Dispersion: $x'' + Kx = f(s)$

Last time: $G(s,s') = S(s)C(s') - C(s)S(s')$

$f(s) = \delta/R$ for dispersion.

$$\mathcal{D}(s) = \int_0^s G(s,s')f(s')\,ds'$$

$$\mathcal{D}''(s) + K(s)\mathcal{D}(s) = f(s)$$

$$\mathcal{D}(0) = \mathcal{D}'(0) = 0$$

$\mathcal{D}$ is one particular solution of Hill's (Inhomogeneous) Eqn.

General Solution:

$$x = x_0 C(s) + x_0' S(s) + \mathcal{D}(s)$$

Can find periodic solution:

$$x_L = x_0 = x_0 C(L) + x_0' S(L) + \mathcal{D}(L)$$

$$x_L' = x_0' = x_0 C'(L) + x_0' S'(L) + \mathcal{D}'(L)$$

If $x'' + Kx = \frac{1}{\rho}$ (set $f(s) = \frac{\delta}{\rho}$ with $\delta = 1$)

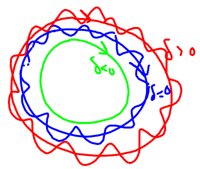
Then periodic solution is call $D(s)$

and is called dispersion function.

Since $D'' + KD = 1/\rho$

can write any solution as

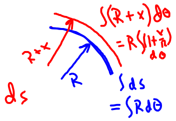
$$x = a\,C(s) + b\,S(s) + \delta D(s)$$



Remember in longitudinal motion : $\frac{\Delta L}{L} \equiv \frac{1}{\gamma_T^2}\frac{\delta p}{p_0} = \frac{\delta}{\gamma_T^2}$

$\mathcal{C}(\delta=0) = \oint ds$

$\mathcal{C}(\delta) = \oint \left(1+\frac{x}{R}\right) ds$



$= \oint ds + \int \frac{x}{R} ds$

$= \mathcal{C}(\delta=0) + \oint \frac{aC(s)+bS(s)+\delta D(s)}{R} ds$

Oscillatory - can ignore

$\mathcal{C}(\delta) - \mathcal{C}(\delta=0) =$

$\delta \oint \frac{D(s)}{R} ds$

$$\frac{\Delta \mathcal{C}}{\mathcal{C}} = \frac{\delta \oint \frac{D(s)}{R} ds}{\oint ds} = \alpha\,\delta$$

momentum compaction factor

$$\frac{\Delta \mathcal{C}}{\mathcal{C}} = \frac{1}{\gamma_T^2}\delta \Rightarrow \frac{1}{\gamma_T^2} = \frac{\oint \frac{D(s)}{R} ds}{\oint ds}$$

$$\int n(x, x', \delta)\, dx\, dx'\, d\delta = 1$$

Say I have a function $h(x, x', \delta)$

Then I can talk about average

$$\langle h \rangle \equiv \int h(x, x', \delta)\, n(x, x', \delta)\, dx\, dx'\, d\delta$$

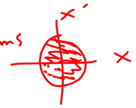
$x = x_\beta + \delta D$ where $x_\beta = aC(s) + bS(s)$

Also $x_\beta = \sqrt{\frac{\varepsilon\beta}{\pi}} \cos(\psi+\delta)$

$x_\beta^2 = \frac{\varepsilon\beta}{\pi}\cos^2(\psi+\delta)$

$\langle x_\beta^2 \rangle = \frac{\beta}{\pi}\langle \varepsilon \cos^2(\psi+\delta)\rangle$

$\langle \cos^2 \rangle = 1/2$ most beams



$$\langle x_\beta^2 \rangle = \frac{\beta}{\pi}\langle \varepsilon \underbrace{\cos^2(\psi+\delta)}_{1/2} \rangle$$

$$= \frac{\beta}{2\pi}\langle \varepsilon \rangle$$

$$\int dx\,dx'\,d\delta\; n(x,x',\delta)\,\varepsilon(x,x')$$

$$\langle x^2 \rangle = \langle (x_\beta + \delta D)^2 \rangle$$

$$= \langle x_\beta^2 + 2\delta x_\beta D + \delta^2 D^2 \rangle$$

$$= \langle x_\beta^2 \rangle + 2\underbrace{\langle \delta x_\beta \rangle}_{\text{uncorrelated} \to 0} D + \langle \delta^2 \rangle D^2$$

$$= \frac{\beta}{2\pi}\langle \varepsilon \rangle + \langle \delta^2 \rangle D^2$$

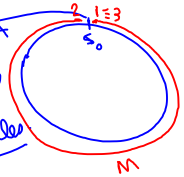
Errors Changing Closed Orbit:

① $\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$

② $\begin{pmatrix} x \\ x' \end{pmatrix} = M \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$

③ $\begin{pmatrix} x \\ x' \end{pmatrix}_3 = \begin{pmatrix} x \\ x' \end{pmatrix}_2 + \begin{pmatrix} 0 \\ \Theta \end{pmatrix}$

Error here that changes x' by Θ for all particles



$$= M\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Theta \end{pmatrix}$$

Closed Orbit: $\begin{pmatrix} x \\ x' \end{pmatrix}_3 = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = M\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Theta \end{pmatrix}$

$$\begin{pmatrix} 0 \\ \Theta \end{pmatrix} = (I - M)\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M)^{-1}\begin{pmatrix} 0 \\ \Theta \end{pmatrix}$$

$M = e^{\mu J}$
$= e^{2\pi\nu J}$
where
$\mu = 2\pi\nu$
ν = Tune
$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$

$$e^{\mu J} = I\cos\mu + J\sin\mu$$

$$e^{\mu_1 J} e^{\mu_2 J} = (I\cos\mu_1 + J\sin\mu_1) \times (I\cos\mu_2 + J\sin\mu_2)$$

$$= I(\cos\mu_1\cos\mu_2 - \sin\mu_1\sin\mu_2) + J(\cos\mu_1\sin\mu_2 + \sin\mu_1\cos\mu_2)$$

$$= I\cos(\mu_1+\mu_2) + J\sin(\mu_1+\mu_2) = e^{(\mu_1+\mu_2)J}$$

Example: Sector Bend

$$x'' + \frac{1}{R^2}x = \frac{\delta}{R}$$

$$y'' = 0$$

$$C(s) = \cos\frac{s}{R} \qquad S = R\sin\frac{s}{R}$$

$$G(s,s') = S(s)C(s') - C(s)S(s')$$
$$= R\left(\sin\frac{s}{R}\cos\frac{s'}{R} - \cos\frac{s}{R}\sin\frac{s'}{R}\right)$$

$$D(s) = \int_0^s R\left(\sin\frac{s}{R}\cos\frac{s'}{R} - \cos\frac{s}{R}\sin\frac{s'}{R}\right)\frac{\delta}{R}\,ds'$$
$$= \delta\sin\frac{s}{R}\int_0^s \cos\frac{s'}{R}\,ds' - \delta\cos\frac{s}{R}\int_0^s \sin\frac{s'}{R}\,ds'$$
$$= \delta\sin\frac{s}{R}\,R\sin\frac{s'}{R}\Big|_0^s + \delta\cos\frac{s}{R}\,R\cos\frac{s'}{R}\Big|_0^s$$
$$= \delta R\left(\underbrace{\sin^2\frac{s}{R} + \cos^2\frac{s}{R}}_{1} - \cos\frac{s}{R}\right)$$
$$= \delta R\left(1 - \cos s/R\right)$$

$$x = \underline{x_0}\cos\frac{s}{R} + \underline{x_0'}\,R\sin\frac{s}{R} + \underline{\delta}\,R\left(1-\cos\frac{s}{R}\right)$$

$$x' = -\frac{1}{R}\underline{x_0}\sin\frac{s}{R} + \underline{x_0'}\cos\frac{s}{R} + \underline{\delta}\sin\frac{s}{R}$$

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} \cos s/R & R\sin s/R & R(1-\cos\frac{s}{R}) \\ -\frac{1}{R}\sin s/R & \cos s/R & \sin s/R \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_0$$

matrix for hom. eqn.   dispers.

Note: $s = L$ for sector bend matrix