$E_z = R(r)\,\Theta(\theta)\,Z(z)\,T(t)$$

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r \frac{dR}{dr}\right) + \frac{1}{r^2}\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = \frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2}$$

$\frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2} = -\omega^2$ because of oscillating nature of solution.

$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -k^2$ where k is determined by B.C.

$\frac{1}{\Theta}\frac{d^2 \Theta}{d\theta^2} = -m^2$ where m = integer

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{m^2}{r^2} - \left(k^2 - \frac{\omega^2}{c^2}\right) = 0$$

There are an infinite number of solutions $\Leftrightarrow$ different m, k, ω

Good acceleration: Want $k=0$

Want symmetry around axis $\Rightarrow m=0$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{\omega^2}{c^2}R = 0$$

$m=0 \Rightarrow \Theta = 1$

$k=0 \Rightarrow Z = 1$

Also know that $T(t) = e^{i\omega t}$

$$E_z = R(r)\, e^{i\omega t} \equiv E_z(r)\, e^{i\omega t}$$

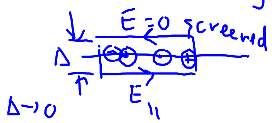
Solutions are Bessel Functions.

For $m=k=0$ solution is

$$J_0\left(\frac{\omega r}{c}\right)$$

Note: $\vec{E}_{\parallel} = 0$ at surface of perfect conductor.

Intuition $\vec{j} = \sigma \vec{E}$, $\sigma = \infty$, $\vec{j}$ finite



$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

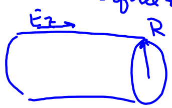
$$\oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

$$= -\frac{d}{dt}\int \vec{B}\cdot d\vec{s}$$

$E_{\parallel} L = 0$ squeeze Δ

$$E_z(r) = J_0\left(\frac{\omega r}{c}\right) e^{i\omega t}$$

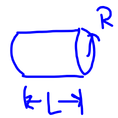
$$E_z(R) = 0 = J_0\left(\frac{\omega R}{c}\right)$$



Choose $\omega \to$ First zero of Bessel fn $\Rightarrow$ Broad region in r of strong E_z.

Transit time factor: T

$E_z(r) = J_0\left(\frac{\omega r}{c}\right)\cos\omega t$



1st Approx to kick:

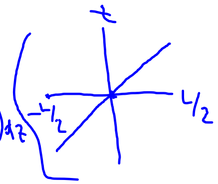
$$\Delta E_{nergy} = eE_z L\cos(\phi)$$

Exact solution:

$$\Delta E_{nergy} = \int_{-L/2}^{L/2} eE\cos(\omega t+\phi)\,dz$$

$$z = vt$$

$$\Delta E_{nergy} = eE\int_{-L/2}^{L/2}\cos\left(\omega\frac{z}{v}+\phi\right)dz$$



$$= \frac{eEv}{\omega}\sin\left(\frac{\omega z}{v}+\phi\right)\Big|_{-L/2}^{L/2}$$

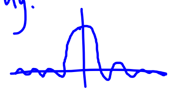
$$= eE\frac{v}{\omega}\left[\sin\left(\frac{\omega L}{2v}+\phi\right) - \sin\left(-\frac{\omega L}{2v}+\phi\right)\right]$$

$$\sin(a\pm b) = \sin a\cos b \pm \cos a\sin b$$

$$\frac{\Delta E_{nergy}}{\Delta E^{Approx}_{nergy}} = \frac{eE\frac{v}{\omega}2\sin\frac{\omega L}{2v}\cos\phi}{eEL\cos\phi}$$

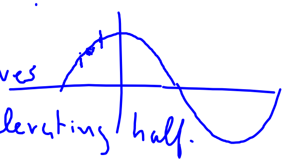
$$T = \frac{2v}{\omega L}\sin\frac{\omega L}{2v} = \frac{\sin\frac{\omega L}{2v}}{\frac{\omega L}{2v}}$$

want cavity not too long.



Why $T<1$?

Sinusoid curves down in accelerating half.



$$Q \equiv \frac{\bar{U}\omega}{P} = \frac{\text{Stored energy}}{\text{Power lost to walls} \times T} \times 2\pi$$

$$= 2\pi \times \frac{\text{Stored energy}}{\text{Energy lost / period}}$$

$$\bar{U} = \frac{1}{2}\varepsilon_0 \int_{\text{cavity}} \bar{E}^2 \, d\bar{V} + \frac{1}{2\mu_0} \int_{\text{Cavity}} \bar{B}^2 \, d\bar{V}$$

$$= \frac{1}{2}\varepsilon_0 \int_{\text{cavity}} E_{max}^2 \, d\bar{V}$$

For TM_{010} only have E_z, B_θ

All other components are 0.

$$U = \frac{1}{2}\varepsilon_0 \int_{\text{Cavity}} E_0^2 \, J_0^2\left(\frac{\omega r}{c}\right) r \, dr \, d\theta \, dz$$

$$= \frac{1}{2}\varepsilon_0 E_0^2 \, 2\pi L \int_0^R J_0^2\left(\frac{\omega r}{c}\right) r \, dr$$

Bessel Fn: $\int_0^1 J_0^2(\alpha u) \, u \, du = \frac{1}{2} J_0'^2(\alpha)$

$$= \frac{1}{2} J_1^2(\alpha)$$

$$\bar{U} = \frac{1}{2}\varepsilon_0 E_0^2 \, 2\pi L R^2 \int_0^1 J_0^2\left(\frac{\omega R}{c}\frac{r}{R}\right) \frac{r}{R} \, d\left(\frac{r}{R}\right)$$

$r/R \sim u$

$\frac{\omega R}{c} \sim \alpha$

$$\bar{U} = \frac{1}{2}\varepsilon_0 E_0^2 \, 2\pi L R^2 \, \frac{1}{2} J_1^2\left(\frac{\omega R}{c}\right)$$

$$= \frac{1}{2}\varepsilon_0 E_0^2 \, \bar{V} \, J_1^2\left(\frac{\omega R}{c}\right)$$