Space Charge:

Beam Distributions $n(x,y,s,t) = \lambda(s,t)\,\rho(x,y)$ $= \lambda(s,t) \times \rho(r,\theta)$

4 Cases: ① Uniform circular cylinder

$$n = \begin{cases} \lambda(s,t)\,\frac{1}{\pi a^2} & (r<a) \\ 0 & (r>a) \end{cases}$$

② 

uniform elliptical cylinder

$$n = \begin{cases} \lambda(s,t)\,\frac{1}{\pi ab} & \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}<1\right) \\ 0 & \text{otherwise} \end{cases}$$

Note: a, b, σ, etc can be fns of s.

③ Circular Gaussian Distribution

$$n = \lambda(s,t)\,\frac{1}{2\pi\sigma^2}\,e^{-r^2/2\sigma^2}$$

④ Elliptical Gaussian Distribution

$$n = \lambda(s,t)\,\frac{1}{2\pi\sigma_a\sigma_b}\,e^{-\frac{x^2}{2\sigma_a^2}-\frac{y^2}{2\sigma_b^2}}$$

work out fields and forces:

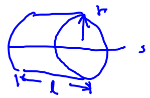
① Uniform Circular $(\vec{E}+\vec{v}\times\vec{B})_\perp =$

$$\vec{E}_\perp/\gamma^2 = \frac{q\lambda}{2\pi\varepsilon_0\gamma^2}\begin{cases} \vec{r}/a^2 & r<a \\ \vec{r}/r^2 & r>a \end{cases}$$

② Uniform Ellipse

$$(\vec{E}+\vec{v}\times\vec{B})_\perp = \vec{E}_\perp/\gamma^2 = \frac{q\lambda}{\pi\varepsilon_0\gamma^2}\left[\frac{x\hat{x}}{a(a+b)} + \frac{y\hat{y}}{b(a+b)}\right]$$

③ Gaussian Beam



$$\int \vec{E}\cdot\vec{ds} = \frac{q}{\varepsilon_0}\int n\,dV$$

$$2\pi r l E_r = \frac{q}{\varepsilon_0}\int \frac{\lambda}{2\pi\sigma^2}\,e^{-\frac{r^2}{2\sigma^2}}\,ds\,r\,dr\,d\theta$$

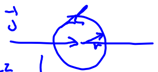
$$= \frac{q}{\varepsilon_0}\frac{\lambda}{2\pi}\,l\,2\pi\int_0^r e^{-\frac{r^2}{2\sigma^2}}\,d\left(\frac{r^2}{2\sigma^2}\right)$$

$$= \frac{q\lambda l}{\varepsilon_0}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$E_r = \frac{q\lambda}{2\pi\varepsilon_0 r}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$E_\theta = 0 \quad \text{(cylindrical symmetry)}$$

$$B_r = 0 = B_s$$



$$\oint \vec{B}\cdot\vec{dl} = \mu_0\int \vec{J}\cdot\vec{ds}$$

$$2\pi r B_\theta = \mu_0\int_0^r q\beta c\,\frac{\lambda}{2\pi\sigma^2}\,e^{-\frac{r^2}{2\sigma^2}}\,r\,dr\,d\theta$$

$\vec{J} = \rho\vec{v}$ $= qn\vec{v}$ $= qn\beta c\hat{s}$

$$= \frac{\mu_0 c\lambda\beta q}{2\pi}\,2\pi\int_0^r e^{-\frac{r^2}{2\sigma^2}}\,d\left(\frac{r^2}{2\sigma^2}\right)$$

$\mu_0 c = \frac{1}{\varepsilon_0 c}$

$$= \frac{\lambda\beta q}{\varepsilon_0 c}\left(1-e^{-r^2/2\sigma^2}\right)$$

$$B_\theta = \frac{\beta}{c}\,\frac{q\lambda}{2\pi\varepsilon_0 r}\left(1-e^{-r^2/2\sigma^2}\right) = \frac{\beta}{c}E_r$$

$$\vec{E}+\vec{v}\times\vec{B} = \hat{r}E_r + \beta c\hat{s}\times\frac{\beta}{c}E_r\hat{\theta}$$

$$= \hat{r}E_r(1-\beta^2) = \vec{E}_\perp/\gamma^2$$

Summary:

Uniform Circle: $\vec{F}_\perp^{sc} = \dfrac{q^2\lambda}{2\pi\varepsilon_0\gamma^2}\dfrac{\vec{r}}{a^2}$

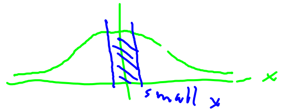
Uniform Ellipse: $\vec{F}_\perp^{sc} = \dfrac{q^2\lambda}{\pi\varepsilon_0\gamma^2}\left\{\dfrac{x\hat{x}}{a(a+b)} + \dfrac{y\hat{y}}{b(a+b)}\right)$

Gaussian (Circular): $\vec{F}_\perp^{sc} = \dfrac{q^2\lambda}{2\pi\varepsilon_0\gamma^2}\dfrac{\hat{r}}{r}\left(1-e^{-\frac{r^2}{2\sigma^2}}\right)$

for $r \ll \sigma$ $\quad 1-e^{-r^2/2\sigma^2} = 1 - 1 + \dfrac{r^2}{2\sigma^2} + \ldots$

$= r^2/2\sigma^2$

for $r \ll \sigma$ $\quad \vec{F}_\perp^{sc} = \dfrac{q^2\lambda}{\pi\varepsilon_0\gamma^2}\dfrac{\vec{r}}{4\sigma^2}$



Tranverse Eq Motion

$x'' + Kx = 0$ where K was force due to magnets (external fields)

$\dfrac{d\vec{p}}{dt} = \vec{F}^{ext} \approx \gamma\beta^2 m c^2 x''$

$\dfrac{d\vec{p}}{dt} = \dfrac{d}{dt}(\gamma m\vec{v}) \sim \dfrac{d}{dt}\left(\gamma m \dfrac{dx}{dt}\right)$

$\dfrac{d}{dt} \sim \beta c \dfrac{d}{ds} \Rightarrow \dfrac{d\vec{p}}{dt} \sim \beta c \dfrac{d}{ds}\left(\gamma m \beta c \dfrac{dx}{ds}\right)$

$\sim \gamma\beta^2 c^2 m x''$

$x'' + Kx = \dfrac{F_x^{sc}}{\gamma\beta^2 m c^2}$

No space charge $\Rightarrow x'' + Kx = 0$

Uniform focusing model $K = \text{constant}$
$= \langle K\rangle$

$x = A\cos\sqrt{K}s + B\sin\sqrt{K}s$

$s \to s + L$, phase advance

$\mu = \sqrt{K}L = 2\pi\nu_0$

$K = \left(\dfrac{2\pi\nu_0}{L}\right)^2$

ν_0 called "bare" tune means no collective effects

$x'' + \left(\dfrac{2\pi\nu_0}{L}\right)^2 x = \dfrac{F_x^{sc}}{\gamma\beta^2 m c^2}$

$= \dfrac{1}{\gamma\beta^2 m c^2}\dfrac{q^2\lambda}{2\pi\varepsilon_0\gamma^2}\dfrac{x}{a^2}$ (uniform circle

$= \dfrac{\lambda}{\gamma^3\beta^2}\underbrace{\dfrac{q^2}{4\pi\varepsilon_0 m c^2}}_{\text{classical particle radius}}\dfrac{2x}{a^2}$

$x'' + \left(\dfrac{2\pi\nu_0}{L}\right)^2 x = \dfrac{\lambda r_0}{\gamma^3\beta^2}\dfrac{2}{a^2}x$

$$x'' + \left(\frac{2\pi\nu_0}{L}\right)^2 x = \frac{\lambda r_0}{\gamma^3\beta^2}\frac{2}{a^2}\, x$$

$$x'' + \left(\frac{2\pi\nu}{L}\right)^2 x = x'' + \left(\frac{2\pi}{L}(\nu_0+\Delta\nu)\right)^2 x$$

$$= 0$$

$$x'' + \left[\left(\frac{2\pi}{L}\right)^2 \nu_0^2 - \frac{\lambda r_0\, 2}{\gamma^3\beta^2 a^2}\right] x = 0$$

$$\left(\frac{2\pi}{L}\right)^2\left[\nu_0^2 - \left(\frac{L}{2\pi}\right)^2 \frac{2r_0\lambda}{\gamma^3\beta^2a^2}\right] = \left(\frac{2\pi}{L}\right)^2(\nu_0+\Delta\nu)^2$$

Assume $\Delta\nu \ll \nu_0$

Neglect

$$\nu_0^2 - \left(\frac{L}{2\pi}\right)^2 \frac{2r_0\lambda}{\gamma^3\beta^2a^2} = \nu_0^2 + 2\nu_0\Delta\nu + \Delta\nu^2$$

$$\boxed{\Delta\nu = -\left(\frac{L}{2\pi}\right)^2 \frac{r_0\lambda}{\gamma^3\beta^2a^2\nu_0}}$$

Space Charge Tune Shift for uniform circular beam.

Homework:

① Problem 6.1

② Calculate Time Shift for

SNS: Assume

$N = 1.5 \times 10^{14}$ protons

$L = 248$ m

$B = 2.0$ (bunch Factor)$^{-1}$

$\lambda = \frac{NB}{L}$

$E_0 = 1.938$ GeV

$V_0 = 6.2$

$a = 3$ cm