Injection and extraction - optics

USPAS

by

Mike Plum and Uli Wienands

Units

Energy and momentum in accelerators is usually expressed in units of "electron Volts":

$1\ \mathrm{eV} = 1.602 \times 10^{-19}$ Joules

We will use energy units:

keV $= 10^3$ eV
MeV $= 10^6$ eV
GeV $= 10^9$ eV

Similarly, the units of momentum, p, are eV/c.
And finally, for mass, the units are eV/c^2. For instance

m_p = mass proton = 938 MeV/c^2
m_e = mass electron = 511 keV/c^2

Relativity

In most accelerators, particles move at relativistic speeds, and therefore we need to use relativistic mechanics to describe particle motion and fields.

Einstein's Special Theory of Relativity:

1) The laws of physics apply in all inertial (non-accelerating) reference frames.
2) The speed of light in vacuum is the same for all inertial observers.

Notice that (1) does not mean that the answer to a physics calculation is the same in all inertial reference frames. It only means that the physics law's governing the calculation are the same.

Relativistic factors

The factors γ and β are commonplace in most relativistic equations:

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$

In fact, the total energy of a particle (sum of kinetic and rest energy), is given by:

$$E = mc^2 = \gamma m_o c^2 = T + m_o c^2$$

For accelerators, it is often convenient to find γ using the kinetic energy, T, of a particle:

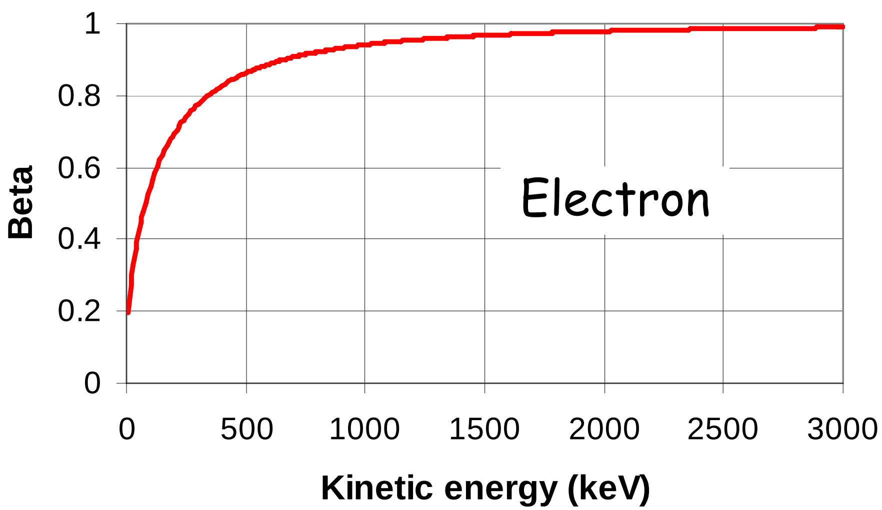
$$T = m_o c^2(\gamma - 1) \Rightarrow \gamma = 1 + \frac{T}{m_o c^2}$$

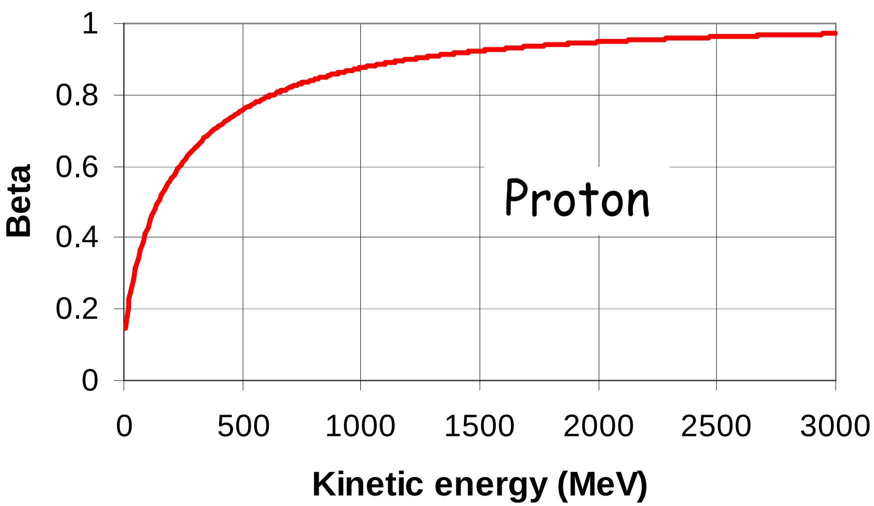
And finally, for the relationship between momentum and energy, we have:

$$\left.\begin{aligned} E &= \gamma m_o c^2 \\ p &= \gamma m_o v = \gamma m_o \beta c \end{aligned}\right\} \Rightarrow cp = \beta E$$

Electrons vs. protons

The β factor is commonly used as a measure of the speed of a particle. As a particle is accelerated, β increases asymptotically towards 1 (speed of light), but never gets there:





- Heavy particles become relativistic at higher energies
- No particle with finite mass can travel at the speed of light in vacuum (β=1)

Maxwell's equations

In accelerators, we typically use electric fields to accelerate particles and magnetic fields to focus particles. The standard equations used to describe the fields are Maxwell's equations:

(For vacuum or "well-behaved" materials)

CGS

$$\nabla \bullet E = \frac{4\pi\rho}{\varepsilon_r}$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial}{\partial t}B$$

$$\nabla \bullet B = 0$$

$$\nabla \times \frac{B}{\mu_r} = \frac{4\pi}{c}J + \frac{\varepsilon_r}{c}\frac{\partial}{\partial t}E$$

MKS

$$\nabla \bullet E = \frac{\rho}{\varepsilon}$$

$$\nabla \times E = -\frac{\partial}{\partial t}B$$

$$\nabla \bullet B = 0$$

$$\nabla \times B = \mu J + \mu\varepsilon\frac{\partial}{\partial t}E$$

$\varepsilon = \varepsilon_r \varepsilon_0, \quad \varepsilon_o =$ permativity of free space

$\mu = \mu_r \mu_o, \quad \mu_0 =$ permeability of free space

The Lorentz Force Equation

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt}$$

A force is the change in momentum with respect to time.

For a charged particle passing through an E or B field the force is governed by the ***Lorentz Force Equation***:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Force from the electric field is in the direction of E

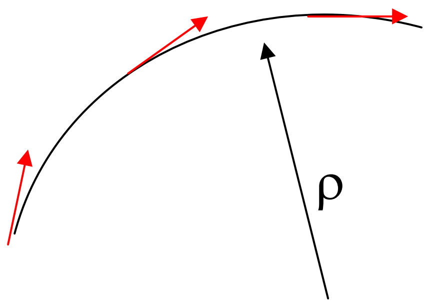
Force from the magnetic field is perpendicular to the direction of v and B, as given by the "Right Hand Rule"

Right Hand Rule for **a = b x c** : Point your right fingers in the direction of **b**, and curl your fingers toward the direction of **c**. Your thumb will point in the direction of **a**.

Accelerator Coordinate Systems

In general, the accelerator will be designed (shaped) to give a "reference trajectory" for particle travel. This reference trajectory usually includes a number of bends:

One axis points in the direction of the reference trajectory at any point (tangent to the reference path).



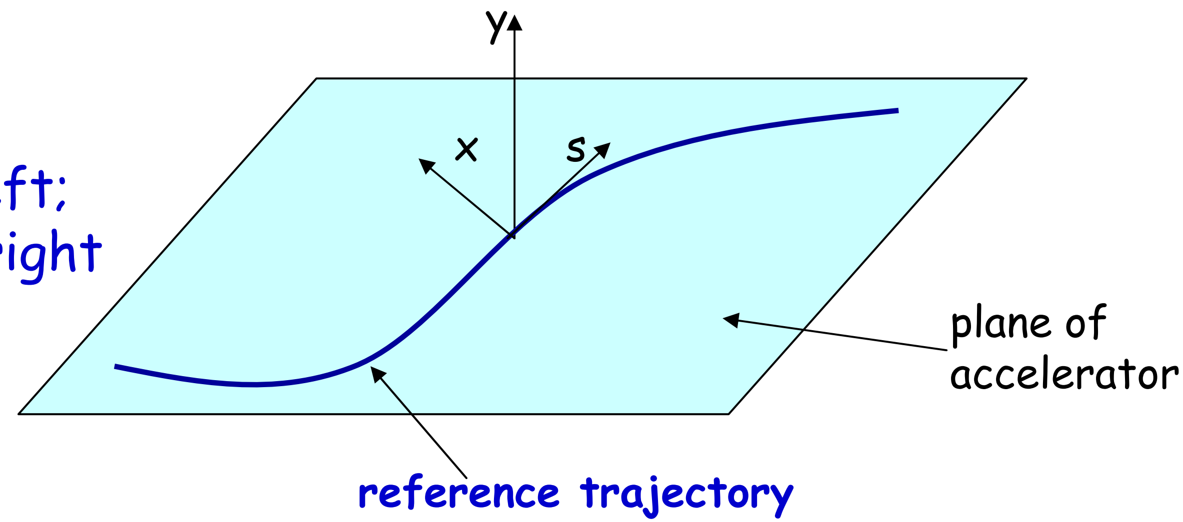
In beam physics, we are generally interested in deviations from the reference trajectory. Therefore it is most convenient to place the coordinate system origin on the reference trajectory, and align one axis of the axes with the reference trajectory.

Accelerator Coordinate Systems (cont.)

In this *curvilinear coordinate system*, the direction of all axes change along the reference path.

Of the two remaining axes, we choose the horizontal axis to go in the plane of the reference trajectory, and the vertical axis to be perpendicular to this plane. This is convenient since most accelerators are laid out entirely in a horizontal plane.

As we look in the direction of s, positive x is to the left; and positive y is up (right handed coordinate system)



Phase Space and Units

In transverse particle dynamics, we are concerned with the effect of an external magnetic field on the phase space coordinates of a particle or beam. The phase space coordinates are called (u, u'), where (u, u') can be either (x, x') or (y, y').

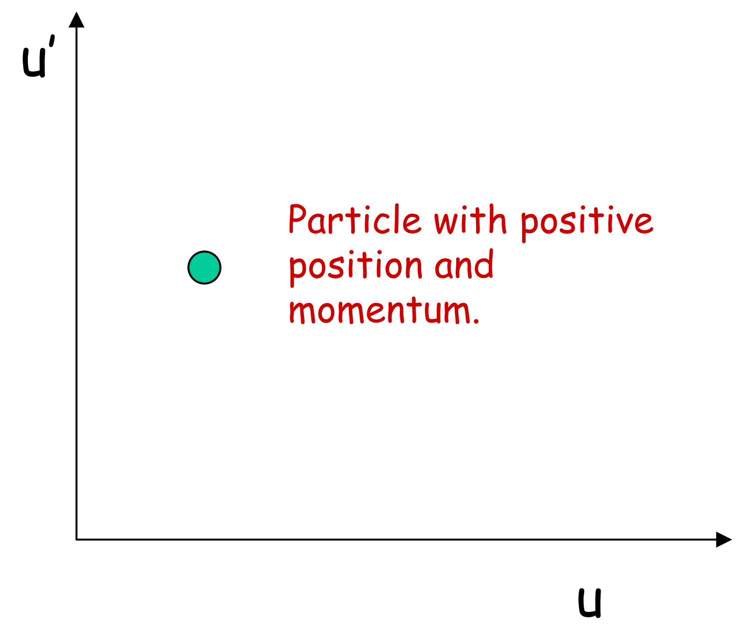
Coordinates and units:

s distance on reference trajectory, [m], ("time" coordinate)

$u \ (x \text{ or } y)$ position [meters]

$u' = \dfrac{du}{ds} \ \left(\dfrac{dx}{ds} \text{ or } \dfrac{dy}{ds}\right)$ transverse momentum [radians]

$u'' = \dfrac{d^2u}{ds^2}$ transverse acceleration [m^{-1}]



What does the phase space path of harmonic oscillator look like? (Mass on a pendulum, child on a swing, etc)?

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Hill's equation

- For linear beam optics (dipoles, quadrupoles, and drifts), the motion of a particle about the reference trajectory can be described by Hill's equation:

$$x'' + K_x(s)x = 0$$

$$x' \equiv dx/ds \qquad K_x \equiv B'/(B\rho) + \rho^{-2} = k_0 + \rho^{-2} \quad B' \equiv \partial B_y / \partial x$$

$$B\rho = \frac{mv}{q} = \frac{p}{q} \quad = \text{magnetic rigidity}$$

Hill's equation

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s) + \delta)$$

$$x'(s) = -\frac{A}{\sqrt{\beta_x(s)}}\left[\alpha(s)\cos(\psi(s)+\delta) + \sin(\psi(s)+\delta)\right]$$

where A and δ are constants, and $\alpha(s) \equiv -\beta'(s)/2$

Solution to Hill's equation

Equation for y is the same – just interchange all x's and y's. Constants will be different.

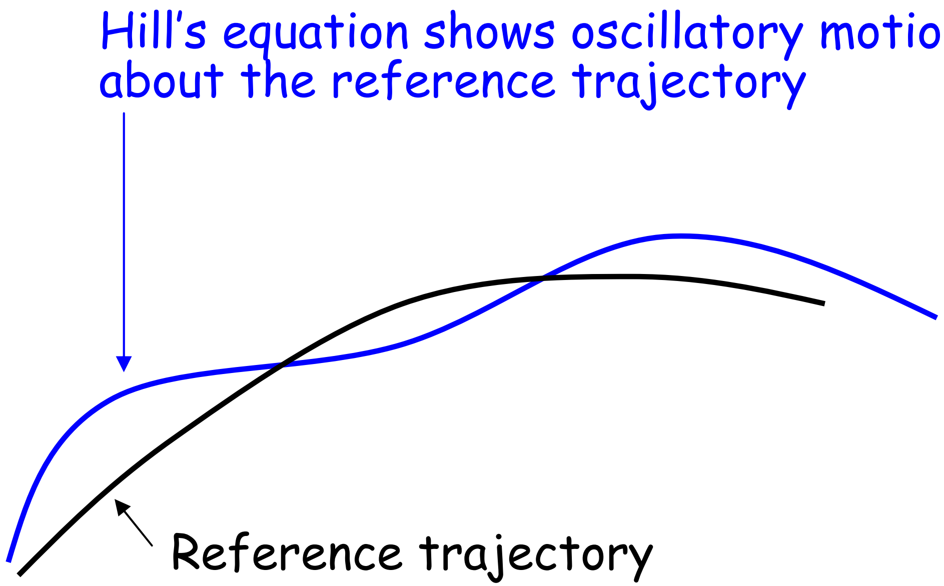
Note: the β(s) on this slide is not the relativistic β factor! There is unfortunate clash in terms in accelerator physics, and you just have to get used to it! Context determines the correct interpretation.

Hill's equation (cont.)

$$x(s) = A\sqrt{\beta_x(s)}\cos(\psi(s)+\delta)$$

$$x'(s) = -\frac{A}{\sqrt{\beta_x(s)}}\left[\alpha(s)\cos(\psi(s)+\delta)+\sin(\psi(s)+\delta)\right]$$

where A and δ are constants, and $\alpha(s) \equiv -\beta'(s)/2$



A Closer Look at Hill's Equation

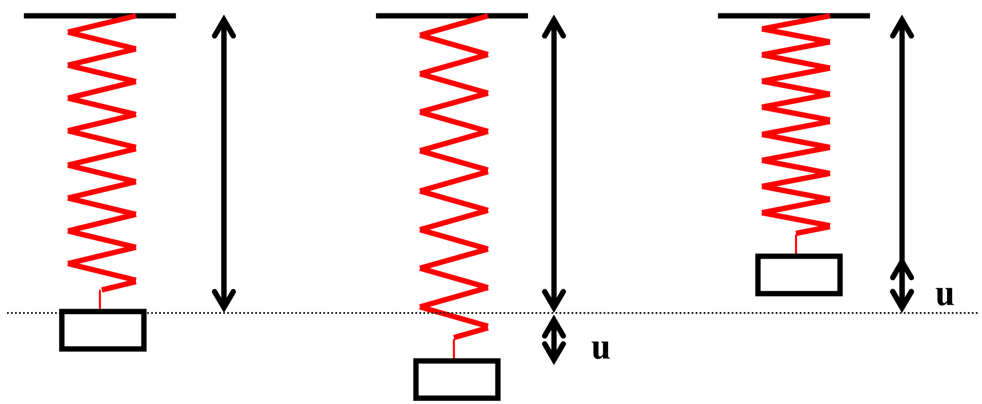
What does it tell us? Look at the form in general.

$$u'' + K(s)u = 0$$

✓ Particle motion about the reference trajectory is caused by dipoles and quadrupoles, whose strength varies with s.

✓ If k_o and ρ are constant with s (or vary slowly), the motion is harmonic. Therefore we won't be surprised later to find that the motion has a "frequency"… The total motion, with s-dependence, is "quasi-harmonic".

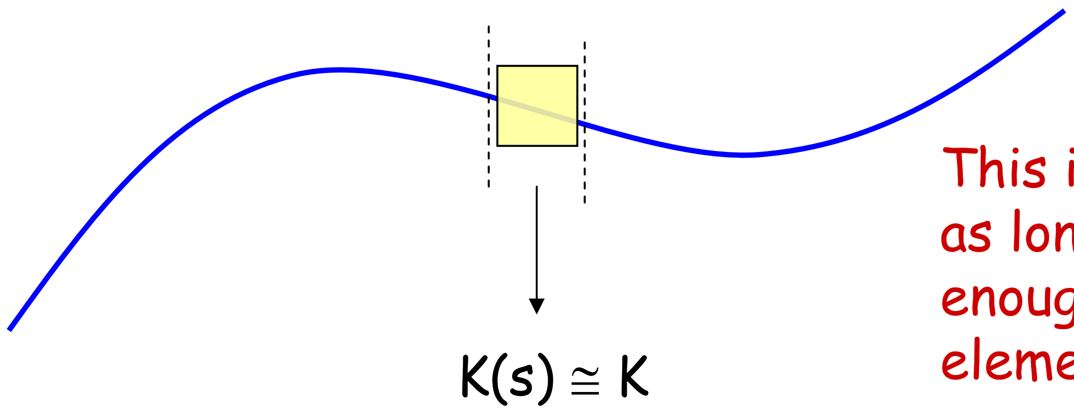
✓ The equation acts like a spring with "spring constant" or restoring force, K(s), which changes over the length of the accelerator.



Hill's equation (cont.)

The variable $K_x(s)$ complicates the normal solution to the harmonic oscillator equation. To make life a little easier, lets consider the transfer matrix for only one piece of the accelerator.

We can approximate that relatively short piece of the accelerator by $K(s) = K$, a constant:



This is a good approximation as long as we pick small enough pieces (<= 1 lattice element)

Hill's equation becomes: $$u'' + Ku = 0$$

And this is a problem we know how to solve!

Hill's equation (cont.)

So, solve the piece-wise constant Hill's equation with appropriate initial conditions:

Solve: $u'' + Ku = 0$ with initial conditions: $u(0) = u_o;\ u'(0) = u_o'$

The solution is: $$u(s) = C(s)u_o + S(s)u_o'$$

<u>K>0:</u>

$$C(s) = \cos(\sqrt{K}s)$$

$$S(s) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}s)$$

<u>K<0:</u>

$$C(s) = \cosh(\sqrt{|K|}s)$$

$$S(s) = \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s)$$

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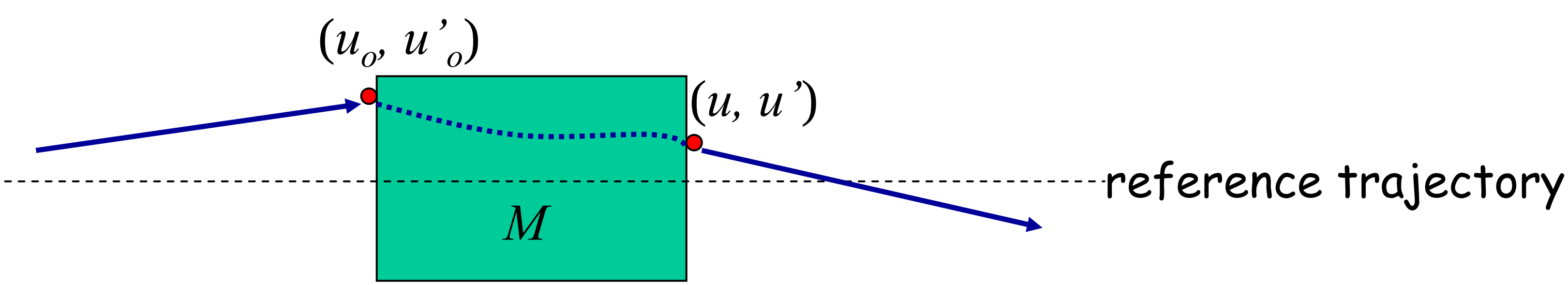
Matrix Representation of Motion

It is convenient to write the transport equation as a matrix:

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \underbrace{\begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}}_{M} \begin{pmatrix} u_o \\ u_o' \end{pmatrix} = M \begin{pmatrix} u_o \\ u_o' \end{pmatrix}$$



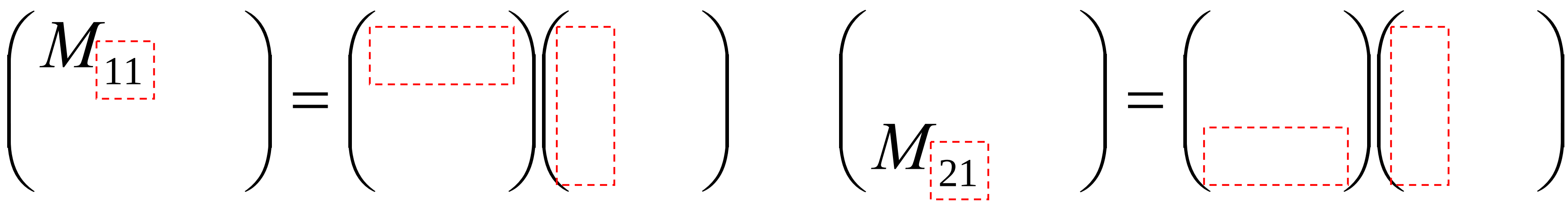
Quick Review of Matrix Multiplication

Suppose we multiply two matrices, M1 and M2. The (row m, column n) element of the final matrix is the vector dot product of the m row from M1 and the n column from m2:

$M_{n,m}$=(row n from M1) x (column m from M2)

Example: M_{11}=(row 1 from M1) x (column 1 from M2)

M_{21}=(row 2 from M1) x (column 1 from M2)

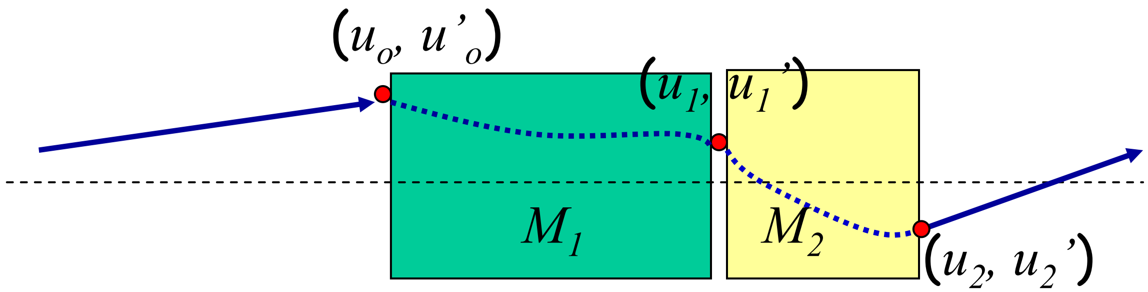


$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

In general, matrix multiplication is not commutative: M1 x M2 $\neq$ M2 x M1 .
The order of multiplication is very important!

Piece-wise Constant Transport: Two Elements

The matrix representation is very convenient. For instance, what if we had two consecutive elements, with strengths K_1 and K_2? What is the final equation of transport for a particle through both elements?



The solution for the first element becomes the initial condition for the second element…

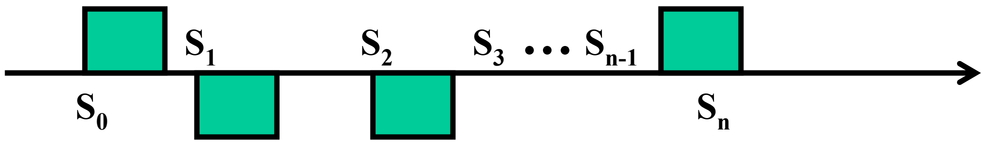
First, $\begin{pmatrix} u_1 \\ u_1' \end{pmatrix} = M_1 \begin{pmatrix} u_o \\ u_o' \end{pmatrix}$, then, $\begin{pmatrix} u_2 \\ u_2' \end{pmatrix} = M_2 \begin{pmatrix} u_1 \\ u_1' \end{pmatrix} = M_2 \left(M_1 \begin{pmatrix} u_o \\ u_o' \end{pmatrix} \right) = M_2 M_1 \begin{pmatrix} u_o \\ u_o' \end{pmatrix}$

And finally, we have $\begin{pmatrix} u_2 \\ u_2' \end{pmatrix} = M(s_2|s_0) \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$, where $M(s_2|s_1) = M_2 M_1$

Piece-wise Constant Transport: N Elements

For an arbitrary number of transport elements, each with a constant, but different K_n, we have:

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \ldots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \mathcal{M}(s_1|s0)$$



from $\mathbf{s_0}$ to $\mathbf{s_1}$

from $\mathbf{s_0}$ to $\mathbf{s_2}$

from $\mathbf{s_0}$ to $\mathbf{s_3}$

from $\mathbf{s_0}$ to $\mathbf{s_n}$

$$\Rightarrow \begin{pmatrix} u_n \\ u_n' \end{pmatrix} = M(s_n|s_o) \begin{pmatrix} u_o \\ u_o' \end{pmatrix}$$

Thus by breaking up the parameter K(s) into piece-wise constant chunks, K(s)={K_1, K_2, … K_n}, we have found a useful method for finding the particle transport equation through a long piece of beam line with many elements.

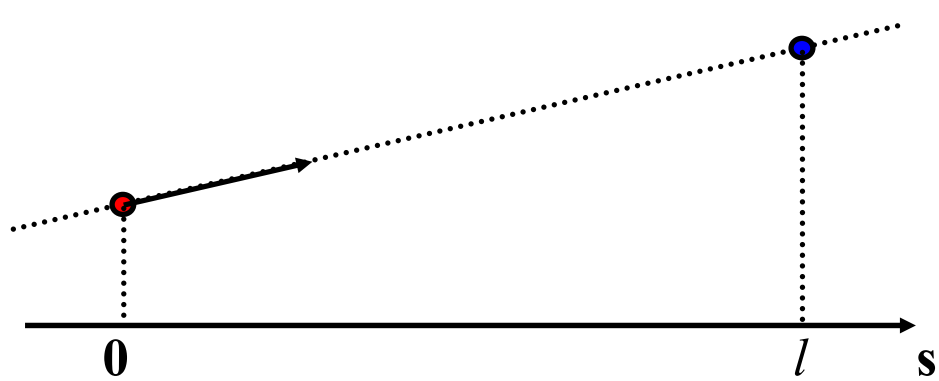
Transport through a Drift

In a drift space, with no B-fields, we take the limit of M as K-> 0.

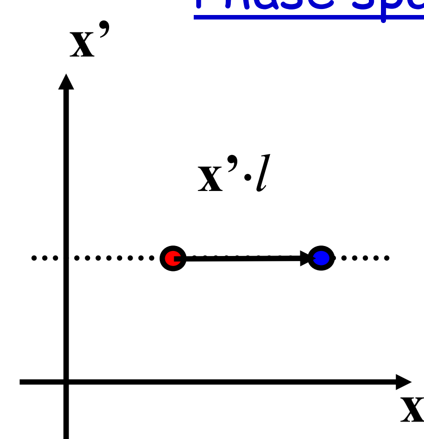
$$M_{drift} = \begin{pmatrix} \cos(\sqrt{K}\,l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,l) \\ -\sqrt{K}\sin(\sqrt{K}\,l) & \cos(\sqrt{K}\,l) \end{pmatrix} \xrightarrow{K=0} M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_o \\ u_o' \end{pmatrix} \qquad \begin{aligned} u &= u_o + l u_o' \\ u' &= u_o' \end{aligned}$$

Real space (s, x)



Phase space (x, x')



Transport through a Quadrupole

In the case of a pure quadrupole, there is no bending so the only remaining term is the quad strength term.

$$K = k_o + \frac{1}{\rho^2} \xrightarrow{\rho=\infty} k_o \qquad k_o[\mathrm{m}^{-2}] = 0.2998 \frac{g[\mathrm{Tesla/m}]}{\beta\, \mathrm{E[GeV]}} \frac{Z}{A}$$

<u>Focusing:</u>

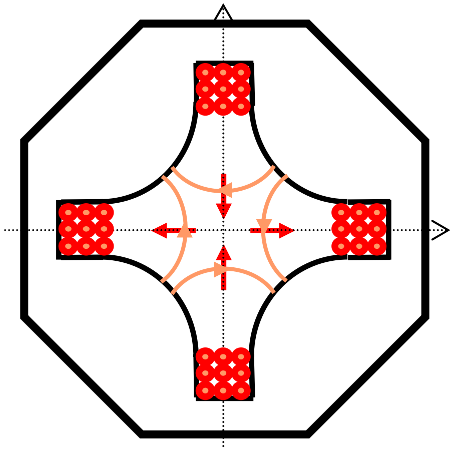
$$\boxed{M_{QF} = \begin{pmatrix} \cos(\sqrt{k_o}\, l) & \frac{1}{\sqrt{k_o}} \sin(\sqrt{k_o}\, l) \\ -\sqrt{k_o}\, \sin(\sqrt{k_o}\, l) & \cos(\sqrt{k_o}\, l) \end{pmatrix}}$$

<u>Defocusing:</u>

$$\boxed{M_{QD} = \begin{pmatrix} \cosh(\sqrt{|k_o|}\, l) & \frac{1}{\sqrt{|k_o|}} \sinh(\sqrt{|k_o|}\, l) \\ \sqrt{|k_o|}\, \sinh(\sqrt{|k_o|}\, l) & \cosh(\sqrt{|k_o|}\, l) \end{pmatrix}}$$

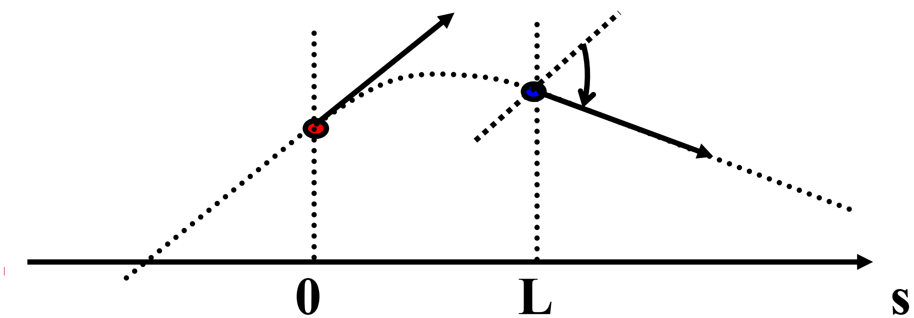
Finite length quad transport (cont.)

The quadrupole has finite length, l. The angle is changed through the length, and the position as well. For instance, for k>0:

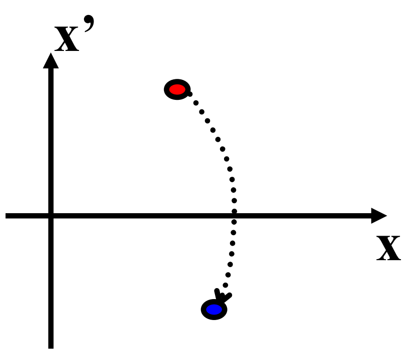


$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k_o}\,l) & \frac{1}{\sqrt{k_o}}\sin(\sqrt{k_o}\,l) \\ -\sqrt{k_o}\sin(\sqrt{k_o}\,l) & \cos(\sqrt{k_o}\,l) \end{pmatrix} \begin{pmatrix} x_o \\ x_o' \end{pmatrix}$$

Real space (s, x):



Phase space (x, x'):



Thin Lens Approximation for a Quadrupole

In the "thin lens approximation", we let the length of the quadrupole approach zero while holding the focal length constant: $l \to 0$ as $1/f = kl$ = constant.

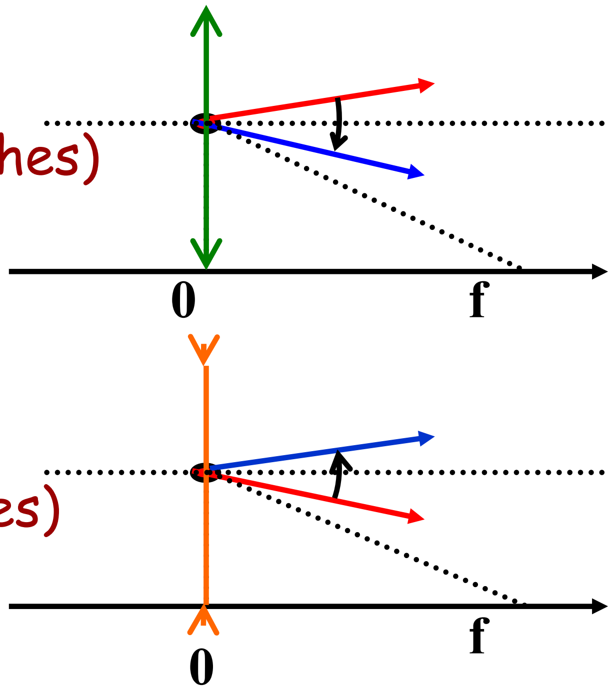
$$M_{Quad} = \begin{pmatrix} 1 & 0 \\ \mp \frac{1}{f} & 1 \end{pmatrix}$$

In this approximation, the position remains fixed, but the angle changes:

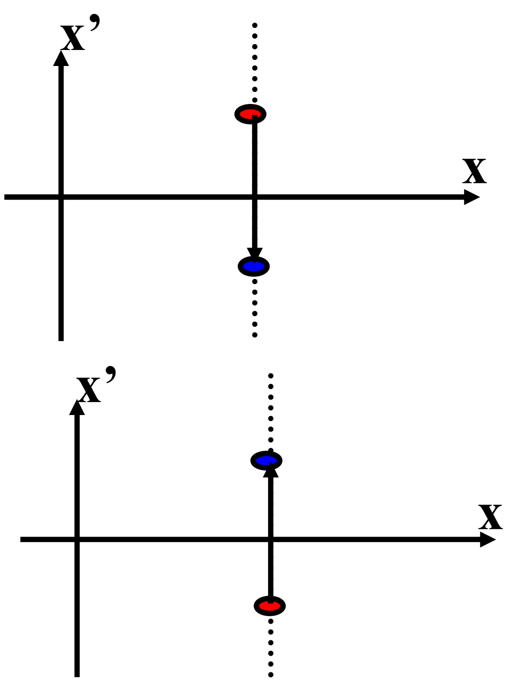
Real space (s, x):

Focusing:
(slope diminishes)

Defocusing:
(slope increases)



Phase space (x, x'):



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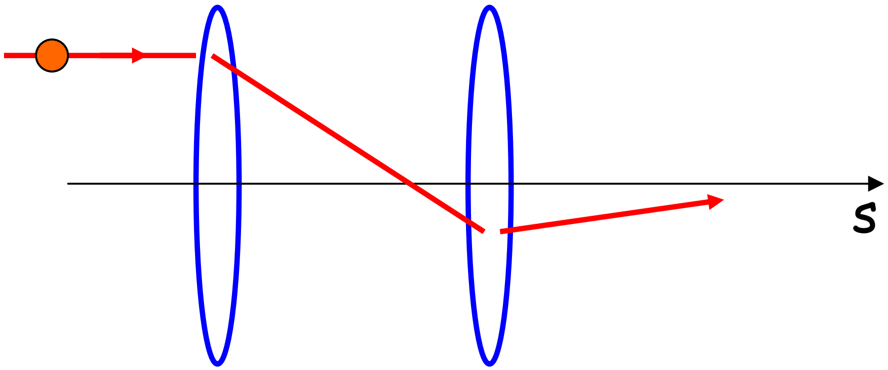
Example: A Quadrupole Focusing Doublet

Let's consider a horizontally focusing quadrupole doublet sequence – FOF - separated by a drift L, in the thin lens approximation:



Answer:

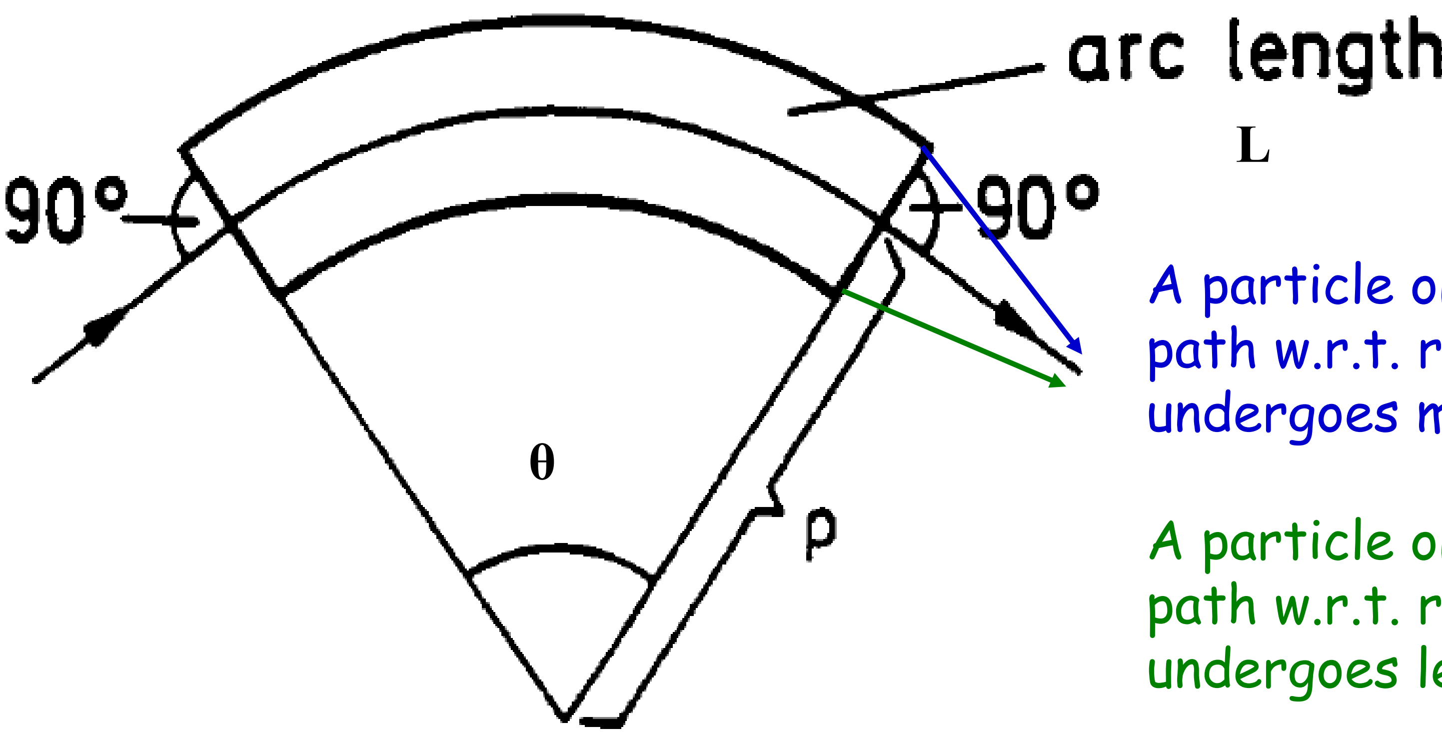
$$M_{\text{Doublet}} = \begin{pmatrix} 1 - \frac{L}{f_1} & L \\ -\frac{1}{f^*} & 1 - \frac{L}{f_2} \end{pmatrix}$$

$$\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$$ is the total focal length of the system.

Why don't we use sequences of …FOFOFOFO… magnets to create lattices in an accelerator?

Focusing in a Sector Dipole

The axis of a sector dipole usually corresponds to the reference trajectory. In the plane of the bend, off-axis particles are focused by the dipole, and the thus the $1/\rho^2$ term which shows up in Hill's equation: $K=k+1/\rho^2$



A particle on an exterior path w.r.t. reference undergoes more bending.

A particle on an interior path w.r.t. reference undergoes less bending.

Transport in Pure Sector Dipole

Here we take the quad strength k, to be zero, k=0. In the deflecting plane, i.e, the plane of the bend (usually horizontal), we have:

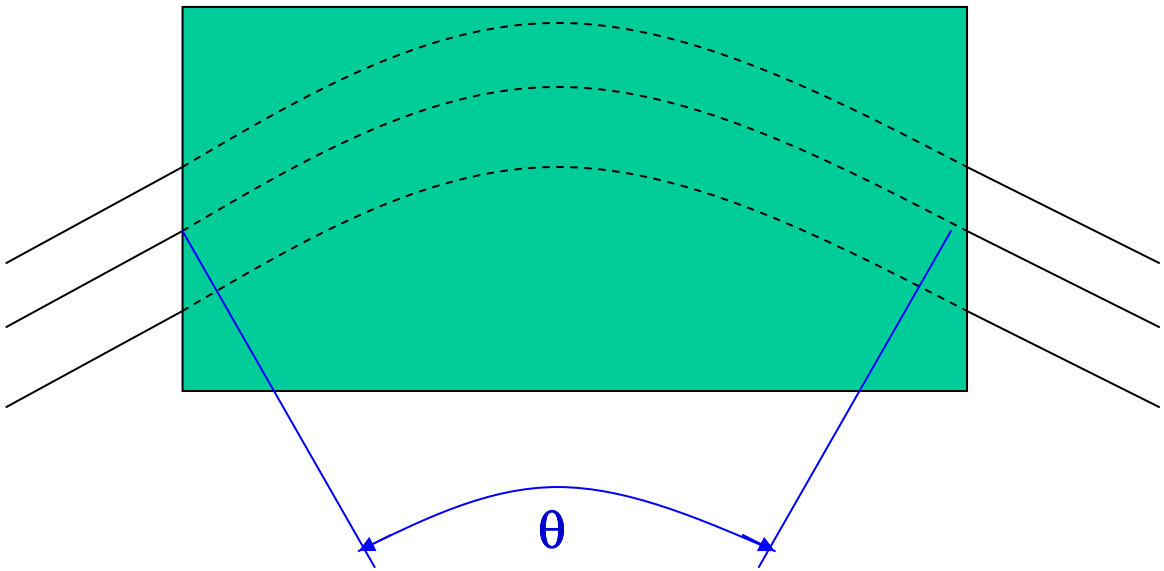
$$M_{\text{x,sector}} = \begin{pmatrix} \cos(\frac{l}{\rho}) & \rho\sin(\frac{l}{\rho}) \\ -\frac{1}{\rho}\sin(\frac{l}{\rho}) & \cos(\frac{l}{\rho}) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \rho\sin(\theta) \\ -\frac{1}{\rho}\sin(\theta) & \cos(\theta) \end{pmatrix}$$

And in the non-deflecting plane, $\rho \to 0$, and we are left with a drift:

$$M_{\text{y,sector}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

Transport in Rectangular Dipoles

In a rectangular dipole, the particle path in the horizontal direction is the same for all trajectories, so there is no focusing in the horizontal direction.



$$M_{\text{x,rect}} = \begin{pmatrix} 1 & \rho \sin\theta \\ 0 & 1 \end{pmatrix}$$

In the horizontal direction the magnet transform like a drift with length equal to $\rho \sin \theta$

The Problem of Real Beam Distributions

So far we have learned how to write the transport equations for a single particle in a beam line.

The problem: In a real machine, we rarely have any information about a single particle!

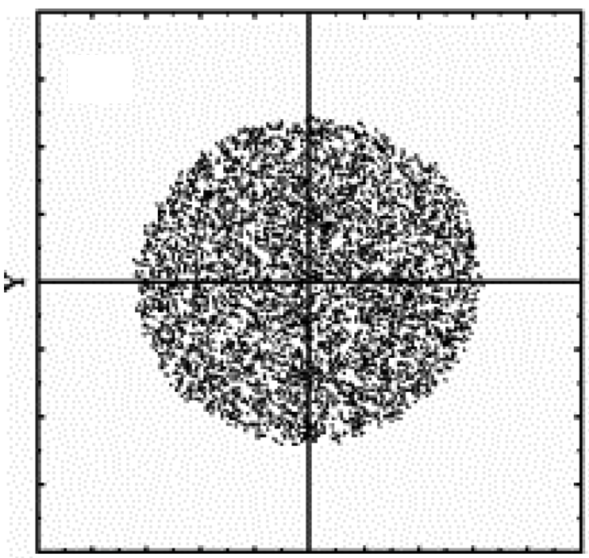
What we do know is some information about the entire beam, i.e., the ensemble of all of the particles. We could do separate transport equations for each particle in the bunch (PIC simulations do something like this)... Very impractical for analytic work!

Solution:
We parameterize the entire particle distribution, and write the transport equations for the parameters. Thus we can write transport equations for the whole beam, not just one particle!

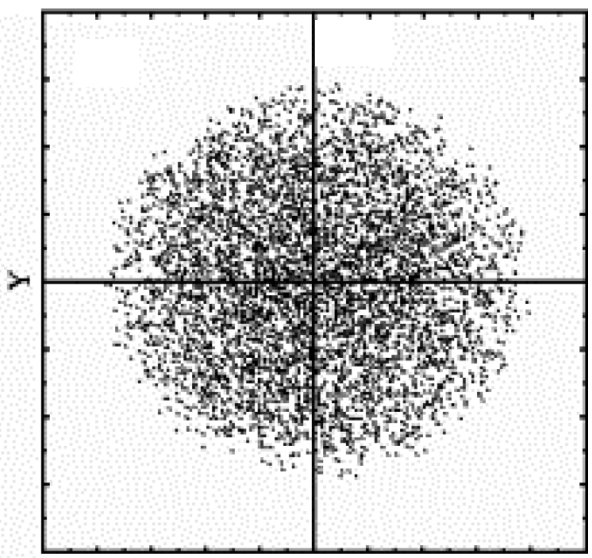
Real Beam Distributions

In reality, real beam distributions are not uniform in phase space, and actually, in practice it can be difficult to locate the beam edge.

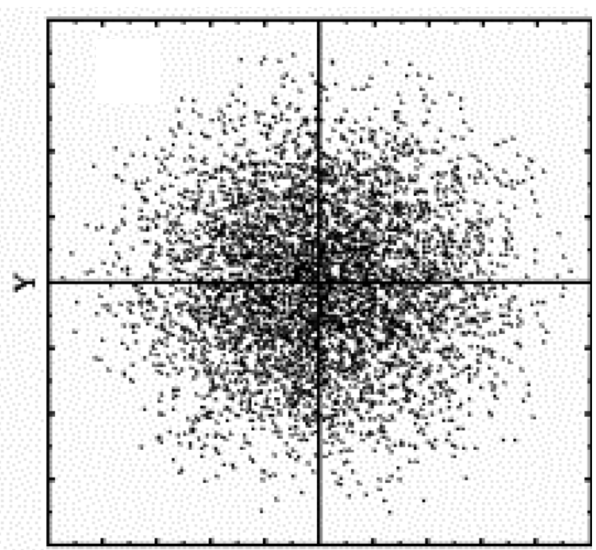
"KV"



"Waterbag"



Gaussian



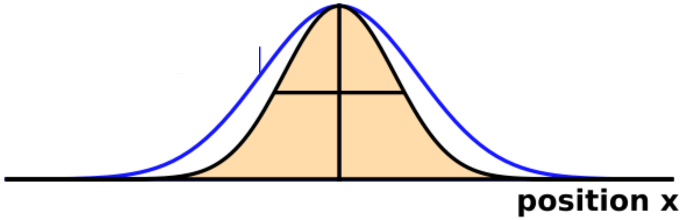
RMS Quantities

Most often, we will deal with RMS quantities. The RMS beam size with N particles is defined as:

$$u_{\mathrm{RMS}} = \sqrt{\frac{1}{N}\sum_{i}^{N}(u_i - u_{\mathrm{ave}})^2}$$

For a Gaussian beam, the rms beam size is just the "sigma" of the beam. To include the tails of the beam we often talk about apertures that are "five sigma", or "seven sigma", etc.

For most common distributions, the RMS represents a % which lies inside this bound. For example, for a 1D Gaussian, 68.3% of particles lie within ±1 RMS.



The beam ellipse Twiss parameters

A good approximation for the beam shape in phase space is an ellipse. Any ellipse can be defined by specifying:

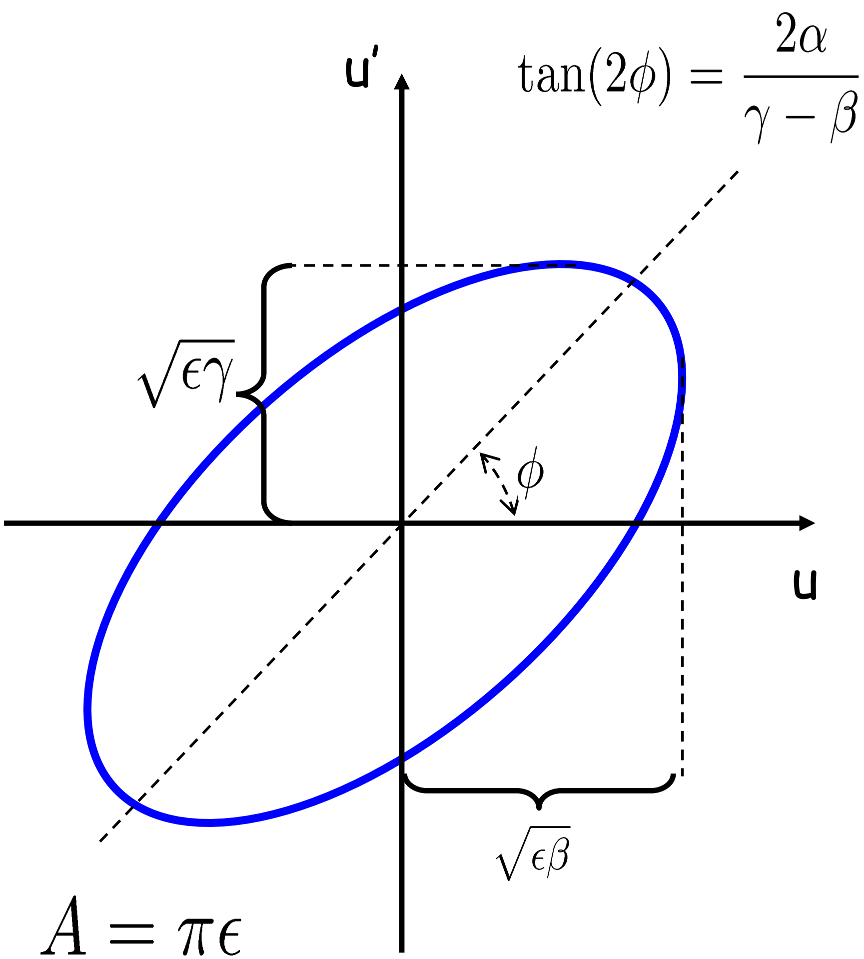
- ✓ Area
- ✓ Shape
- ✓ Orientation

We choose 4 parameters – 3 independent, 1 dependent:

α - related to beam tilt
β - related to beam shape, width
ε - related to beam area
γ - dependent on the other 3.

These are the "Twiss Parameters" (or "Courant-Snyder Parameters")

Beam Ellipse in Phase Space:



Transverse Beam "Emittance"

The equation for the beam ellipse, with our Twiss parameterization can be written as:

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

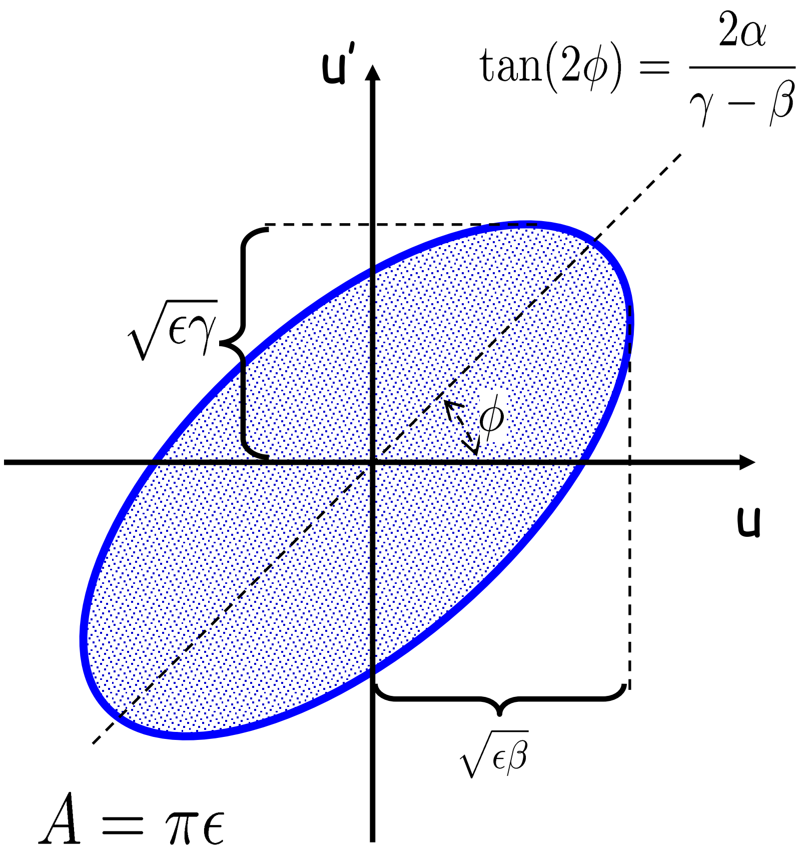
And the ellipse has area:

$$\int dx'\,dx = \pi\varepsilon$$

$$\varepsilon = \text{beam emittance}$$

The beam emittance is the phase space area of the beam (to within π). Emittance is a parameter used to gauge beam quality.

Beam Ellipse in Phase Space:



RMS Quantities

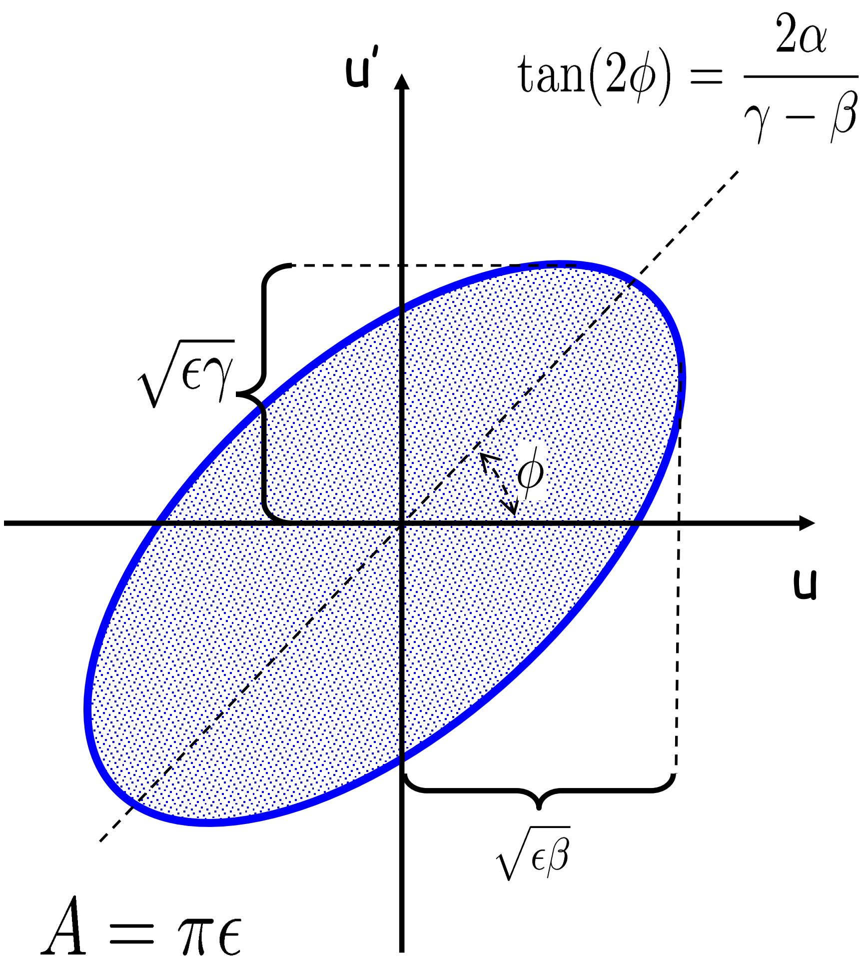
We can relate our Twiss Parameters for the beam to RMS quantities, as well:

$$\beta = \frac{u_{\mathrm{RMS}}^2}{\varepsilon_{\mathrm{RMS}}}$$

$$\gamma = \frac{u'^2_{\mathrm{RMS}}}{\varepsilon_{\mathrm{RMS}}}$$

$$\alpha = \frac{(uu')_{\mathrm{RMS}}}{\varepsilon_{\mathrm{RMS}}}$$

Beam Ellipse in Phase Space:



RMS beam size

$$u_{\text{RMS}} = \sqrt{\beta \varepsilon_{\text{RMS}}}$$

Example:

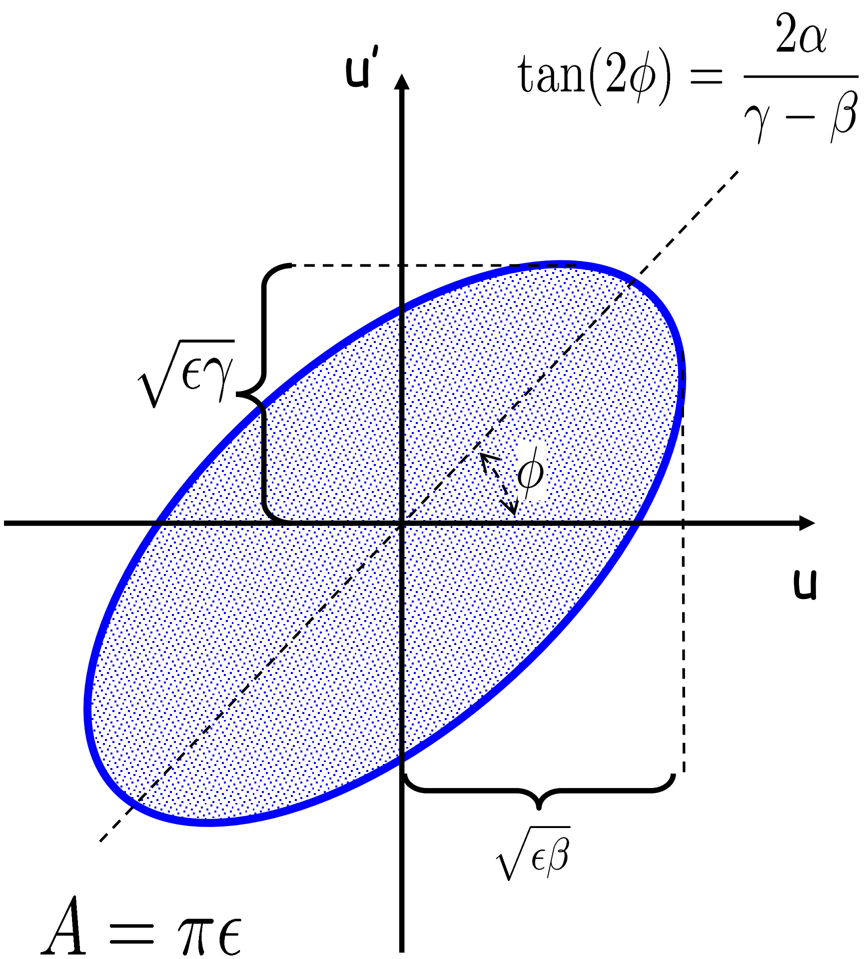
$$\beta = 10\ \text{m}$$

$$\varepsilon_{RMS} = 3.6\,\pi\ \text{mm mrad}$$

$$u_{\text{RMS}} = \sqrt{(10\ \text{m})(3.6 \cdot 10^{-6}\ \text{m rad})} = 6\ \text{mm}$$

Note: for the conventions used in this course, leave off the π in the emittance number when calculating beam parameters

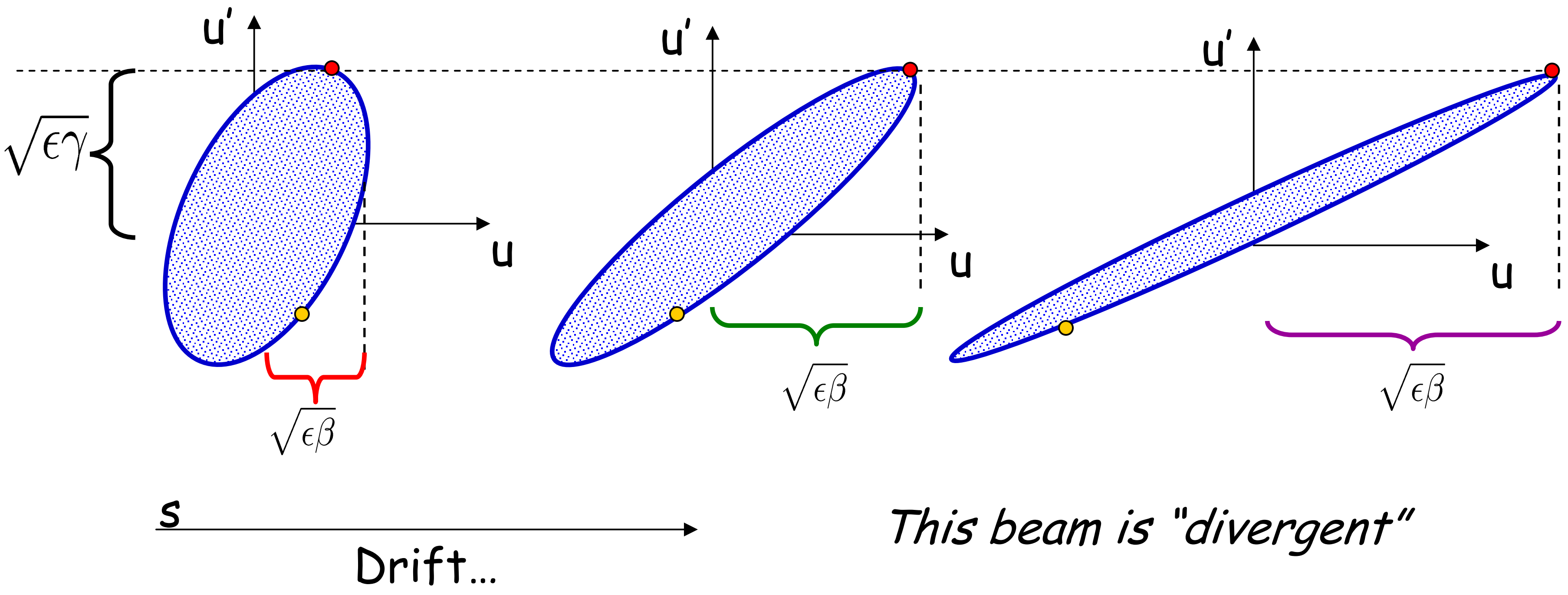
Beam Ellipse in Phase Space:



The Beam Ellipse in a Drift

How does this "beam ellipse" transform through a drift space?

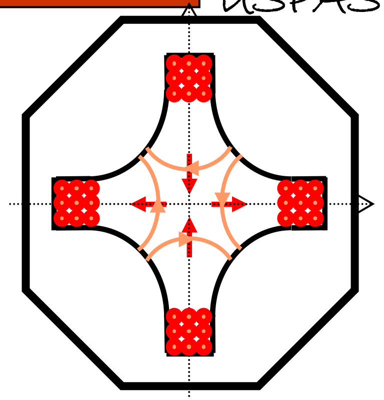
Drift: $u=u_o+u_o'l$



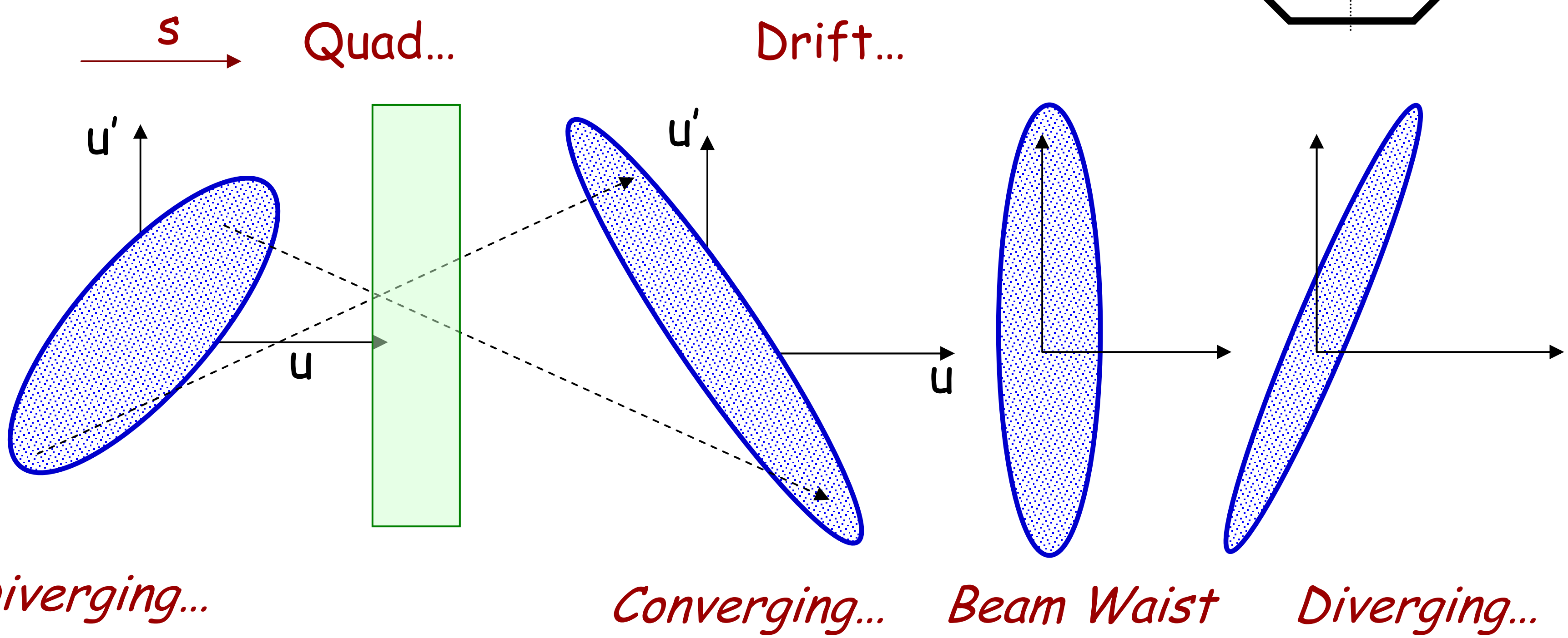
This beam is "divergent"

Analogous with a single particle, u is increasing while u' remains fixed.
Observation: Without focusing, any beam would spread out…

The Beam Ellipse in a Quadrupole



Recall that for a (focusing) quadrupole, the force of the *kick is opposite to the sign of the particle's position, and proportional to the distance from the axis.* So for a distribution of particles:



The quadrupole causes a diverging beam to converge. In reality, the scenario is more complicated because we focus in one plane while defocusing in the other.

Transporting Twiss Parameters

According to *Louiville's Theorem*, the phase space area of the beam does not change under linear transformations (i.e. just dipoles and quadrupoles). This means the beam emittance is conserved in a linear transport system.

For our homogenous Hill's equation, the emittance between two points is conserved, regardless of the change in beam shape and orientation.

With this fact, we find that for a piece-wise constant lattice, the Twiss parameters transform as:

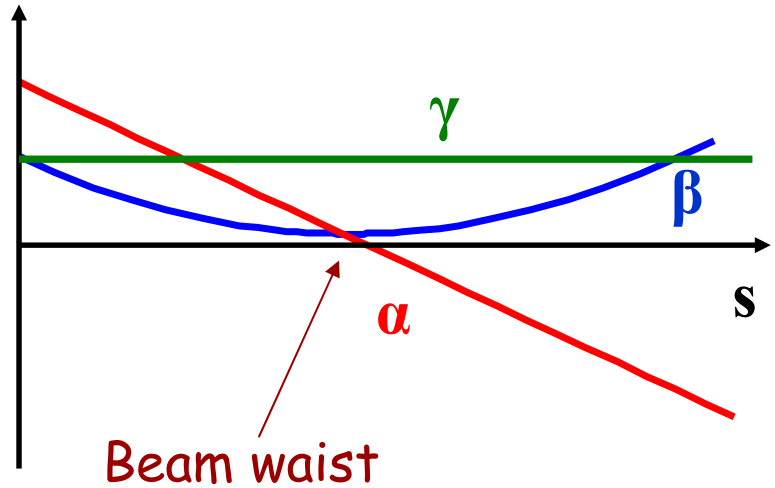
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = M_{twiss} \begin{pmatrix} \beta_o \\ \alpha_o \\ \gamma_o \end{pmatrix}$$

Twiss Parameters through a Drift

$$M_{\text{twiss, drift}} = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix}$$

Recall that $\sqrt{\beta\varepsilon}$ is a measure of beam size. So clearly, the beam size always eventually grows in the absence of focusing.

$$\beta = \beta_o - 2l\alpha_o + l^2\gamma_o$$
$$\alpha = \alpha_o - l\gamma_o$$
$$\gamma = \gamma_o$$



We will attach more physical meaning to these parameters soon!

*

Some more propagation matrices

$$M_{\text{twiss, drift}} = \begin{pmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{pmatrix}$$

Drift of length l

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = M_{twiss} \begin{pmatrix} \beta_o \\ \alpha_o \\ \gamma_o \end{pmatrix}$$

$$M_{\text{twiss, quad}} = \begin{pmatrix} 1 & 0 & 0 \\ \pm\frac{1}{f} & 1 & 0 \\ \frac{1}{f^2} & \pm\frac{2}{f} & 1 \end{pmatrix}$$

Quad of
focal length f

$$M_{\text{twiss, dipole}} = \begin{pmatrix} \cos^2\theta & -\rho\sin 2\theta & \rho^2\sin^2\theta \\ \frac{1}{2\rho}\sin 2\theta & \cos 2\theta & -\frac{1}{2}\rho\sin 2\theta \\ \frac{1}{\rho^2}\sin^2\theta & \frac{1}{\rho}\sin 2\theta & \cos^2\theta \end{pmatrix}$$

Sector bending
magnet of radius
ρ and angle θ

Recap:

- ✓ We found the equation of motion, w.r.t. the reference trajectory
- ✓ We solved the equation of motion for the case of K=constant.
- ✓ We learned how to represent the solution in matrix form.
- ✓ We learned how to transport a particle through an arbitrarily long piece-wise constant lattice, by multiplying the individual transport matrices in the right order.
- ✓ We parameterized the entire distribution of particles using Twiss parameters.
- ✓ We learned how to transport the Twiss parameters – and therefore the shape and orientation of the beam - through a piece-wise constant lattice.

- We would like to thank Sarah Cousineau and Stuart Henderson, who supplied many of these slides from their USPAS course. Some of their slides were based on USPAS slides from Y. Papaphilippou, N.Catalan-Lasheras, F. Sannibale, and D. Robin

- Extra slides

Stability Condition

Most of the time we deal with lattices where the arrangements of magnets repeat themselves, i.e., periodic systems.

Q: How do we know if our lattice produces stable particle motion?
A: After finding the composite transport Matrix, M, of the lattice, we can find a "stability condition" on this matrix:

If: $$M_{\text{Lattice}} = M_n M_{n-1} M_{n-2} \ldots M_2 M_1$$

Stability condition: $$\left| Tr(M_{\text{Lattice}}) \right| < 2$$

Where "Tr(M)" means the "Trace" of the matrix, which is the sum of the diagonal elements. And for N repititions of this lattice sequence, we generalize to:

$$\left| Tr(M_{Lattice}^{N}) \right| < 2$$

What is the stability condition for a FODO lattice?

Stability Condition for a FODO Lattice (f1=-f2)

USPAS

The stability condition for a FODO lattice is found by taking the trace and applying the stability condition. So, for the thin lens approximation of a FODO cell with equal focal length quadrupoles:

Transfer matrix:

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 - 2\frac{L^2}{f^2} & \ldots \\ \ldots & 1 - 2\frac{L^2}{f^2} \end{pmatrix}$$

Stability condition:

$$\left| Tr\left(M_{FODO}\right)\right| = \left| 2 - 4\frac{L^2}{f^2} \right| < 2$$

Result for FODO:

$$\boxed{0 < \frac{L}{f} < 1}$$

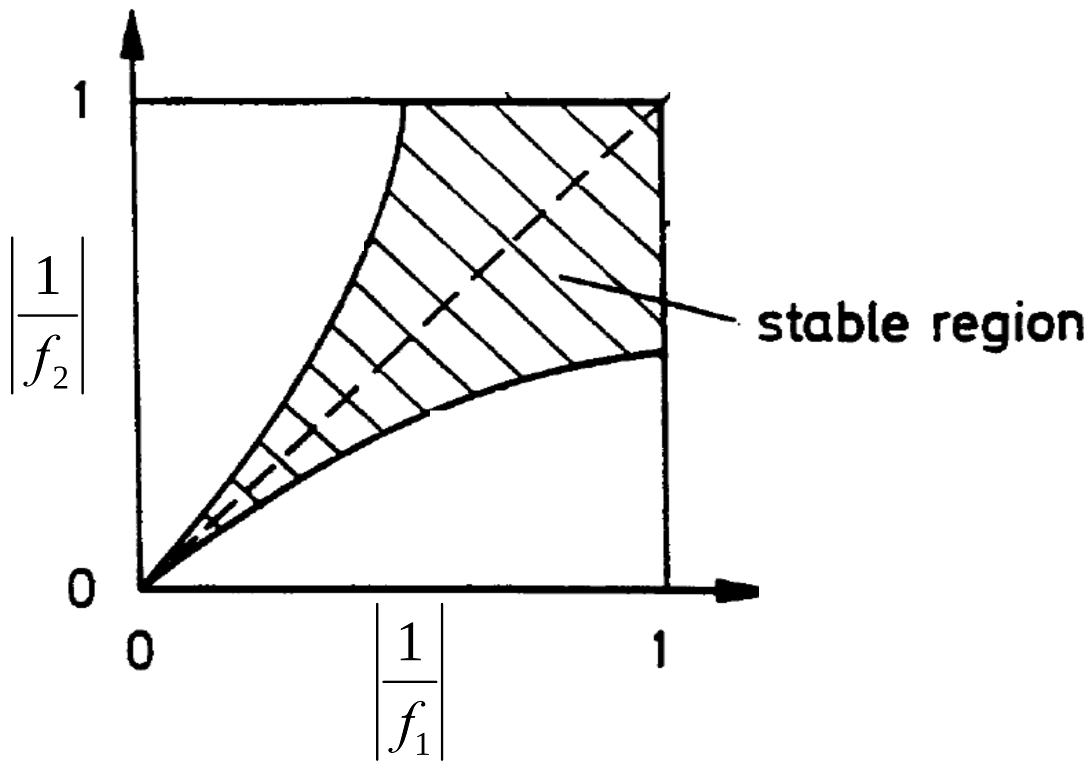
For a thin lens FODO lattice, the focal length should be greater than the distance between magnets.

Stability Condition for a FODO Lattice (general)

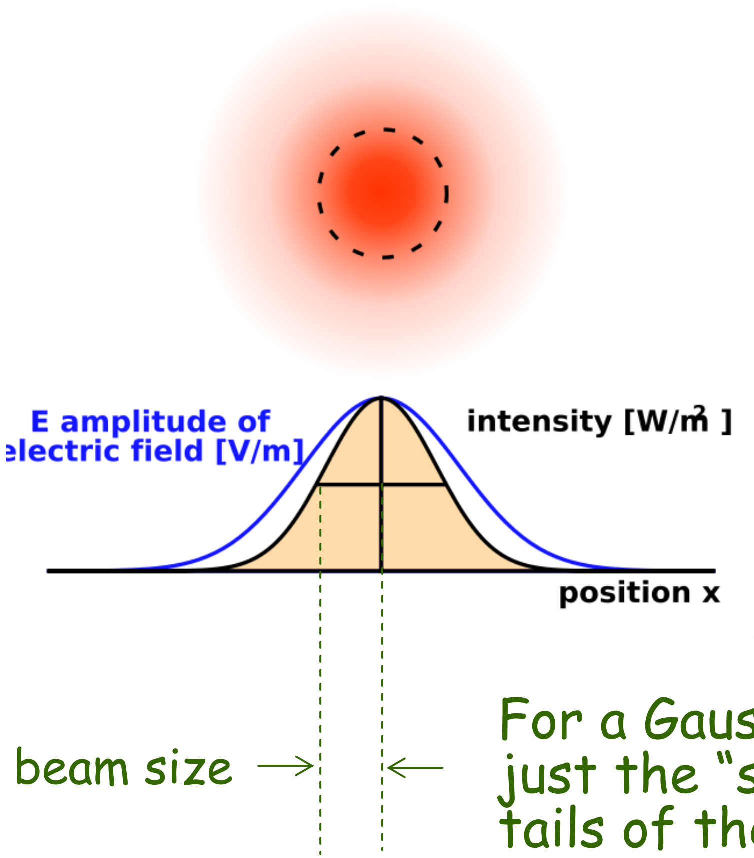
For the general FODO cell with unequal focal lengths, the condition is more complicated. We have:

$$0 < \frac{L}{f_1} + \frac{L}{f_2} - \frac{L}{f_1 f_2} < 1$$

The allowed magnitudes are given by the "Necktie Diagram"



RMS beam size



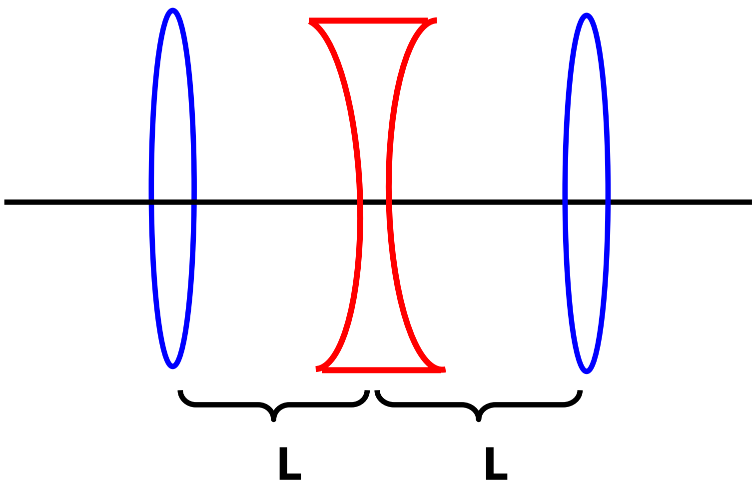
$$x_{\mathrm{rms}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2} = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}$$

rms beam size → ←

For a Gaussian beam, the rms beam size is just the "sigma" of the beam. To include the tails of the beam we often talk about apertures that are "five sigma", or "seven sigma", etc.

(Image is from Wikipedia)

Example: FODO Channel



- Consider a defocusing quadrupole "sandwiched" by two focusing quadrupoles with focal lengths **f**.
- The symmetric transfer matrix is from center to center of focusing quads (thus one full focus and one full defocus quad)

$$M_{\text{FODO}} = M_{\text{Half QF}} M_{\text{Drift}} M_{\text{QD}} M_{\text{Drift}} M_{\text{Half QF}}$$

This arrangement is very common in beam transport lines.

For a symmetric lattice sequence, we have the special property:

If $M_{\text{first half}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then automatically, $M_{\text{second half}} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$

and finally, multiplying the two:

$$M_{\text{Full}} = M_{\text{second half}} M_{\text{first half}} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + bc \end{pmatrix}$$

Example: FODO Channel

The general expression for a FODO lattice with focal lengths f1 and f2, separated by a distance L is:

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 - 2\dfrac{L}{f^*} & 2L\left(1 + \dfrac{L}{f_2}\right) \\ -\dfrac{2}{f^*}\left(1 - \dfrac{L}{f_1}\right) & 1 - 2\dfrac{L}{f^*} \end{pmatrix}$$

with, $$\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$$

And the special case where f1=-f2=f is:

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 - 2\dfrac{L^2}{f^2} & 2L\left(1 + \dfrac{L}{2f}\right) \\ -\dfrac{2L}{f^2}\left(1 - \dfrac{L^2}{f^2}\right) & 1 - 2\dfrac{L^2}{f^2} \end{pmatrix}$$

This arrangement is very common in beam transport lines.