Transverse Uncoupled beam Transport with $\delta = 0$:

M: Eigenvalues

$$\lambda^2 - \mathrm{Trace}\,M\,\lambda + \det M = 0$$

$$\lambda_\pm = \frac{1}{2}\mathrm{Trace}\,M \pm \sqrt{\left(\frac{\mathrm{Trace}\,M}{2}\right)^2 - \det M}$$

$\det M = 1$

$$\lambda_+ + \lambda_- = \mathrm{Trace}\,M$$

$$\lambda_+ \cdot \lambda_- = \det M = 1$$

$$\lambda_\pm = \frac{1}{2}\mathrm{Tr}\,M \pm i\sqrt{1 - \left(\frac{1}{2}\mathrm{Tr}\,M\right)^2}$$

$$\cos\mu \equiv \frac{1}{2}\mathrm{Tr}\,M$$

$$\lambda_\pm = \cos\mu \pm i\underbrace{\sqrt{1-\cos^2\mu}}_{\sin\mu}$$

$$\lambda_\pm = \cos\mu \pm i\sin\mu = e^{\pm i\mu}$$

Matrices are nonsingular.
Can be diagonalized.
Can write solutions to $x'' + Kx = 0$ in terms of eigenvectors: $\begin{pmatrix} x \\ x' \end{pmatrix}_\pm$

$$M\begin{pmatrix} x \\ x' \end{pmatrix}_\pm = \lambda_\pm \begin{pmatrix} x \\ x' \end{pmatrix}_\pm$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = a_+\begin{pmatrix} x \\ x' \end{pmatrix}_+ + a_-\begin{pmatrix} x \\ x' \end{pmatrix}_-$$

Consider periodic system with period L: $x'' + K(s)x = 0$

$$K(s+L) = K(s)$$

$$M_{0L} = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix}$$

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$$M_{0L} = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix}$$

$$\det M_{0L} = \cos^2\mu + (\beta\gamma - \alpha^2)\sin^2\mu = 1$$

$$\beta\gamma - \alpha^2 = 1 \qquad \gamma = \frac{1+\alpha^2}{\beta}$$

$$M_{0L} = I\cos\mu + \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\sin\mu$$

$$J \equiv \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix} \qquad M_{0L} = I\cos\mu + J\sin\mu$$

$$J^2 = J\cdot J = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix} = \begin{bmatrix} \alpha^2-\beta\gamma & 0 \\ 0 & \alpha^2-\beta\gamma \end{bmatrix}$$

$$= -I$$

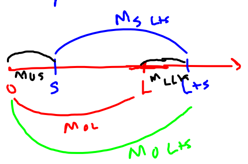
$$M_{0L} = e^{\mu J} = I + \mu J + \frac{1}{2}\mu^2 J^2 + \frac{1}{3!}\mu^3 J^3 + \frac{1}{4!}\mu^4 J^4 + \ldots$$

$$= I + \mu J - \frac{1}{2}\mu^2 I - \frac{1}{3!}\mu^3 J + \frac{1}{4!}\mu^4 I + \ldots$$

$$= I\left(1 - \frac{1}{2}\mu^2 + \frac{1}{4!}\mu^4 - \ldots\right)$$

$$+ J\left(\mu - \frac{1}{3!}\mu^3 + \frac{1}{5!}\mu^5 \ldots\right)$$

$$= I\cos\mu + J\sin\mu$$



$$M_{0\to L+s}$$

$$= M_{L\,L+s} M_{0L}$$

$$= M_{s\,L+s} M_{0s}$$

$$M_{s\,L+s} = \overset{M_{0s}}{\overset{\shortparallel}{M_{L\,L+s}}} M_{0L} M_{0s}^{-1}$$

$$= M_{0s} M_{0L} M_{0s}^{-1}$$

$$\text{Trace } M_{s\,L+s} = \text{Trace } M_{0L}$$

$$= 2\cos\mu$$

So μ independent of starting point

Eigenfunctions: $M_{0L}\begin{pmatrix} x \\ x' \end{pmatrix}_\pm = e^{\pm i\mu}\begin{pmatrix} x \\ x' \end{pmatrix}_\pm$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_\pm (L) = e^{\pm i\mu}\begin{pmatrix} x \\ x' \end{pmatrix}_\pm (0)$$

Define $p_\pm(s) \equiv e^{\mp i\mu s/L}\, x_\pm(s)$

$$x_\pm(s) = p_\pm(s)\, e^{\pm i\mu s/L}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_\pm (s+L) = M_{0\,s+L} \begin{pmatrix} x \\ x' \end{pmatrix}_\pm (0)$$

$$= \underbrace{M_{L\,s+L}}_{M_{0s}} \underbrace{M_{0L} \begin{pmatrix} x \\ x' \end{pmatrix}_\pm (0)}_{e^{\pm i\mu} \begin{pmatrix} x \\ x' \end{pmatrix}_\pm (0)}$$

$$= e^{\pm i\mu} M_{0s} \begin{pmatrix} x \\ x' \end{pmatrix}_\pm (0)$$

$$= e^{\pm i\mu} \begin{pmatrix} x \\ x' \end{pmatrix}_\pm (s)$$

$$p_\pm(s+L) \underbrace{e^{\pm i\mu \frac{s+L}{L}}}_{e^{\pm i\mu} e^{\pm i\mu s/L}} = e^{\pm i\mu} e^{\pm i\mu \frac{s}{L}} p_\pm(s)$$

$$p_\pm(s+L) = p_\pm(s)$$

So $\begin{pmatrix} x \\ x' \end{pmatrix}_\pm (s) = p_\pm(s)\, e^{\pm i\mu s/L}$

where $p_\pm(s)$ is periodic.

Shift gears:

Let's solve $x'' + K(s)\, x = 0$

$$\boxed{x(s) = x_0\, w(s) \cos(\psi(s) + \delta)}$$

$$x'(s) = x_0 w' \cos(\ ) - x_0 w \psi' \sin(\ )$$

$$x''(s) = x_0 w'' \cos(\ ) - 2 x_0 w' \psi' \sin(\ ) - x_0 w \psi'' \sin(\ ) - x_0 w \psi'^2 \cos(\ )$$

$$x_0 \left[w'' - w\psi'^2 + K w \right] \cos(\ )$$

$$- x_0 \left[2 w' \psi' + w \psi'' \right] \sin(\ ) = 0$$

$$\frac{1}{w} \left[2 w w' \psi' + w^2 \psi'' \right] = 0$$

$$\text{"} \quad \frac{1}{w} (w^2 \psi')' = 0 \quad \Rightarrow \quad \boxed{w^2 \psi' = \text{const}}$$

$$\psi' = \frac{k}{w^2} \qquad \psi = k \int \frac{ds}{w^2}$$

$$w'' - \frac{k^2}{w^3} + K(s)\, w = 0$$

Require w = periodic function : Motivate from form of eigenfns.

$$x = x_0\, w(s) \cos(\psi(s) + \delta)$$

$$= A_1 w \cos\psi + A_2 w \sin\psi$$

Require w = periodic.

We know $$w'' - \frac{k^2}{w^3} + K w = 0$$

$$\psi' = \frac{k}{w^2}$$

$$x' = (A_1 w' + A_2 w \psi') \cos\psi + (A_2 w' - A_1 w \psi') \sin\psi$$

$$= \left(A_1 w' + A_2 \frac{k}{w}\right) \cos\psi + \left(A_2 w' - A_1 \frac{k}{w}\right) \sin\psi$$

At $s = 0$: $x = x_0 = A_1 w_0$

$$x' = x_0' = A_1 w_0' + A_2 \frac{k}{w_0}$$

$$A_1 = x_0 / w_0 \qquad A_2 = \frac{w_0 x_0' - x_0 w_0'}{k}$$

Plug in:

$$x_s = \frac{x_0}{w_0} w \cos\psi + \frac{w_0 x_0' - x_0 w_0'}{k} w \sin\psi$$

$$x_s' = \left[-\left(\frac{k}{w_0 w} + \frac{w_0' w'}{k}\right) \sin\psi + \left(\frac{w'}{w_0} - \frac{w_0'}{w}\right) \cos\psi\right] x_0$$

$$+ \left[\frac{w_0 w'}{k} \sin\psi + \frac{w_0}{w} \cos\psi\right] x_0'$$