$$K_x \equiv \frac{1}{R^2} + \frac{B_N'}{RB_0} \qquad K_{\delta x} \equiv \frac{2}{R^2} + \frac{B_N'}{RB_0}$$

$$= K_x + 1/R^2$$

$$K_y = -\frac{B_N'}{RB_0} = -K_x + \frac{1}{R^2} \equiv K_{\delta y}$$

$$S \equiv \frac{B_s'}{RB_0}$$

Finally:

$$x'' + \frac{(\gamma\beta)'}{\gamma\beta}x' + (K_x - K_{\delta x}\delta)x + S(1-\delta)y =$$

$$= \frac{\delta}{R} - \frac{\Delta B_{0y}}{RB_0} - \left(\frac{2}{R^2}\frac{\Delta B_{0y}}{B_0} + \frac{\Delta B_N'}{RB_0}\right)x - \frac{\Delta B_s'}{RB_0}y$$

$$y'' + \frac{(\gamma\beta)'}{\gamma\beta}y' + K_y(1-\delta)y + S(1-\delta)y =$$

$$= \frac{\Delta B_{0x}}{RB_0} + \frac{\Delta B_N'}{RB_0}y - \left(\frac{\Delta B_s'}{RB_0} - \frac{2\Delta B_{0x}}{R^2 B_0}\right)x$$

Title: Nov 2 - 3:50 PM (2 of 2)

$$(K + \Delta K)L = \frac{1}{f} + \Delta\left(\frac{1}{f}\right)$$

$$\Delta\nu = \frac{\beta}{4\pi}\Delta\left(\frac{1}{f}\right) = \frac{\beta L}{4\pi}\Delta K$$

$$K = \frac{B_N'}{RB_0} \qquad \Delta K = \frac{\Delta B_N'}{RB_0}$$

$$RB_0 \equiv P_0/q$$

Chromaticity: Before, with magnetic focusing errors

$$\Delta\frac{1}{f} = \frac{\Delta B_N'}{RB_0}$$

Now, want to see effects of momentum change on focusing. Looking at basic equations for transverse motion we get

$$\Delta\frac{1}{f} \cong -K\delta$$

$$\Delta\nu = \frac{1}{4\pi}\oint \beta\left(\Delta\frac{1}{f}\right)ds$$

$$= \frac{-\delta}{4\pi}\oint \beta K\, ds$$

$$\mathcal{E} \equiv \frac{\Delta\nu}{\delta} = -\frac{1}{4\pi}\oint \beta K\, ds$$

Chromaticity: Bad - Tune spread means some particles have rational tunes $\Rightarrow$ single particles resonance.

Good - Stabilizes collective instabilities.

Damped, driven SHO:

$$m\ddot{x} + \beta\dot{x} + kx = A\cos\omega t$$

Homogeneous solution:

$$m\ddot{x} + \beta\dot{x} + kx = 0 \qquad \boxed{e^{\alpha t}}$$

$$e^{\alpha t}(m\alpha^2 + \beta\alpha + k) = 0$$

$$\alpha = -\frac{\beta}{2m} \pm \sqrt{\left(\frac{\beta}{2m}\right)^2 - \frac{k}{m}}$$

(1) $\left(\frac{\beta}{2m}\right)^2 > \frac{k}{m} \Rightarrow$ 2 damped exponentials

Overdamped

(2) $\left(\frac{\beta}{2m}\right)^2 = \frac{k}{m} \Rightarrow \alpha = -\frac{\beta}{2m}$ Critially damped

(3) $\left(\frac{\beta}{2m}\right)^2 < \frac{k}{m} \Rightarrow \alpha = -\frac{\beta}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{\beta}{2m}\right)^2}$

underdamped

Inhomogeneous solution:

Try $x = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t$

$$m\ddot{x} + \beta\dot{x} + kx = A\cos\omega t$$

$$-m x_0 \omega^2 \cos\omega t + \beta\dot{x}_0\cos\omega t + kx_0\cos\omega t$$

$$-m\frac{\dot{x}_0}{\omega}\omega^2\sin\omega t - \beta\omega x_0\sin\omega t + k\frac{\dot{x}_0}{\omega}\sin\omega t$$

$$= A\cos\omega t$$

Not instructive unless add homogeneous

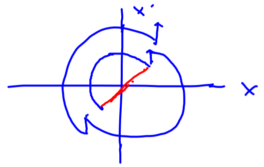
Try $m\ddot{x} + \beta\dot{x} + kx = Ae^{i\omega t}$

$$x = x_0 e^{i\omega t}$$

$$-\omega^2 m x_0 + i\omega\beta x_0 e^{i\omega t} + kx_0 e^{i\omega t} = Ae^{i\omega t}$$

$$x_0 = \frac{A}{k - \omega^2 m + i\omega\beta}$$

$$= \frac{A(k - \omega^2 m - i\omega\beta)}{(k-\omega^2 m)^2 + \omega^2\beta^2}$$



Chromaticity correction with Sextupoles.

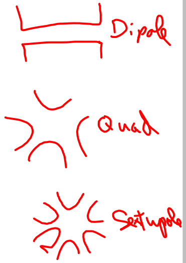
$$B_y = k_{sext} x^2$$

$$B' = 2k_{sext} x$$

$$\langle B' \rangle = 2k_{sext} \langle x \rangle$$

$$= 2k_{sext} \langle \delta \rangle D$$

$$\Delta \nu_{sext} = \frac{2\delta}{4\pi} \oint \frac{\beta k_{sext} D}{R B_0} ds$$



Space Charge (Chapter 6)

Particle Density

$$n = n(x,y,s,t) = n(r,\theta,s,t)$$

where $r = (x^2+y^2)^{1/2}$ $\theta = \tan^{-1}\frac{y}{x}$

$x = r\cos\theta$ $y = r\sin\theta$

$$\int n(r,\theta,s,t)\, r\,dr\,d\theta\,ds \equiv N = \text{total \# particles}$$

$$= \int n(x,y,s,t)\,dx\,dy\,ds$$

$$\int n(r,\theta,s,t)\, r\,dr\,d\theta \equiv \lambda(s,t)$$

$$= \int n(x,y,s,t)\,dx\,dy$$

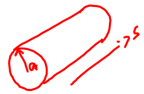
$= \frac{\text{\# particles}}{\text{length in } s}$

Assume primary motion of all particles is $\vec{v} = \beta c \hat{s}$

Consider 4 Simple Cases:

① Uniform Cylindrical Dist:

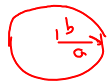
Radius $a(s)$



$$n = \begin{cases} \dfrac{\lambda(s,t)}{\pi a^2} & r < a \\ 0 & r > a \end{cases}$$

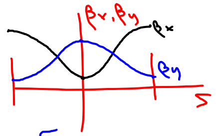
② Uniform Ellipse: $a(s), b(s)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$



$$n = \begin{cases} \dfrac{\lambda(s,t)}{\pi a b} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

Why ellipse?



③ Round Gaussian Beam $\sigma(s)$

$$n = \lambda(s,t) \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2}{2\sigma^2} + \frac{y^2}{2\sigma^2}\right)}$$

$$= \frac{\lambda}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

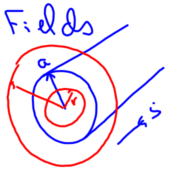
④ 2-D Gaussian Beam: $\sigma_x(s)$, $\sigma_y(s)$

$$n = \lambda(s,t) \frac{1}{2\pi\sigma_x\sigma_y} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)}$$

Case 1: Cylinder get Fields

Gauss Law:

$$\oint_S \vec{E} \cdot d\vec{s} = \int_V \frac{\rho \, dV}{\varepsilon_0}$$



Inside

$$\int \vec{E}\cdot d\vec{s} = 2\pi r l E_r$$

$$= \frac{\rho}{\varepsilon_0}\pi r^2 l$$

$$\rho = qn = q\lambda/\pi a^2$$

$$E_r = \frac{1}{2\pi r l}\frac{q\lambda}{\varepsilon_0 \pi a^2}\pi r^2 l = \frac{q\lambda r}{2\pi\varepsilon_0 a^2}$$

$$E_\theta = 0$$

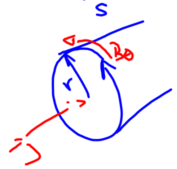
Outside: $2\pi r l E_r = \frac{\rho}{\varepsilon_0}\pi a^2 l = \frac{q\lambda l}{\varepsilon_0}$

$$E_r = \frac{q\lambda}{2\pi\varepsilon_0 r}$$

B-field:

Ampere's Law $\oint \vec{B}\cdot d\vec{l} = \mu_0\int_S \vec{J}\cdot d\vec{s}$

$$B_r = B_s = 0$$



Inside

$$2\pi r B_\theta = \mu_0 J_s \pi r^2$$

$$J_s = \rho v = \rho\beta c$$

$$= \frac{q\lambda}{\pi a^2}\beta c$$

$$\mu_0 c = \frac{1}{\varepsilon_0 c}$$

$$B_\theta = \frac{\mu_0}{2\pi r}\frac{q\lambda\beta c}{\pi a^2}\pi r^2 = \frac{\beta}{\varepsilon_0 c}\frac{q\lambda r}{2\pi a^2} = \frac{\beta}{c}E_r$$

Outside: $B_\theta = \frac{\mu_0}{2\pi r}\frac{q\lambda\beta c}{\pi a^2}\pi a^2$

$$= \frac{\beta}{\varepsilon_0 c}\frac{q\lambda}{2\pi r} = \frac{\beta}{c}E_r$$

$$\left(\vec{E} + \vec{v}\times\vec{B}\right)_\perp = \hat{r}E_r + \beta c\underbrace{\hat{s}\times\hat{\theta}}_{-\hat{r}}B_\theta$$

$$= \hat{r}E_r + \beta c(-\hat{r})\frac{\beta}{c}E_r$$

$$= \hat{r}\underbrace{(1-\beta^2)}_{1/\gamma^2}E_r$$

$$= \hat{r}E_r/\gamma^2$$