From Last Time:

$$x'' + \frac{(\gamma\beta)'}{\gamma\beta}x' + (K_x - K_{\delta x}\delta)x + S(1-\delta)y = \frac{\delta}{R}$$

$$y'' + \frac{(\gamma\beta)'}{\gamma\beta}y' - K_y(1-\delta)y + S(1-\delta)x = 0$$

$$K_x = \frac{1}{R^2} + \frac{B_N'}{RB_0} \qquad K_{\delta x} = K_x + \frac{1}{R^2}$$

$$K_y = \frac{B_N'}{RB_0} = K_{\delta y} = K_x - \frac{1}{R^2}$$

$$S = B_S'/RB_0$$

Physics in equations:

Bending: R, Note $B_0 R = \frac{p_0}{q}$

So $R \to \infty \Leftrightarrow B_0 \to 0$ such that $B_0 R = p_0/q$

Oscillations: K_x, K_y

Coupling: S

Chromaticity: $K_{\delta x}, K_{\delta y}$

Dispersion: δ/R

Acceleration: $(\gamma\beta)'$

Design Accelerator: 1st Order

Assume $S = 0$

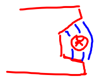
Take Reference energy $\Rightarrow \delta = 0$

Neglect acceleration $\frac{(\gamma\beta)'}{\gamma\beta} \ll 1$

$$x'' + K_x x = 0$$

$$y'' - K_y y = 0$$

Weak Focusing:



$$B_y = \frac{B_0 R^n}{r^n} \quad @\ y=0$$

$$= B_0\left(1+\frac{x}{R}\right)^{-n} \approx B_0\left(1 - \frac{nx}{R}\right)$$

$$= B_0 + B_{yx}x$$

$$B_{yx} = -\frac{B_0 n}{R} = B_N'$$

$$K_x = \frac{B_N'}{RB_0} + \frac{1}{R^2} = (1-n)R^2 > 0$$

$$K_y = \frac{B_N'}{RB_0} = \frac{-n}{R^2} < 0$$

$0 < n < 1$

Strong Focussing:
Separate focusing and bending into Different Magnets.

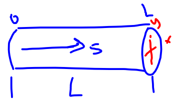
① Drift Space : $K_x = K_y = 0$

$x'' = y'' = 0$

$x' = x_0' = \text{const.}$ $\quad y' = y_0' = \text{const}$

$x = x_0 + x_0' s$ $\quad y = y_0 + y_0' s$

Length L



$x_L = x_0 + x_0' L$

$x_L' = x_0'$

$y_L = y_0 + y_0' L$

$y_L' = y_0'$

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_L = \left[\begin{array}{cc|cc} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_0$$

$$M_D = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

② Normal Quadrupole length L:

$R = \infty$, $B_0 R = \frac{p_0}{q}$,

$K_x = K_y = \frac{B_N}{RB_0} \equiv K$

$$\left.\begin{aligned} x'' + Kx &= 0 \\ y'' - Ky &= 0 \end{aligned}\right\}$$ Can't focus both directions at once

$K > 0$:

$x = x_0 \cos(\sqrt{K}s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}s)$

$x' = -\sqrt{K} x_0 \sin(\sqrt{K}s) + x_0' \cos(\sqrt{K}s)$

$y = y_0 \cosh(\sqrt{K}s) + y_0' \frac{1}{\sqrt{K}} \sinh(\sqrt{K}s)$

$y' = y_0 \sqrt{K} \sinh(\sqrt{K}s) + y_0' \cosh(\sqrt{K}s)$

Convention:

$K > 0$ focusing Quad
- focuses x
- defocuses y

$K < 0$ defocusing Quad
- defocuses x
- focuses y

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_L = \left[\begin{array}{cc|cc} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L & 0 & 0 \\ -\sqrt{K}\sin\sqrt{K}L & \cos\sqrt{K}L & 0 & 0 \\ \hline 0 & 0 & \cosh\sqrt{K}L & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L \\ 0 & 0 & \sqrt{K}\sinh\sqrt{K}L & \cosh\sqrt{K}L \end{array}\right] \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_0$$

$$M_Q^f = \begin{bmatrix} \cos\sqrt{K}L & \frac{1}{\sqrt{K}}\sin\sqrt{K}L \\ -\sqrt{K}\sin\sqrt{K}L & \cos\sqrt{K}L \end{bmatrix}$$

$$M_Q^d = \begin{bmatrix} \cosh\sqrt{K}L & \frac{1}{\sqrt{K}}\sinh\sqrt{K}L \\ \sqrt{K}\sinh\sqrt{K}L & \cosh\sqrt{K}L \end{bmatrix}$$

③ Thin lens: $L \to 0$, $\sqrt{K}L \to 0$, $KL \to \frac{1}{f}$

$$M_Q^f = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \qquad M_Q^D = \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}$$

$\Downarrow$

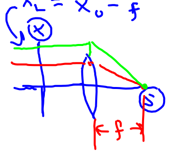
$x_L = x_0$

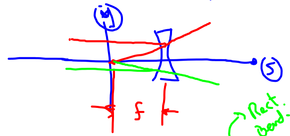
$x_L' = x_0' - \frac{x}{f}$

$\Downarrow$

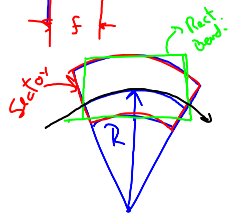
$y_L = y_0$

$y_L' = y_0' + \frac{y_0}{f}$





④ Sector Bend:



Assume $B = \hat{y} B_0$

$B_N' = 0$

$K_x = \frac{1}{R^2} \quad K_y = 0$

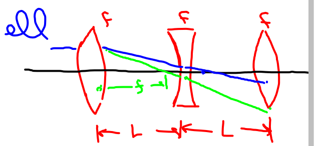
$x'' + \frac{1}{R^2} = 0$

$y'' = 0$

$$M_{SB} = \left[\begin{array}{cc|cc} \cos\frac{L}{R} & R\sin\frac{L}{R} & 0 & 0 \\ -\frac{1}{R}\sin\frac{L}{R} & \cos\frac{L}{R} & 0 & 0 \\ \hline 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{array}\right]$$

⑤ FODO Cell

$f > L$



$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

⑥ Skew Quad:

$S \neq 0$ everything else $= 0$

$$\left.\begin{aligned} x'' + S y &= 0 \\ y'' + S x &= 0 \end{aligned}\right\} \begin{aligned} (x+y)'' + S(x+y) &= 0 \\ (x-y)'' - S(x-y) &= 0 \end{aligned}$$

$$\left.\begin{aligned} u &\equiv x+y \\ v &\equiv x-y \end{aligned}\right\} \begin{aligned} u'' + S u &= 0 \\ v'' - S v &= 0 \end{aligned}$$

$$u = u_0 \cos\sqrt{S}\,L + \frac{u_0'}{\sqrt{S}} \sin\sqrt{S}\,L$$

$$u' = -\sqrt{S}\,u_0 \sin\sqrt{S}\,L + u_0' \cos\sqrt{S}\,L$$

$$v = v_0 \cosh\sqrt{S}\,L + \frac{v_0'}{\sqrt{S}} \sinh\sqrt{S}\,L$$

$$v' = \sqrt{S}\,v_0 \sin\sqrt{S}\,L + v_0' \cosh\sqrt{S}\,L$$

$x = \frac{u+v}{2}$ $\quad y = \frac{u-v}{2}$ $\quad u_0 = x_0 + y_0$ $\quad v_0 = x_0 - y_0$

Plug in to get matrix.

Where we're going:

$$x'' + K(s)\,x = 0$$

K-piecewise continuous periodic in rings.