Synchrotron Oscillations:

Last time: $\frac{d}{dn}\Delta\phi = \frac{\eta \omega_{rf} T_s}{\beta_s^2 E_s} \Delta E$

$\frac{d}{dn}\Delta E = e\bar{V}(\sin\phi - \sin\phi_s)$

Small Oscillations $\Rightarrow$

$$\frac{d^2\Delta\phi}{dn^2} + (2\pi\nu_s)^2\Delta\phi = 0$$

$\nu_s \equiv$ synchrotron tune

$$= \sqrt{\frac{-\eta \omega_{rs} T_s E \bar{V} \cos\phi_s}{4\pi^2\beta_s^2 E_s}}$$

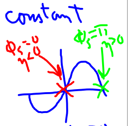
Integrate Equations $\Longrightarrow$

$$\Delta E^2 + \frac{2\beta^2 E_s e\bar{V}}{\eta \omega_{rf} T_s}(\cos\phi + \phi\sin\phi_s)$$

$= \text{constant}$

Pure focusing
No acceleration

This time: Suppose $\phi_s = 0$



This $\Rightarrow \eta < 0$ (For $\eta > 0$ use $\phi_s = \pi$)

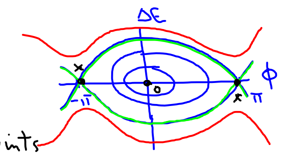
Longitudinal phase space is given by $\Delta\phi$ or ϕ vs ΔE. Particles follow contours of const "Energy" given above.

separatrix

Fixed points

Stable - O points

Untable - X points



$$\frac{2\beta^2 E_s e\bar{V}}{\eta \omega_{rf} T_s} \equiv -D^2$$

$4\sqrt{2} D \sin\frac{\phi}{2}\Big|_{-\pi/2}^{\pi/2}$

Area $= 8\sqrt{2} D$

$\Delta E^2 - D^2\cos\phi = \text{Const}$

x-points $\Rightarrow \Delta E = 0,\ \phi = \pm\pi$

On separatrix $-D^2\cos(\pm\pi) = \text{const} = D^2$

Separtrix: $\Delta E^2 = D^2(1+\cos\phi)$

Area $= \int_{-\pi}^{\pi} 2\Delta E\, d\phi = 2D\int_{-\pi}^{\pi}(1+\cos\phi)^{1/2}\, d\phi$

$= 2D\sqrt{2}\int_{-\pi/2}^{\pi/2}\cos(\phi/2)\, 2\, d(\phi/2) =$

Longitudinal Emittance:

$$\Delta E^2 - D^2(\cos\phi + \phi\sin\phi_s) = \text{const}$$

Synchronous particle: $\Delta E = 0$

$\Delta\phi = 0$ or $\phi = \phi_s$

$$\text{const} = -D^2(\cos\phi_s + \phi_s\sin\phi_s)$$

$$\cos\phi + \phi\sin\phi_s = \cos(\phi_s + \Delta\phi) + (\phi_s + \Delta\phi)\sin\phi_s$$

$$= \cos\phi_s\cos\Delta\phi - \sin\phi_s\sin\Delta\phi + \phi_s\sin\phi_s + \Delta\phi\sin\phi_s$$

$$= \cos\phi_s\underbrace{\left(1 - \tfrac{1}{2}\Delta\phi^2 + \ldots\right)}_{\cos\Delta\phi} + \phi_s\sin\phi_s + \sin\phi_s\left(\Delta\phi - \underbrace{\Delta\phi + \tfrac{1}{3!}\Delta\phi^3 - \ldots}_{\sin\Delta\phi}\right)$$

$$= \cos\phi_s + \phi_s\sin\phi_s - \frac{\cos\phi_s}{2}\Delta\phi^2 + \mathcal{O}(\Delta\phi^3)$$

$$\Delta E^2 - D^2(\cos\phi_s + \phi_s\sin\phi_s) + \frac{D^2\cos\phi_s}{2}\Delta\phi^2 = -D^2(\cos\phi_s + \phi_s\sin\phi_s)$$

This 0 point. move away from 0 point $\Rightarrow$

$$\Delta E^2 + \frac{D^2}{2}\cos\phi_s\,\Delta\phi^2 = \frac{D^2}{2}\cos\phi_s\,\Delta\phi_M^2$$

$$\frac{2}{D^2}\frac{\Delta E^2}{\cos\phi_s\,\Delta\phi_m^2} + \frac{\Delta\phi^2}{\Delta\phi_{MAX}^2} = 1$$

Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Area of ellipse $= \pi ab$

Area in phase space $=$

$$\pi D\sqrt{\frac{\cos\phi_s}{2}}\,\Delta\phi_{MAX}^2$$

$=$ emittance

Adiabatic Damping:

Assume particle "slowly" accelerating in the sense that fractional changes in $E_s, \tau_s, \eta, \gamma, \beta$ are small in passing through a cavity.

$$\frac{dE}{dn} = e\tilde{V}\sin\phi = \tau(E)\frac{dE}{dt}$$

$$\frac{dE_s}{dn} = e\tilde{V}\sin\phi_s = \tau(E_s)\frac{dE_s}{dt}$$

$$\frac{d}{dn} = \frac{dt}{dn}\frac{d}{dt} = \tau(E)\frac{d}{dt}$$

$$\tau(E)\frac{dE}{dt} - \tau(E_s)\frac{dE_s}{dt} = e\tilde{V}(\sin\phi - \sin\phi_s)$$

$$= e\tilde{V}\cos\phi_s\,\Delta\phi$$

$$\tau(E) = \tau(E_s) + \frac{d\tau}{dE}(E-E_s) + \dots$$

$E-E_s = \Delta E$

$$\tau(E_s)\left(\frac{dE}{dt} - \frac{dE_s}{dt}\right) + \frac{d\tau}{dE}(E-E_s)\frac{dE}{dt}$$

$$= e\tilde{V}\cos\phi_s\,\Delta\phi$$

$\frac{d\tau}{dE}\frac{dE}{dt} = \frac{d\tau}{dt}$

$$\tau(E_s)\frac{d}{dt}\Delta E + \Delta E\frac{d\tau}{dt} = e\tilde{V}\cos\phi_s\,\Delta\phi$$

$$\boxed{\frac{d}{dt}(\tau_s\Delta E) = e\tilde{V}\cos\phi_s\,\Delta\phi}$$

$$\frac{d\phi}{dn} = \frac{\eta\omega_{rf}\tau_s}{\beta^2 E_s}\Delta E = \tau_s\frac{d\phi}{dt}$$

$$\frac{d\phi_s}{dn} = 0 = \tau_s\frac{d\phi_s}{dt}$$

$$\tau_s\frac{d}{dt}(\phi-\phi_s) = \frac{\eta\omega_{rf}\tau_s}{\beta^2 E_s}\Delta E$$

$$\boxed{\frac{d}{dt}(\Delta\phi) = \frac{\eta\omega_{rf}}{\beta^2 E_s}\Delta E}$$

$$\frac{d}{dt}(T_s \Delta E) = e\tilde{V}\cos\phi_s \,\Delta\phi$$

$$\frac{d}{dt}(\Delta\phi) = \frac{\eta w_{rf}}{\beta^2 E_s} \Delta E$$

Define: $\lambda \equiv \frac{\eta w_{rf}}{\beta^2 E_s}$

$$\mu \equiv \frac{e\tilde{V}\cos\phi_s}{T_s}$$

$$\frac{d}{dt}(T_s \Delta E) = T_s \,\mu\, \Delta\phi$$

$$\frac{d}{dt}(\Delta\phi) = (\lambda / T_s)\, T_s \Delta E$$

$$\frac{d^2 \Delta\phi}{dt^2} = \frac{d}{dt}\left(\frac{\lambda}{T_s}\right) T_s \Delta E + \frac{\lambda}{T_s} \frac{d}{dt}(T_s \Delta E)$$

$$= \frac{1}{(\lambda / T_s)} \frac{d}{dt}\left(\frac{\lambda}{T_s}\right) \frac{d}{dt}(\Delta\phi) + \frac{\lambda}{T_s} T_s \mu \Delta\phi$$

$$\boxed{\frac{d^2 \Delta\phi}{dt^2} - \frac{1}{\lambda / T_s} \frac{d}{dt}\left(\frac{\lambda}{T_s}\right) \frac{d}{dt}(\Delta\phi) - \lambda\mu\Delta\phi = 0}$$

$$\frac{d^2}{dt^2}(T_s \Delta E) = \frac{d}{dt}(T_s \mu) \Delta\phi + T_s \mu \frac{d}{dt}\Delta\phi$$

$$= \frac{1}{T_s \mu} \frac{d}{dt}(T_s \mu) \frac{d}{dt}(T_s \Delta E) + T_s \mu \frac{\lambda}{T_s} T_s \Delta E$$

$$\boxed{\frac{d^2}{dt^2}(T_s \Delta E) - \frac{1}{T_s \mu} \frac{d}{dt}(T_s \mu) \frac{d}{dt}(T_s \Delta E) - \lambda\mu T_s \Delta E = 0}$$

Define $-\lambda\mu = \Omega_s^2$

$$= -\frac{w_{rf}\,\eta\; e\tilde{V}\cos\phi_s}{\beta^2 E_s \quad T_s}$$

$$\lambda = \frac{\eta \omega_{rf}}{\beta^2 E_s} \qquad \mu = \frac{eV \cos\phi_s}{T_s}$$

$$\frac{d^2 \Delta\phi}{dt^2} - \frac{1}{\frac{\lambda}{T_s}} \frac{d}{dt}\left(\frac{\lambda}{T_s}\right) \frac{d}{dt}\Delta\phi + \Omega_s^2 \Delta\phi = 0$$

$$\frac{d^2 (T_s \Delta E)}{dt^2} - \frac{1}{T_s \mu} \frac{d}{dt}(T_s \mu) \frac{d}{dt}(T_s \Delta E) + \Omega_s^2 T_s \Delta E = 0$$

$$\ddot{\eta} - \frac{1}{g}\dot{g}\dot{\eta} + \Omega^2 \eta = 0$$

Assume $\dot{\Omega} \ll \Omega / T$ =

$\dot{g} \ll g/T$

Let $\eta \equiv uv$

$$\dot{\eta} = \dot{u}v + u\dot{v}$$

$$\ddot{\eta} = \ddot{u}v + 2\dot{u}\dot{v} + u\ddot{v}$$

$$\ddot{u}v + 2\dot{u}\dot{v} + u\ddot{v} - \frac{1}{g}\dot{g}(\dot{u}v + \dot{v}u) + \Omega^2 uv = 0$$