A Course in Mechanics by Dr. J. Tinsley Oden
Part II - Homework 1

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Exercise Set II.1

1. (a) The probability that a team selected randomly has j wins is

$$\mathbb{P}(j) = \frac{\text{Number of teams with } j \text{ wins}}{\text{Total number of teams}} = \frac{N(j)}{N}.$$

(b)

$$\sum_{j=0}^{\infty} \mathbb{P}(j) = \sum_{j=0}^{\infty} \frac{N(j)}{N} = \frac{1}{N} \sum_{j=0}^{\infty} N(j) = \frac{N}{N} = 1.$$

The probabilities in the problem are as follows:
$\mathbb{P}(7) = 0.3$, $\mathbb{P}(8) = 0.4$, $\mathbb{P}(9) = 0.2$, $\mathbb{P}(11) = 0.1$, and $\mathbb{P}(j) = 0$ for $j \neq 7, 8, 9, 11$. We now show the above, i.e.,

$$\sum_{j=0}^{\infty} \mathbb{P}(j) = 0.3 + 0.4 + 0.2 + 0.1 = 1.0.$$

(c)

$$\langle j \rangle = \frac{\sum_{j=0}^{\infty} jN(j)}{N} = \sum_{j=0}^{\infty} j \frac{N(j)}{N} = \sum_{j=0}^{\infty} j\mathbb{P}(j).$$

We now calculate explicitly:

$$\langle j \rangle = \frac{1}{10}(7 \times 3 + 8 \times 4 + 9 \times 2 + 11 \times 1) = \frac{82}{10} = 8.2.$$

$$\sum_{j=0}^{\infty} j\mathbb{P}(j) = 7 \times 0.3 + 8 \times 0.4 + 9 \times 0.2 + 11 \times 0.1 = 8.2 = \langle j \rangle.$$

(d) The variance is

$$\sigma^2 = \sum_{j=0}^{\infty} (j - \langle j \rangle)^2 \mathbb{P}(j) = (7 - 8.2)^2 \times 0.3 + (8 - 8.2)^2 \times 0.4 + (9 - 8.2)^2 \times 0.2 + (11 - 8.2)^2 \times 0.1$$
$$= (-1.2)^2 \times 0.3 + (-0.2)^2 \times 0.4 + (0.8)^2 \times 0.2 + (2.8)^2 \times 0.1 = 1.36.$$

The standard deviation is $\sigma = \sqrt{1.36} \approx 1.17$.

2. (a) We determine C as follows:

$$1 = \int_{-\infty}^{\infty} \rho(x)dx = \int_{-\infty}^{\infty} Ce^{-\frac{\alpha}{2}(x-x_0)^2} dx.$$

Assume the change of variables $y = \sqrt{\frac{\alpha}{2}}(x - x_0)$. Then $dx = \sqrt{\frac{2}{\alpha}}dy$ and (see Appendix)

$$1 = C\sqrt{\frac{2}{\alpha}} \int_{-\infty}^{\infty} e^{-y^2} dy = C\sqrt{\frac{2\pi}{\alpha}}.$$

Then, we obtain

$$\boxed{C = \sqrt{\frac{\alpha}{2\pi}}}.$$

(b) We obtained in (a) that the Gaussian probability is

$$\rho(x) = \sqrt{\frac{\alpha}{2\pi}} e^{-\frac{\alpha}{2}(x-x_0)^2}$$

We now calculate $\langle x \rangle$:

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx = \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} xe^{-\frac{\alpha}{2}(x-x_0)^2} dx$$

As before, let $y = \sqrt{\frac{\alpha}{2}}(x - x_0)$, then

$$\langle x \rangle = \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\alpha}} y + x_0 \right) e^{-y^2} \sqrt{\frac{2}{\alpha}} dy = \sqrt{\frac{2}{\alpha}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ye^{-y^2} dy + \frac{1}{\sqrt{\pi}} x_0 \int_{-\infty}^{\infty} e^{-y^2} dy$$
$$= \frac{1}{\sqrt{\pi}} x_0 \sqrt{\pi} = x_0.$$

(see Appendix for the integrals)

$$\boxed{\langle x \rangle = x_0}.$$

We now calculate $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \rho(x)dx = \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{\alpha}{2}(x-x_0)^2} dx = \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\alpha}} y + x_0 \right)^2 e^{-y^2} \sqrt{\frac{2}{\alpha}} dy$$
$$= \frac{2}{\alpha} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy + 2\sqrt{\frac{2}{\alpha}} \frac{1}{\sqrt{\pi}} x_0 \int_{-\infty}^{\infty} ye^{-y^2} dy + \frac{1}{\sqrt{\pi}} x_0^2 \int_{-\infty}^{\infty} e^{-y^2} dy.$$

Then,

$$\boxed{\langle x^2 \rangle = \frac{1}{\alpha} + x_0^2.}.$$

(c)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{\alpha} + x_0^2 - x_0^2 = \frac{1}{\alpha}.$$

Then,

$$\boxed{\sigma = \frac{1}{\sqrt{\alpha}}}.$$

Note: we recovered the standard form of a Gaussian function:

$$\rho(x) = \sqrt{\frac{1}{2\pi\sigma^2}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}.$$

(d) In order to sketch a graph of $\rho(x)$ we use the following properties:

- $\rho(x_0) = \rho_0 := \sqrt{\frac{1}{2\pi\sigma^2}}$
- $\lim_{x \to \pm\infty} \rho(x) = 0$

The function $\rho(x)$ will look as follows:

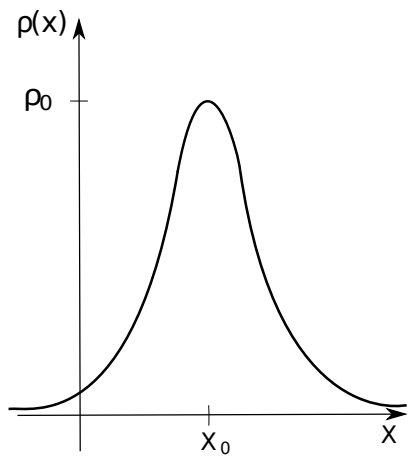

Figure 1: Gaussian probability distribution.

3. (a) In order to determine A we use the normalization property of the wave function:

$$1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = \int_{-1}^{0} A^2(1+x)^2 dx + \int_{0}^{1} A^2(1-x)^2 dx$$
$$= A^2 \left(\frac{(1+x)^3}{3}\bigg|_{-1}^{0} - \frac{(1-x)^3}{3}\bigg|_{0}^{1} \right) = A^2 \frac{2}{3}.$$

Then, we have

$$\boxed{A = \pm\sqrt{\frac{3}{2}}}.$$

(b)

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x)x\psi(x)dx = \frac{3}{2}\left(\int_{-1}^{0} x(1+x)^2 dx + \int_{0}^{1} x(1-x)^2 dx \right)$$
$$= \frac{3}{2}\left(\int_{-1}^{0} x^3 + 2x^2 + x\, dx + \int_{0}^{1} x^3 - 2x^2 + x\, dx \right)$$
$$= \frac{3}{2}\left\{ \left(\frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} \right)\bigg|_{-1}^{0} + \left(\frac{x^4}{4} - 2\frac{x^3}{3} + \frac{x^2}{2} \right)\bigg|_{0}^{1} \right\}$$
$$= \frac{3}{2}\left\{ \left(-\frac{1}{4} + \frac{2}{3} - \frac{1}{2} \right) + \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) \right\} = 0.$$

Then, we have

$$\boxed{\langle x \rangle = 0}.$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^*(x)x^2\psi(x)dx = \frac{3}{2}\left(\int_{-1}^{0} x^2(1+x)^2 dx + \int_{0}^{1} x^2(1-x)^2 dx \right)$$
$$= \frac{3}{2}\left(\int_{-1}^{0} x^4 + 2x^3 + x^2\, dx + \int_{0}^{1} x^4 - 2x^3 + x^2\, dx \right)$$
$$= \frac{3}{2}\left\{ \left(\frac{x^5}{5} + 2\frac{x^4}{4} + \frac{x^3}{3} \right)\bigg|_{-1}^{0} + \left(\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3} \right)\bigg|_{0}^{1} \right\}$$
$$= \frac{3}{2}\left\{ \left(\frac{1}{5} - \frac{2}{4} + \frac{1}{3} \right) + \left(\frac{1}{5} - \frac{2}{4} + \frac{1}{3} \right) \right\} = \frac{1}{10}.$$

Then, we have

$$\boxed{\langle x^2 \rangle = \frac{1}{10}}.$$

(c)

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{10},$$

$$\boxed{\sigma_x = \frac{1}{\sqrt{10}}}.$$

(d)

$$\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty} \psi^*(x) p \psi(x) dx = \int_{-\infty}^{\infty} \psi^*(x) \frac{\hbar}{i}\frac{d}{dx}\psi(x) dx \\
&= \frac{3}{2}\frac{\hbar}{i}\left(\int_{-1}^{0}(1+x)dx - \int_{0}^{1}(1-x)dx\right) = \frac{3}{2}\frac{\hbar}{i}\left\{\left(x+\frac{x^2}{2}\right)\Big|_{-1}^{0} - \left(x-\frac{x^2}{2}\right)\Big|_{0}^{1}\right\} \\
&= \frac{3}{2}\frac{\hbar}{i}\left\{\left(1-\frac{1}{2}\right) - \left(1-\frac{1}{2}\right)\right\} = 0.
\end{aligned}$$

Then, we have

$$\boxed{\langle p\rangle = 0}.$$

4. (a) The Shrödinger equation is

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) + \frac{\hbar}{i}\frac{\partial}{\partial t}\right)\Psi(x,t) = 0.$$

We now calculate the different components for $\Psi(x,t) = \alpha e^{-\Lambda}$ with $\Lambda = \beta\hbar^{-1}(mx^2 + i\gamma t)$.

$$\begin{aligned}
\frac{\hbar}{i}\frac{\partial}{\partial t}\Psi(x,t) &= -\frac{\hbar}{i}\Psi(x,t)\beta\hbar^{-1}i\gamma = -\beta\gamma\Psi(x,t), \\
-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) &= \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\Psi(x,t)\beta\hbar^{-1}2mx\right) = \beta\hbar\frac{\partial}{\partial x}\left(\Psi(x,t)x\right) \\
&= -2mx^2\beta^2\Psi(x,t) + \beta\hbar\Psi(x,t)
\end{aligned}$$

Then,

$$\begin{aligned}
V(x)\Psi(x,t) &= \left(\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar}{i}\frac{\partial}{\partial t}\right)\Psi(x,t) \\
&= 2mx^2\beta^2\Psi(x,t) - \beta\hbar\Psi(x,t) + \beta\gamma\Psi(x,t) = \beta\left(2mx^2\beta + \gamma - \hbar\right)\Psi(x,t).
\end{aligned}$$

Then, we have

$$\boxed{V(x) = \beta\left(2mx^2\beta + \gamma - \hbar\right)}.$$

(b)

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty}\Psi^*(x,t)x\Psi(x,t)dx = \int_{-\infty}^{\infty}\alpha e^{-\beta\hbar^{-1}(mx^2+i\gamma t)}x\alpha e^{-\beta\hbar^{-1}(mx^2-i\gamma t)}dx \\
&= \alpha^2\int_{-\infty}^{\infty} x e^{-2\beta\hbar^{-1}mx^2}dx.
\end{aligned}$$

Let $y = \sqrt{2\beta\hbar^{-1}m}x$ then

$$\langle x\rangle = \alpha^2\int_{-\infty}^{\infty}\frac{y}{\sqrt{2\beta\hbar^{-1}m}}e^{-y^2}\frac{1}{\sqrt{2\beta\hbar^{-1}m}}dy = \frac{\alpha^2}{2\beta\hbar^{-1}m}\int_{-\infty}^{\infty} y e^{-y^2}dy = 0.$$

Then, we get

$$\boxed{\langle x\rangle = 0}.$$

Similarly, we calculate

$$\begin{aligned}
\langle x^2\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)x^2\Psi(x,t)dx = \alpha^2 \int_{-\infty}^{\infty} x^2 e^{-2\beta\hbar^{-1}mx^2}dx \\
&= \alpha^2 \int_{-\infty}^{\infty} \frac{y^2}{2\beta\hbar^{-1}m} e^{y^2} \frac{1}{\sqrt{2\beta\hbar^{-1}m}} dy = \frac{\alpha^2}{(2\beta\hbar^{-1}m)^{3/2}} \int_{-\infty}^{\infty} y^2 e^{y^2} dy \\
&= \frac{\alpha^2}{(2\beta\hbar^{-1}m)^{3/2}} \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

$$\boxed{\langle x^2\rangle = \frac{\alpha^2}{8}\sqrt{\frac{2\pi\hbar^3}{(m\beta)^3}}}.$$

We now calculate

$$\begin{aligned}
\langle p\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)p\Psi(x,t)dx = \int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x,t)dx \\
&= \int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\hbar}{i}(-\beta\hbar^{-1}2mx)\Psi(x,t)dx \\
&= i\beta 2m \int_{-\infty}^{\infty} \Psi^*(x,t)x\Psi(x,t)dx = i2m\beta\langle x\rangle = 0.
\end{aligned}$$

Then, we get

$$\boxed{\langle p\rangle = 0}.$$

$$\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)p^2\Psi(x,t)dx = \int_{-\infty}^{\infty} \Psi^*(x,t)(-\hbar^2)\frac{\partial^2}{\partial x^2}\Psi(x,t)dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial}{\partial x}(-\beta\hbar^{-1}2mx\Psi(x,t))dx \\
&= 2m\beta\hbar \int_{-\infty}^{\infty} \Psi^*(x,t)\frac{\partial}{\partial x}(x\Psi(x,t))dx = 2m\beta\hbar \int_{-\infty}^{\infty} \Psi^*(x,t)\left(\Psi(x,t) + x(-\beta\hbar^{-1}2mx\Psi(x,t))\right)dx \\
&= 2m\beta\hbar \int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)dx - 4m^2\beta^2 \int_{-\infty}^{\infty} \Psi^*(x,t)x^2\Psi(x,t)dx \\
&= 2m\beta\hbar \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx - 4m^2\beta^2\langle x^2\rangle
\end{aligned}$$

We calculate

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \alpha^2 \int_{-\infty}^{\infty} e^{-2\beta\hbar^{-1}mx^2}dx = \alpha^2 \int_{-\infty}^{\infty} e^{-y^2}\frac{1}{\sqrt{2\beta\hbar^{-1}m}}dy = \alpha^2\sqrt{\frac{\pi\hbar}{2\beta m}}.$$

Then, we get

$$\langle p^2\rangle = 2m\beta\hbar\alpha^2\sqrt{\frac{\pi\hbar}{2\beta m}} - 4m^2\beta^2\frac{\alpha^2}{8}\sqrt{\frac{2\pi\hbar^3}{(m\beta)^3}} = \alpha^2\sqrt{\frac{\pi\beta m\hbar^3}{2}}.$$

$$\boxed{\langle p^2\rangle = \frac{\alpha^2}{2}\sqrt{2\pi\beta m\hbar^3}}.$$

(c)

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle = \frac{\alpha^2}{8}\sqrt{\frac{2\pi\hbar^3}{(m\beta)^3}}$$

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2 = \langle p^2 \rangle = \frac{\alpha^2}{2}\sqrt{2\pi\beta m\hbar^3}$$

$$\sigma_x^2\sigma_p^2 = \frac{\alpha^2}{8}\sqrt{\frac{2\pi\hbar^3}{(m\beta)^3}}\frac{\alpha^2}{2}\sqrt{2\pi\beta m\hbar^3} = \left(\alpha^2\sqrt{\frac{\pi\hbar}{2m\beta}}\right)^2\frac{\hbar^2}{4} = \left(\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx\right)^2\frac{\hbar^2}{4} = \frac{\hbar^2}{4},$$

where we used the normalization of $\Psi(x,t)$. Then,

$$\boxed{\sigma_x^2\sigma_p^2 = \left(\frac{\hbar}{2}\right)^2},$$

consistent with the Heisenberg principle.

(d) From the normalization we get

$$\boxed{\alpha = \pm\sqrt[4]{\frac{2\beta m}{\pi\hbar}}},$$

Appendix

1. Define

$$I := \int_{-\infty}^{\infty} e^{-x^2} dx.$$

Then,

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-y^2} e^{-x^2} dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy.$$

We now use polar coordinates so that $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = r^2$; futhermore, $dxdy = rd\theta dr$, then

$$I^2 = \int_0^{\infty} \int_0^{2\pi} e^{-r^2} r d\theta dr = 2\pi \int_0^{\infty} r e^{-r^2} dr = 2\pi \left(\frac{e^{-r^2}}{-2} \right) \Bigg|_0^{\infty} = \pi.$$

Then $I = \sqrt{\pi}$, i.e.,

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}.$$

2.

$$\int_{-\infty}^{\infty} y e^{-y^2} dy = \frac{e^{-y^2}}{-2} \Bigg|_{-\infty}^{\infty} = 0.$$

$$\boxed{\int_{-\infty}^{\infty} y e^{-y^2} dy = 0}.$$

3.

$$\begin{aligned} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy &= -\lim_{\beta \to 1} \int_{-\infty}^{\infty} \frac{\partial}{\partial \beta} e^{-\beta y^2} dy = -\lim_{\beta \to 1} \frac{\partial}{\partial \beta} \int_{-\infty}^{\infty} e^{-\beta y^2} dy \\ &= -\lim_{\beta \to 1} \frac{\partial}{\partial \beta} \frac{1}{\sqrt{\beta}} \int_{-\infty}^{\infty} e^{-z^2} dz = -\lim_{\beta \to 1} \frac{\partial}{\partial \beta} \frac{\sqrt{\pi}}{\sqrt{\beta}} = \lim_{\beta \to 1} \frac{1}{2} \frac{\sqrt{\pi}}{\beta^{3/2}} = \frac{\sqrt{\pi}}{2}. \end{aligned}$$

$$\boxed{\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{2}}.$$