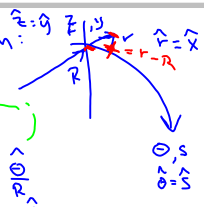
Transverse Motion:



Last time:

$$\beta c \approx r\dot{\Theta}\left(\sqrt{1+\frac{\dot{r}^2+\dot{z}^2}{r^2\dot{\Theta}^2}}\right)$$

$$\frac{d}{dt} = \frac{\beta c R}{r}\frac{d}{ds} \qquad \frac{d\hat{r}}{ds} = \frac{\hat{\Theta}}{R} \qquad \frac{d\hat{\Theta}}{ds} = -\frac{\hat{r}}{R}$$

$$\vec{r} = \hat{r}r + \hat{z}z$$

$$\frac{d\vec{r}}{ds} = \hat{r}r' + \hat{\Theta}\frac{r}{R} + \hat{z}z'$$

$$\frac{d^2\vec{r}}{ds^2} = \hat{r}\left(r'' - \frac{r}{R^2}\right) + 2\hat{\Theta}\frac{r'}{R} + \hat{z}z''$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{ds}{dt}\frac{d\vec{r}}{ds} = \frac{\beta c R}{r}\left[\hat{r}r' + \hat{\Theta}\frac{r}{R} + \hat{z}z'\right]$$

$$\frac{d\vec{p}}{dt} = \vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

$$= \frac{ds}{dt}\frac{d\vec{p}}{ds} = \frac{\beta c R}{r}\frac{d}{ds}\left[\gamma m\frac{\beta c R}{r}\left(\hat{r}r' + \hat{\Theta}\frac{r}{R} + \hat{z}z'\right)\right]$$

$$\text{LHS} = \frac{\beta c R}{r} mcR\Big[(\gamma\beta)'\frac{1}{r}\left(\hat{r}r' + \hat{\Theta}\frac{r}{R} + \hat{z}z'\right)$$
$$- \frac{1}{r^2}r'\gamma\beta\left(\hat{r}r' + \hat{\Theta}\frac{r}{R} + \hat{z}z'\right)$$
$$+ \frac{\gamma\beta}{r}\left(\hat{r}\left(r'' - \frac{r}{R^2}\right) + 2\hat{\Theta}\frac{r'}{R} + \hat{z}z''\right)\Big]$$

$$= \text{RHS} = q\vec{E} + q\frac{\beta c R}{r}\left[\hat{r}r' + \hat{\Theta}\frac{r}{R} + \hat{z}z'\right]\times$$
$$\left[\hat{r}B_r + \hat{\Theta}B_\Theta + \hat{z}B_z\right]$$

$$\frac{\gamma\beta^2 mc^2R^2}{r^2}\left[\begin{pmatrix}\frac{(\gamma\beta)'}{\gamma\beta} \\ -\frac{r'}{r}\end{pmatrix}\left(\hat{r}r' + \hat{\Theta}\frac{r}{R} + \hat{z}z'\right)\right.$$
$$\left. + \hat{r}\left(r'' - \frac{r}{R^2}\right) + 2\hat{\Theta}\frac{r'}{R} + \hat{z}z''\right)\right]$$

$$= q\vec{E} + q\frac{\beta c R}{r}\Big[\hat{r}\left(B_\Theta z' - \frac{B_z r}{R}\right)$$
$$\hat{z}\left(B_r\frac{r}{R} - B_\Theta r'\right.$$
$$\hat{\Theta}\left(B_z r' - B_r z'\right)\Big]$$

Usually: $\vec{E} = E_\Theta\hat{\Theta}$ (cavities)

$\vec{B} = B_r\hat{r} + B_z\hat{z}$ (except solenoids + fringes)

$\hat{r}$ component: $$\frac{\gamma\beta^2 mc^2R^2}{r^2}\left(r'' - \frac{r}{R^2} + \frac{(\gamma\beta)'}{\gamma\beta}r' - \frac{r'^2}{r}\right)$$

$$= -q\frac{\beta c R}{r}\frac{B_z r}{R} \qquad \frac{q\beta c R}{r}B_r\frac{r}{R}$$

$\hat{z}$ component: $$\frac{\gamma\beta^2 mc^2R^2}{r^2}\left(z'' + \frac{(\gamma\beta)'}{\gamma\beta}z' - \frac{r'z'}{r}\right) = \uparrow$$

$\hat{r}$ component:

$$\left(r'' - \frac{r}{R^2} + \frac{(\gamma\beta)'}{\gamma\beta} r' - \frac{r'^2}{r}\right) = -\frac{q \beta c \dot{r}^2 B_z}{\gamma \beta^2 R^2 m c^2}$$

$\hat{z}$ component:

$$\left(z'' + \frac{(\gamma\beta)'}{\gamma\beta} z' - \frac{r' z'}{r}\right) = \frac{q \beta c \dot{r}^2 B_r}{\gamma \beta^2 m c^2 R^2}$$

Don't need θ component: Longitudinal

$$\frac{r}{R} = 1 + \frac{x}{R}$$

Next step: $x = r - R$ or $r = R + x$

$y = z$

$$B_z = B_y = B_0\left(1 + \frac{1}{B_0} B_{yx} x + \frac{1}{B_0} B_{yy} y\right)$$

$$B_r = B_x = B_0\left(\frac{1}{B_0} B_{xx} x + \frac{1}{B_0} B_{xy} y\right)$$

Equations again: Go to x and y

$\hat{x}$:

$$x'' - \frac{R+x}{R^2} + \frac{(\gamma\beta)' x'}{\gamma\beta} - \frac{x'^2}{R+x} = -\frac{q B_0}{\gamma\beta m c}\left(1 + \frac{x}{R}\right)^2 \left(1 + B_{yx} x + B_{yy} y\right)$$

$\hat{y}$:

$$y'' + \frac{(\gamma\beta)'}{\gamma\beta} y' - \frac{x' y'}{R+x} = +\frac{q B_0}{\underbrace{\gamma\beta m c}_{\gamma m v = p}}\left(1 + \frac{x}{R}\right)^2 \left(B_{xx} x + B_{xy} y\right)$$

$$\frac{dp}{dt} = q v \times B = \omega p_0 = q \omega R B_0$$

$$\frac{p}{p_0} = \frac{\gamma m v}{\gamma_0 m v_0} = \frac{\gamma\beta m c}{\gamma_0 \beta_0 m c} = \quad 1 + \delta = \frac{p_0 + \Delta p}{p_0} = 1 + \frac{\Delta p}{p_0}$$

$$\delta \equiv \Delta p / p_0$$

$$\frac{q B_0}{\gamma\beta m c} = \frac{q B_0}{p_0(1+\delta)} = \frac{1}{R}\frac{1}{1+\delta}$$

Small quantities: $\frac{x}{R}, \frac{y}{R}, x', y', \delta$

$\vec{B}$ fields:

$$\vec{B} = \vec{B}_0 + \sum_{ij} \vec{B}_{ij} \frac{x^i}{i!} \frac{y^j}{j!}$$

$$= \vec{B}_0 + \vec{B}_{10} x + \vec{B}_{01} y$$

Aside on $\vec{B}$ field expansion