$$\left.\begin{aligned} &x'' + K(s)\,x = 0 \\ &K(s+L) = K(s) \end{aligned}\right\} \Rightarrow$$

$$x = x_0\, w(s) \cos(\psi(s) + \delta)$$

$$= A_1\, w(s) \cos\psi(s) + A_2 w_s \sin\psi(s)$$

$$\psi' = k/w^2$$

$$w'' + K(s)\,w - \frac{k^2}{w^3} = 0$$

$$x = \left[\frac{w}{w_0}\cos\psi - \frac{w_0' w}{k}\sin\psi\right] x_0 + \frac{w_0 w}{k}\sin\psi\, x_0'$$

$$x' = \left[\left(\frac{w'}{w_0} - \frac{w_0'}{w}\right)\cos\psi - \left(\frac{k}{w_0 w} + \frac{w_0' w'}{k}\right)\sin\psi\right] x_0$$

$$+ \left[\frac{w_0}{w}\cos\psi + \frac{w_0 w'}{k}\sin\psi\right] x_0'$$

$$M_{0s} = \begin{bmatrix} \frac{w}{w_0}\cos\psi - \frac{w_0' w}{k}\sin\psi & \frac{w_0 w}{k}\sin\psi \\ \left(\frac{w'}{w_0} - \frac{w_0'}{w}\right)\cos\psi - \left(\frac{k}{w_0 w} + \frac{w_0' w'}{k}\right)\sin\psi & \frac{w_0}{w}\cos\psi + \frac{w_0 w'}{k}\sin\psi \end{bmatrix}$$

Consider 1 period: M_{0L}, $\left\{\begin{aligned} &\psi = \psi_L \\ &w = w_L = w_0 \\ &w' = w_L' = w_0' \end{aligned}\right.$

$$M_{0L} = \begin{bmatrix} \cos\psi_L - \frac{w_0 w_0'}{k}\sin\psi_L & \frac{w_0^2}{k}\sin\psi_L \\ -\left(\frac{k}{w_0^2} + \frac{w_0'^2}{k}\right)\sin\psi_L & \cos\psi_L + \frac{w_0 w_0'}{k}\sin\psi_L \end{bmatrix}$$

$$= \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix}$$

So $\psi_L = \mu$

$$\beta = \frac{w^2}{k} \qquad \alpha = \frac{-w w'}{k} = -\frac{\beta'}{2}$$

$$\beta\gamma - \alpha^2 = 1$$

$$\psi' = \frac{k}{w^2} = \frac{1}{\beta} \Rightarrow \psi = \int \frac{ds}{\beta}$$

$\nu \equiv \mu/2\pi$ Betatron Tune

$$x = x_0\, w(s) \cos(\psi + \delta)$$
$$= A\sqrt{\beta}\, \cos(\psi + \delta)$$

$$w'' + K w - \frac{k^2}{w^3} = 0$$

$$w = \sqrt{k}\sqrt{\beta} \qquad w' = \sqrt{k}\, \frac{1}{2\sqrt{\beta}}\, \beta'$$
$$= -\frac{\sqrt{k}}{\sqrt{\beta}}\, \alpha$$

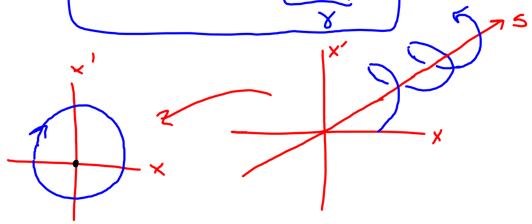
$$w'' = \frac{\sqrt{k}\, \beta''}{2\sqrt{\beta}} - \frac{\sqrt{k}\, \beta'^2}{4\beta^{3/2}}$$

$$\left\{ \frac{\sqrt{k}\, \beta''}{2\sqrt{\beta}} - \frac{\sqrt{k}\, \beta'^2}{4\beta^{3/2}} + K\sqrt{k}\sqrt{\beta} - \frac{k^2}{k^{3/2}\beta^{3/2}} = 0 \right\} \times \frac{4\beta^{3/2}}{\sqrt{k}}$$

$$\boxed{2\beta\beta'' - \beta'^2 + 4K\beta^2 - 4 = 0}$$ Envelope Eqn.

or $$K\beta - \frac{(2\alpha)^2}{4\beta} - \alpha' - \frac{1}{\beta} = 0$$

$$\boxed{\alpha' = K\beta - \underbrace{\frac{\alpha^2 + 1}{\beta}}_{\gamma}}$$



$$w = \sqrt{k}\,\sqrt{\beta}\ ,\quad w' = \sqrt{k}\,\frac{1}{2}\frac{\beta'}{\sqrt{\beta}} = -\sqrt{\frac{k}{\beta}}\,\alpha$$

$$M_{0s} = \begin{bmatrix} \sqrt{\frac{\beta}{\beta_0}}\cos\psi + \sqrt{\frac{\beta}{\beta_0}}\,\alpha_0 \sin\psi & \sqrt{\beta\beta_0}\,\sin\psi \\ \left(\frac{\alpha_0}{\sqrt{\beta\beta_0}} - \frac{\alpha}{\sqrt{\beta\beta_0}}\right)\cos\psi - \left(\frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\right)\sin\psi & \sqrt{\frac{\beta_0}{\beta}}\cos\psi - \sqrt{\frac{\beta_0}{\beta}}\,\alpha \sin\psi \end{bmatrix}$$

Emittance: Area of phase space ellipse for particle undergoing betatron oscillations.

(I) $x = A\sqrt{\beta}\cos(\psi+\delta)$

$$x' = A\frac{1}{2\sqrt{\beta}}\beta'\cos(\psi+\delta) - A\sqrt{\beta}\,\psi'\sin(\psi+\delta)$$

$$= -\frac{\alpha x}{\beta} - A\sqrt{\beta}\,\frac{1}{\beta}\sin(\psi+\delta)$$

(II) $A\sqrt{\beta}\sin(\psi+\delta) = -(\alpha x + \beta x')$

$$(I)^2 + (II)^2 = A^2\beta = x^2 + (\alpha x + \beta x')^2$$

$$\boxed{\frac{x^2 + (\alpha x + \beta x')^2}{\beta} = A^2 \text{ constant}}$$

Define Emittance $= \epsilon = \pi A^2$

How do I show emittance is an area?

$p_x \equiv x' + \frac{\alpha}{\beta}x$ Then

$$\frac{x^2}{\beta} + \beta p_x^2 = \frac{\epsilon}{\pi}$$

$\frac{x^2}{a^2} + \frac{p^2}{b^2} = 1$
$\Rightarrow$ Area $= \pi ab$

$$\frac{x^2}{(\beta\epsilon/\pi)} + \frac{p_x^2}{(\epsilon/\beta\pi)} = 1 \Rightarrow \text{Area} = \pi\left(\frac{\beta\epsilon}{\pi}\right)^{1/2}\left(\frac{\epsilon}{\beta\pi}\right)^{1/2}$$

$$= \epsilon$$

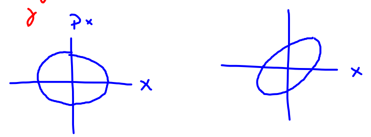
Area in x-p_x space = emittance

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\alpha}{\beta} & 1 \end{pmatrix}\begin{pmatrix} x \\ p_x \end{pmatrix}$$

$$\text{Area in } x\text{-}x' \text{ space} = \underbrace{\begin{vmatrix} 1 & 0 \\ \frac{\alpha}{\beta} & 1 \end{vmatrix}}_{=1} \times \text{Area in } x\text{-}p_x \text{ space}$$

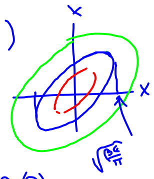
In x-x' space

$$x^2\underbrace{\frac{1+\alpha^2}{\beta}}_{\gamma} + 2\alpha x x' + \beta x'^2 = \frac{\epsilon}{\pi}$$



Admittance:

$$x(s) = \sqrt{\frac{\epsilon}{\pi}\beta(s)}\cos(\psi + \delta)$$



$$x_{max}(s) = \sqrt{\frac{\beta\epsilon}{\pi}}$$

Say beam pipe has radius $a(s)$

$$\epsilon_{max}(s) = \frac{\pi x_{max}^2}{\beta} \leq \frac{\pi a^2(s)}{\beta(s)}$$

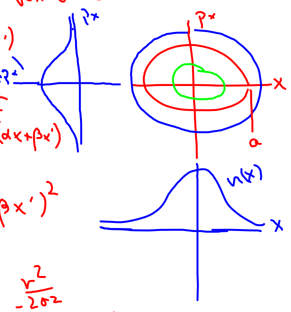
$$\epsilon_{max}(\text{overall}) = \text{Min}\left(\frac{\pi a^2(s)}{\beta_s}\right)$$

Beam Emittances: (Gaussian)

$$n(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2} dx$$

$$n(\alpha x + \beta x')\,d(\alpha x + \beta x')$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\alpha x + \beta x')^2}{2\sigma^2}} d(\alpha x + \beta x')$$



$$r^2 = x^2 + (\alpha x + \beta x')^2$$

$$= \frac{\beta\epsilon}{\pi}$$

$$n(r)\,dr = \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r\,dr\,d\theta \rightarrow 2\pi$$

$$n(r)\,dr = \frac{1}{\sigma^2} e^{-r^2/2\sigma^2} r\,dr$$

$$\int_0^a n(r)\,dr = -e^{-r^2/2\sigma^2}\Big|_0^a = 1 - e^{-a^2/2\sigma^2}$$

$= F(a) =$ fraction of beam inside a

$$1 - e^{-\frac{\left(a^2 = \frac{\beta\epsilon}{\pi}\right)}{2\sigma^2}} = F(a)$$

$$e^{-\frac{\beta\epsilon}{2\pi\sigma^2}} = 1 - F(a)$$

$$\boxed{\epsilon = -\frac{2\pi\sigma^2}{\beta}\ln(1 - F(a))}$$