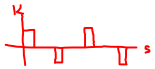
Transverse <u>Decoupled</u> Motion.

Diff. eq. $x'' + K(s)x = 0$



$K(s)$ piecewise continuous (constant)

Matrix solutions: $\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{0\to s} \begin{pmatrix} x \\ x' \end{pmatrix}_0$

Suppose we have 2 independent solutions to differential equation $x'' + Kx = 0$

$$C_0(s): \quad C_0(s_0) = 1, \; C_0'(s_0) = 0$$
$$S_0(s): \quad S_0(s_0) = 0, \; S_0'(s_0) = 1$$
$$C_1(s): \quad C_1(s_1) = 1, \; C_1'(s_1) = 0$$
$$S_1(s): \quad S_1(s_1) = 0, \; S_1'(s_1) = 1$$

General solution

$$x(s) = x_0 C_0(s) + x_0' S_0(s) = x_1 C_1(s) + x_1' S_1(s)$$
$$x'(s) = x_0 C_0'(s) + x_0' S_0'(s) = x_1 C_1'(s) + x_1' S_1'(s)$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C_0(s) & S_0(s) \\ C_0'(s) & S_0'(s) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 = M_{0\to s} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$\det M = C_0 S_0' - C_0' S_0$$
$$\frac{d}{ds}(\det M) = C_0' S_0' + C_0 S_0'' - C_0'' S_0 - C_0' S_0'$$
$$= C_0(-K S_0) - (-K C_0) S_0$$
$$= -K C_0 S_0 + K C_0 S_0$$
$$= 0$$

$\det M$ = constant as fn. of s

$$M_{0\to 0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det M =$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C_1(s) & S_1(s) \\ C_1'(s) & S_1'(s) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}$$

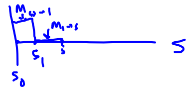
$$\begin{pmatrix} C_0 \\ C_0' \end{pmatrix}_s = \begin{pmatrix} C_1(s) & S_1(s) \\ C_1'(s) & S_1'(s) \end{pmatrix} \begin{pmatrix} C_0 \\ C_0' \end{pmatrix}_{s_1}$$

$$\begin{pmatrix} S_0 \\ S_0' \end{pmatrix}_s = \begin{pmatrix} C_1(s) & S_1(s) \\ C_1'(s) & S_1'(s) \end{pmatrix} \begin{pmatrix} S_0 \\ S_0' \end{pmatrix}_{s_1}$$

$$\begin{pmatrix} C_0 & S_0 \\ C_0' & S_0' \end{pmatrix}_s = \begin{pmatrix} C_1(s) & S_1(s) \\ C_1'(s) & S_1'(s) \end{pmatrix} \begin{pmatrix} C_0 & S_0 \\ C_0' & S_0' \end{pmatrix}_{s_1}$$

$$\begin{pmatrix} C_0 & S_0 \\ C_0' & S_0' \end{pmatrix}_s = \begin{pmatrix} C_1 & S_1 \\ C_1' & S_1' \end{pmatrix}_s \begin{pmatrix} C_0 & S_0 \\ C_0' & S_0' \end{pmatrix}_{s_1}$$

$$\boxed{M_{s_0 \to s} = M_{s_1 \to s} \, M_{s_0 \to s_1}}$$



$$M_{0n} = M_{n-1\,n} \cdot M_{n-2\,n-1} \cdots M_{12} \cdot M_{01}$$

Symplecticity?

Case: Single particle in 1 dimension.
Same case we're studing
$x'' + K(s)\,x = 0$

Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}K x^2$

$$\left.\begin{aligned} x' &= \frac{\partial H}{\partial p} = p \\ p' &= -\frac{\partial H}{\partial x} = -Kx \end{aligned}\right\} \; p' = x'' = -Kx$$

Write $H = \frac{1}{2}(x\ p)\begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$

$$\equiv \frac{1}{2}(x p)\, H_0 \begin{pmatrix} x \\ p \end{pmatrix}$$

$$H_0 \equiv \begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix} = H_0^T$$

Define $S \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$S^T = -S \qquad S \cdot S = -I$$

$$SS^T = S^T S = I$$

$$\frac{d}{ds}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ -Kx \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$$

$$= S H_0 \begin{pmatrix} x \\ p \end{pmatrix}$$

Define independent solutions

$$u = \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \text{ and } v = \begin{pmatrix} x_2 \\ x_2' \end{pmatrix}$$

$$u_s = M_{0s} u_0 \qquad v_s = M_{0s} v_0$$

Form product $u^T S v$

Note $\frac{d}{ds}(u^T S v) = x_1' x_2' + x_1 x_2'' - x_1'' x_2 - x_1' x_2'$

$= x_1(-Kx_2) - (-Kx_1)x_2$
$= 0$

$= (x_1, x_1') \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_2' \end{pmatrix}$

$= x_1 x_2' - x_1' x_2$

$$\frac{d}{ds}(u^T S v) = \left(\frac{d}{ds}u\right)^T S v + u^T S \frac{d}{ds} v$$

$$= (S H_0 u)^T S v + u^T \underbrace{S S}_{-I} H_0 v$$

$$= u^T \underbrace{H_0^T}_{H_0} \underbrace{S^T S}_{I} v - u^T I H_0 v$$

$$= u^T H_0 v - u^T H_0 v = 0$$

$u^T S v = \text{constant}$

$(u^T S v)_s = (u^T S v)_0$

$u_s^T S v_s \qquad = u_0^T S v_0$

$= (M_{0s} u_0)^T S (M_{0s} v_0) = u_0^T S v_0$

$u_0^T M_{0s}^T S M_{0s} v_0 = u_0^T S v_0$

$$\boxed{M_{0s}^T S M_{0s} = S} \quad \text{Symplecticity!}$$

True for all Ham. Systems!

Take determinant:

$$\underbrace{(\det M_{0s}^T)}_{\det M_{0s}} \cdot \det S \cdot \det M_{0s} = \det S$$

$(\det M_{0s})^2 = 1 \Rightarrow \det M_{0s} = \pm 1$

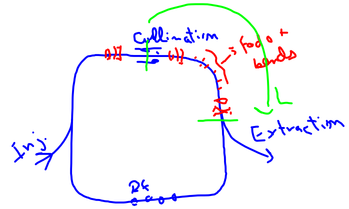
$= \text{select} +1$

Stability:

Repeating sequences of matrices (elements)

Whole thing called "lattice".

Usually repeats with period L.



Suppose n repetions of period:

and matrix for $\underline{1}$ repetition is M.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_n = M^n \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad Tr(M) = a + d$$

$$\text{Det } M = ad - bc = 1$$

Eigenvectors and Eigenvalues:

$$M\begin{pmatrix} x \\ x' \end{pmatrix} = \lambda\begin{pmatrix} x \\ x' \end{pmatrix} = \lambda I \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$(\lambda I - M)\begin{pmatrix} x \\ x' \end{pmatrix} = 0 \Rightarrow \det(\lambda I - M) = 0$$

$$\det\begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} = 0 = \lambda^2 - (a+d)\lambda + ad - bc$$

$$0 = \lambda^2 - Tr(M)\lambda + \overset{\to 1}{\det M}$$

$$\lambda_\pm = \left(\frac{Tr(M)}{2}\right) \pm \sqrt{\left(\frac{Tr(M)}{2}\right)^2 - \underbrace{\det M}_{1}}$$

$$\lambda_+\lambda_- = \det M = 1$$

$$\lambda_+ + \lambda_- = Tr(M)$$

If $|Tr(M)| > 2$ then $|\lambda_+| > 1$ or $|\lambda_-| > 1$

$$\left.\begin{aligned} c x_\pm + (d - \lambda_\pm) x'_\pm &= 0 \\ (a - \lambda_\pm) x_\pm + b x'_\pm &= 0 \end{aligned}\right\} \quad M\begin{pmatrix} x \\ x' \end{pmatrix}_\pm = \lambda_\pm \begin{pmatrix} x \\ x' \end{pmatrix}_\pm$$