$$Q = \frac{\bar{U}\omega}{P}$$

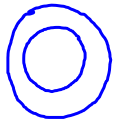
Last time: $\bar{U} = \frac{1}{2}\varepsilon_0^2 E_0^2 V J_1^2(2.405)$

$$\omega = 2.405\, c/R$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

First get $\vec{B}$:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -i\omega\vec{B}$$



$$\hat{r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}E_z - \frac{\partial}{\partial z}E_\theta\right) + \hat{\theta}\left(\frac{\partial}{\partial z}E_r - \frac{\partial}{\partial r}E_z\right) + \hat{z}\left(\frac{1}{r}\frac{\partial}{\partial r}rE_\theta - \frac{1}{r}\frac{\partial}{\partial\theta}E_r\right) = -\hat{\theta}\frac{\partial}{\partial r}E_z$$

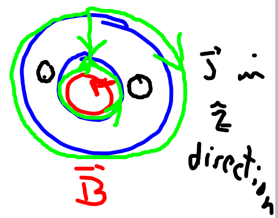
$$\vec{E} = \vec{E}_0(r)e^{i\omega t}$$
$$\vec{B} = \vec{B}_0(r)e^{i\omega t}$$

$$\vec{B} = \frac{i}{\omega}(-i\omega\vec{B}) = -\hat{\theta}\frac{i}{\omega}\frac{\partial}{\partial r}E_z = -\hat{\theta}\frac{i}{\omega}\frac{\partial}{\partial r}E_0 J_0\left(\frac{\omega r}{c}\right)e^{i\omega t}$$
$$= -\hat{\theta}\frac{i}{\omega}\frac{\omega}{c}E_0 J_0'\left(\frac{\omega r}{c}\right)e^{i\omega t}$$
$$= \hat{\theta}\frac{E_0}{c}J_1\left(\frac{\omega r}{c}\right)e^{i(\omega t + \pi/2)}$$

$$i = e^{i\pi/2}$$

Now get current in cavity walls.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{s} + \frac{1}{c^2}\int \frac{\partial \vec{E}}{\partial t} \cdot d\vec{s}$$

can ignore conductors $\Rightarrow \vec{E}$ small

$$\vec{B} = \hat{\theta}\, \frac{E_0}{c} J_1\left(\frac{\omega r}{c}\right) e^{i(\omega t + \frac{\pi}{2})}$$

$$\oint_{\text{inside loop}} \vec{B} \cdot d\vec{l} = \frac{2\pi R E_0}{c} J_1\left(\frac{\omega R}{c}\right) e^{i(\omega t + \frac{\pi}{2})}$$

Assume skin depth $\delta(\omega) = \delta$

Assume resistivity ρ in δ and treat material as uniform.

$\varepsilon_0 \mu_0 = \frac{1}{c^2}$

$$\int \mu_0 \vec{J} \cdot d\vec{s} = \mu_0 J_z 2\pi R \delta$$
$$= \frac{2\pi R E_0}{c} J_1\left(\frac{\omega R}{c}\right) e^{i(\omega t + \pi/2)}$$

$$\vec{J} = \hat{z} J_z = \varepsilon_0 c \frac{E_0}{\delta} J_1\left(\frac{\omega R}{c}\right) e^{i(\omega t + \frac{\pi}{2})}$$

Surface $\vec{J}_s \equiv \delta \vec{J}$

$\vec{E} = \rho \vec{J}$

work done by field on charges in current/time $= \vec{J} \cdot \vec{E} = \rho J^2$

$$P_{outside} = \int \rho J^2 \, dV = \int \rho J_{max}^2 \sin^2 \omega t \, dV$$

$\langle \sin^2 \omega t \rangle = 1/2$ $\quad\quad$ $\delta <<$ small

$$= \frac{1}{2} \int \rho J_{max}^2 \, dV$$

$$= \frac{1}{2} \rho \frac{\epsilon_0^2 E_0^2 c^2}{\delta^2} J_1^2\left(\frac{\omega R}{c}\right) 2\pi R \delta L$$

$$= \frac{1}{2} \rho_s 2\pi R L \times \left(\epsilon_0 E_0 c J_1\left(\frac{\omega R}{c}\right)\right)^2$$

Define $\rho_s = \rho / \delta$

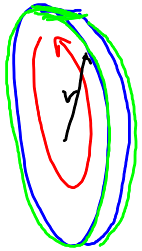
$$Z_0^2 = \frac{\mu_0}{\epsilon_0} = \frac{1}{\epsilon_0} \frac{1}{c^2 \epsilon_0} = \frac{1}{\epsilon_0^2 c^2} \quad\quad \epsilon_0 \mu_0 = 1/c^2$$

$\epsilon_0 c = 1/Z_0$

$$P_{outside} = \frac{1}{2} \rho_s \left(\frac{E_0}{Z_0} J_1\left(\frac{\omega R}{c}\right)\right)^2 2\pi R L$$

End:

$$\oint \vec{B}\cdot d\vec{l} = 2\pi r \frac{E_0}{c} J_1\left(\frac{\omega r}{c}\right) e^{i(\omega t+\pi/2)}$$



$$= \int \vec{J}\cdot d\vec{s}\, \mu_0$$

$$= J_r\, 2\pi r\, \delta\, \mu_0$$

$$\vec{J}_{ends} = \hat{r}\, \frac{E_0}{\delta \mu_0 c} J_1\left(\frac{\omega r}{c}\right) e^{i(\omega t+\pi/2)}$$

$$P_{ends} = 2\int_{end} \rho J^2\, dV = 2\times\frac{1}{2}\int_{end} \rho J_{max}^2\, dV$$

$$= \int_{end} \rho \left(\frac{E_0 J_1\left(\frac{\omega r}{c}\right)}{\mu_0 c}\right)^2 \frac{1}{\delta^2}\, r\, dr\, d\theta\, \delta$$

$$= \frac{\rho}{\delta}\left(\frac{E_0}{\mu_0 c}\right)^2 \int_0^R J_1^2\left(\frac{\omega r}{c}\right) r\, dr\; 2\pi$$

$$= \rho_s \left(\frac{E_0}{Z_0}\right)^2 2\pi\, \frac{c^2}{\omega^2} \int_0^{\frac{\omega R}{c}} J_1^2(u)\, u\, du$$

$$= 2\pi \rho_s \left(\frac{E_0}{Z_0}\right)^2 \frac{c^2}{\omega^2} \left.\underbrace{\frac{u^2}{2}}_{\text{all } 0 \,@\, 0}\left(J_1^2(u) - \underbrace{J_0(u) J_2(u)}_{0 \,@\, \frac{\omega R}{c}}\right)\right|_0^{\omega R/c}$$

$$= 2\pi \rho_s \left(\frac{E_0}{Z_0}\right)^2 \frac{c^2}{\omega^2}\, \frac{1}{2}\, \frac{\omega^2 R^2}{c^2} J_1^2\left(\frac{\omega R}{c}\right)$$

$$= \pi \rho_s \left(\frac{E_0}{Z_0}\right)^2 R^2 J_1^2\left(\frac{\omega R}{c}\right)$$

$$P_{TOTAL} = P_{outside} + P_{ends}$$

$$= \frac{1}{2} \rho_s \left(\frac{E_0}{Z_0}\right)^2 2\pi R L \left(1 + \frac{R}{L}\right) J_1^2\left(\frac{\omega R}{c}\right)$$

$$Q = \frac{\omega U}{P} = \frac{\omega \frac{\varepsilon_0 E_0^2}{2} \underbrace{V}_{=(\pi R^2 L)} J_1^2\left(\frac{\omega R}{c}\right)}{\frac{1}{2} \rho_s \frac{E_0^2}{Z_0^2} 2\pi R L \left(1+\frac{R}{L}\right) J_1^2\left(\frac{\omega R}{c}\right)}$$

$$= \frac{\omega \varepsilon_0 Z_0^2 \; R}{\rho_s \; 2\left(1+R/L\right)}$$

$$= \frac{2.405\, c\, \mu_0}{2 \rho_s (1+R/L)}$$

$$\frac{\omega R}{c} = 2.405$$

$$\varepsilon_0 c = \frac{1}{\mu_0 c}$$

$$Z_0^2 = \frac{\mu_0}{\varepsilon_0}$$

$$\text{Shunt Impedance} = \frac{(\text{energy gain/charge})^2}{P}$$

$$\text{energy gain/charge} = E_0 L T$$

$$R_s = \frac{(E_0 L T)^2}{\frac{1}{2} \rho_s \frac{E_0^2}{Z_0^2} 2\pi R L \left(1+R/L\right) J_1^2(2.405)}$$

$$= \frac{Z_0^2 L T^2}{\pi \rho_s R \left(1+R/L\right) J_1^2(2.405)}$$