# Adiabatic Damping - Continued

Last time: $\dot{} \equiv d/dt$

$$\ddot{\Delta\phi} - \frac{1}{(\lambda/T_s)}\left(\dot{\overline{\lambda/T_s}}\right)\dot{\Delta\phi} + \Omega_s^2\Delta\phi = 0$$

$$(\ddot{\tilde{T}_s\Delta E}) - \frac{1}{T_s\mu}(\dot{\overline{T_s\mu}})(\dot{\overline{\tilde{T}_s\Delta E}}) + \Omega_s^2\tilde{T}_s\Delta E = 0$$

where $\lambda \equiv \frac{\eta\omega_{rf}}{\beta^2 E_s}$ $\quad \mu = \frac{e\bar{V}\cos\phi_s}{T_s}$

and $T_s$ are slowly changing.

Both $\Delta\phi$ and $\tilde{T}_s\Delta E$ equations take the form

$$\ddot{\eta} - g\dot{\eta} + \Omega^2\eta = 0 \quad \text{where}$$

$g$ small $\qquad$ $g$ and $\Omega$ change slowly.

$\eta \equiv uv$

$\dot{\eta} = \dot{u}v + \dot{v}u \qquad \ddot{\eta} = \ddot{u}v + 2\dot{u}\dot{v} + \ddot{v}u$

$$\ddot{u}v + 2\dot{u}\dot{v} + \ddot{v}u - g\dot{u}v - g\dot{v}u + \Omega^2 vu = 0$$

$$\ddot{u}v + \underbrace{(2\dot{v} - gv)}_{\text{want zeroed out}}\dot{u} + (\cancel{\ddot{v}} - \cancel{g\dot{v}} + \Omega^2 v)u = 0$$

$\frac{\dot{v}}{v} = \frac{g}{2} \qquad v = e^{\frac{1}{2}\int g\,dt}$ $\quad$ Note $\dot{v} = \frac{g}{2}v \ll v$

So $g\dot{v} = \frac{g^2}{2}v \ll \Omega^2 v$

$$\ddot{v} = \frac{g^2}{4}v + \frac{\dot{g}}{2}v \ll \Omega^2 v$$

$$\ddot{u}v + \Omega^2 vu = 0 \Rightarrow \ddot{u} + \Omega^2 u = 0$$

$$u = f(t)\cos(\Omega t + K)$$

$$\dot{u} = \dot{f}\cos(\Omega t + K) - \Omega f\sin(\Omega t + K)$$

$$\ddot{u} = \ddot{f}\cos(\Omega t + K) - 2\dot{f}\Omega\sin(\Omega t + K)$$
$$- \dot{\Omega} f\sin(\Omega t + K) - \Omega^2 f\cos(\Omega t + K)$$

$$= (\cancel{\ddot{f}} - \Omega^2 f)\cos(\Omega t + K)$$
$$- \underbrace{(2\Omega\dot{f} + \dot{\Omega}f)}_{\text{solve for } f \text{ to zero out}}\sin(\Omega t + K) = -\Omega^2 f\cos(\Omega t + K)$$

$\ddot{f} \ll \Omega^2 f$

$$\frac{\dot{\Omega}}{\Omega} + \frac{2\dot{f}}{f} = 0 \Rightarrow \ln\Omega + 2\ln f = \text{const}$$

$$\ln \Omega + \ln f^2 = \text{const}$$

$$\Omega f^2 = \text{const} \qquad f = A/\sqrt{\Omega}$$

$$\eta = f \cos(\Omega t + K)\, e^{\int g/2\, dt}$$

$$= \frac{A}{\sqrt{\Omega}} e^{\int g/2\, dt} \cos(\Omega t + K)$$

$$\Delta\phi : \quad \Omega = \Omega_s, \; g = \frac{(\dot{\lambda/T_s})}{\lambda/T_s} = \dot{\ln}(\lambda/T_s)$$

$$T_s \Delta E : \quad \Omega = \Omega_s, \; g = \frac{(\dot{\mu T_s})}{\mu T_s} = \dot{\ln}(\mu T_s)$$

$$\Delta\phi : \quad e^{\int g/2\, dt} = e^{\int \frac{1}{2}(\dot{\ln} \lambda/T_s)\, dt} = e^{\frac{1}{2}\ln \lambda/T_s}$$

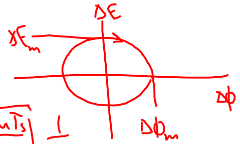
$$= e^{\ln(\lambda/T_s)^{1/2}} = \sqrt{|\lambda/T_s|}$$

$$T_s \Delta E : \quad e^{\int g/2\, dt} = \sqrt{|\mu T_s|}$$

$$\Delta\phi = A \frac{1}{\sqrt{\Omega_s}} \sqrt{\left|\frac{\lambda}{T_s}\right|} \cos(\Omega_s t + K_\phi)$$

$$T_s \Delta E = B \frac{1}{\sqrt{\Omega_s}} \sqrt{|\mu T_s|} \cos(\Omega_s t + K_{\Delta E})$$

Area $\equiv$ longitudinal emittance




$$= \pi A \sqrt{\left|\frac{\lambda}{\Omega_s T_s}\right|}\, B \sqrt{\left|\frac{\mu T_s}{\Omega_s}\right|} \frac{1}{T_s}$$

$$= \pi A B \sqrt{\left|\frac{\lambda \mu}{\Omega_s^2}\right|} \frac{1}{T_s} \qquad \lambda\mu = -\Omega_s^2 \;\Rightarrow\; \sqrt{1} = 1$$

$$= \pi A B / T_s$$

Conjugate variables: $\left\{ \begin{array}{l} \text{Either} \\ \text{Hamilt. Dyn.} \\ \text{or} \\ \text{Q.M.} \end{array} \right.$

$x \leftrightarrow p$

$E \leftrightarrow t$

$$\phi = \omega_{rf} t$$

$$\Delta E_m \Delta\phi_m = \frac{\pi A B}{T_s} \qquad \Delta t = \frac{\Delta\phi}{\omega_{rf}}$$

$$\Delta E_m \Delta t_m = \frac{\pi A B}{\omega_{rf} T_s} = \frac{AB}{2h} \quad \left\{ \begin{array}{l} \text{Long. Emit} \\ \text{in } E\text{-}t \end{array} \right.$$

Scaling of $\Delta E$, $\Delta\phi$ with energy:

$$\Delta E = \frac{1}{T_s} B \sqrt{\left|\frac{\mu}{T_s \Omega_s}\right|} \cos(\quad)$$

$$\Delta\phi = A \sqrt{\left|\frac{\lambda}{T_s \Omega_s}\right|} \cos(\quad)$$

$$\Delta E_m \sim \frac{1}{T_s} \sqrt[4]{\left|\frac{\lambda}{T_s^2 \mu}\right|} \qquad \Omega_s = \sqrt{\lambda\mu}$$

$$\propto \frac{1}{T_s} \left| \frac{\eta \overbrace{\omega_{rf}}^{2\pi h} T_s}{\beta^2 E_s T_s^2 e\bar{V} \cos\phi_s} \right|^{1/4}$$

$$\propto \frac{1}{T_s} \left| \frac{\eta}{\beta^2 E_s T_s^2} \right|^{1/4}$$

Can take limits $\gamma \ll \gamma_T \Rightarrow$

$\beta T_s = \text{const}$

$\eta \sim 1/\gamma^2 \sim 1/E_s^2$

$\gamma \gg \gamma_T \Rightarrow$ $\beta = 1$

$T_s = \text{const}$

$\eta = \frac{1}{\gamma_T^2} = \text{const}$

$$\Delta\phi_m = A \sqrt{\left|\frac{\lambda}{\Omega_s T_s}\right|} = A \left|\frac{\lambda^2}{\Omega_s^2 T_s^2}\right|$$

$$\propto \left|\frac{\lambda}{\mu T_s^2}\right|^{1/4} = \text{you get the idea!}$$

Crossing Transition: Ring radius $R$

$$\frac{d^2\Delta\phi}{dt^2} - \frac{1}{\lambda/\tau}\frac{d(\lambda/\tau)}{dt} + \Omega_s^2\Delta\phi = 0$$

$$\boxed{\begin{aligned} \omega_{rf} &= \frac{hv}{R} \\ &= h\beta c/R \\ \tau &= \frac{2\pi R}{\beta c} \end{aligned}}$$

$$\frac{\lambda}{\tau} = \frac{\eta\,\omega_{rf}}{\beta^2 E_s \tau}$$

$$= \frac{\eta}{E_s}\frac{1}{\beta^2}\frac{h\beta c}{R}\frac{\beta c}{2\pi R}$$

$$= \frac{\eta}{E_s}\frac{hc^2}{2\pi R^2}$$

$$\frac{1}{\lambda/\tau}\dot{(\lambda/\tau)} = \frac{2\pi R^2}{hc^2}\frac{1}{\eta/E_s}\frac{hc^2}{2\pi R^2}\dot{\left(\frac{\eta}{E_s}\right)} = \frac{\dot{(\eta/E_s)}}{\eta/E_s}$$

$$\Omega_s^2 = -\lambda\mu = -\lambda\frac{e\bar{V}\cos\phi_s}{\tau} = -\frac{hc^2 e\bar{V}\cos\phi_s}{2\pi R^2}\frac{\eta}{E_s}$$

$$\ddot{\Delta\phi} - \frac{1}{(\eta/E_s)}\dot{(\eta/E_s)}\,\dot{\Delta\phi} - \frac{hc^2 e\bar{V}\cos\phi_s}{2\pi R^2}\frac{\eta}{E_s}\Delta\phi = 0$$

Problem: $\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_2} = 0$ @ transition.

Assume transition @ $t=0$ and expand

$\eta/E_s$ in time $\Rightarrow$ $\frac{\eta}{E_s} = kt + k_2 t^2 + \dots$

$\approx kt$ for near transition

$$\ddot{\Delta\phi} - \frac{1}{kt}\underbrace{\dot{(kt)}}_{k}\dot{\Delta\phi} - \frac{hc^2 e\bar{V}\cos\phi_s}{2\pi R^2}kt\,\Delta\phi = 0$$

$$t\cos\phi_s = -|t\cos\phi_s| = -|t|\,|\cos\phi_s|$$

$t>0$ $\quad d/dt = d/d|t|, \quad d^2/dt^2 = d^2/d|t|^2$

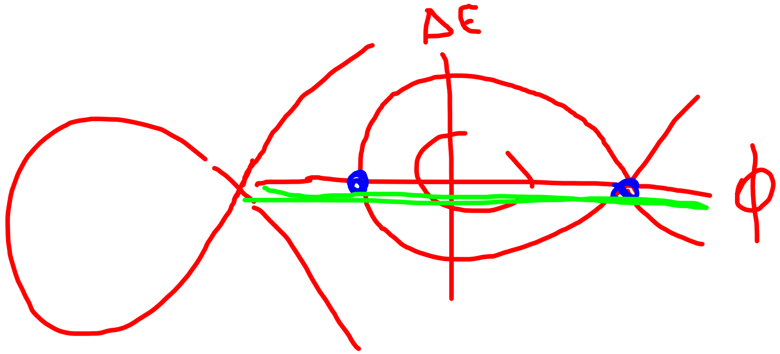
$t<0$ $\quad d/dt = -d/d|t|, \quad d^2/dt^2 = d^2/d|t|^2$

$$\frac{d^2\phi}{d|t|^2} - \frac{1}{|t|}\frac{d\Delta\phi}{d|t|} + K|t|\Delta\phi = 0$$

Book take from here.

# Hint on #5

$$\Delta E^2 - D^2(\cos\phi + \phi\sin\phi_s) = \text{const.}$$




For $\Delta E = 0$ have

$$\Delta E^2 - D^2(\cos\phi - \phi\sin\phi_s) = -D^2(\cos\phi - \phi\sin\phi_s)$$

At x point $\cos\phi - \phi\sin\phi_s =$

$= \text{extremum}$