A Course in Mechanics by Dr. J. Tinsley Oden
Part II - Homework 2 - Solutions

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Set II.2:

1. Let $u = \left(\tilde{Q} - \langle Q \rangle\right)\Psi$ and $v = \left(\tilde{M} - \langle M \rangle\right)\Psi$.

   (a) We now calculate

$$\begin{aligned}\sigma_Q^2 = \langle Q^2 \rangle - \langle Q \rangle^2 &= \langle \Psi, \tilde{Q}^2 \Psi \rangle - \langle Q \rangle^2 \langle \Psi, \Psi \rangle = \langle \Psi, \tilde{Q}\tilde{Q}\Psi \rangle - \langle \Psi, \langle Q \rangle^2 \Psi \rangle \\ &= \langle \Psi, \tilde{Q}(\tilde{Q}\Psi) \rangle - \langle \langle Q \rangle \Psi, \langle Q \rangle \Psi \rangle.\end{aligned}$$

   Where we use the fact that $\langle Q \rangle$ is a scalar. An Hermitian operator $\tilde{Q}$ satisfies $\langle \Psi, \tilde{Q}\varphi \rangle = \langle \tilde{Q}\Psi, \varphi \rangle$, then

$$\begin{aligned}\sigma_Q^2 &= \langle \tilde{Q}\Psi, \tilde{Q}\Psi \rangle - \langle \langle Q \rangle \Psi, \langle Q \rangle \Psi \rangle \\ &= \langle \tilde{Q}\Psi, \tilde{Q}\Psi \rangle - \langle \langle Q \rangle \Psi, \langle Q \rangle \Psi \rangle + \langle \langle Q \rangle \Psi, \langle Q \rangle \Psi \rangle - \langle \langle Q \rangle \Psi, \langle Q \rangle \Psi \rangle \\ &= \langle \tilde{Q}\Psi, \tilde{Q}\Psi \rangle - \langle \langle Q \rangle \Psi, \tilde{Q}\Psi \rangle + \langle \langle Q \rangle \Psi, \langle Q \rangle \Psi \rangle - \langle \langle Q \rangle \Psi, \tilde{Q}\Psi \rangle \\ &= \langle (\tilde{Q} - \langle Q \rangle)\Psi, \tilde{Q}\Psi \rangle - \langle \langle Q \rangle \Psi, (\tilde{Q} - \langle Q \rangle)\Psi \rangle \\ &= \langle (\tilde{Q} - \langle Q \rangle)\Psi, \tilde{Q}\Psi \rangle - \langle (\tilde{Q} - \langle Q \rangle)\Psi, \langle Q \rangle \Psi \rangle \\ &= \langle (\tilde{Q} - \langle Q \rangle)\Psi, (\tilde{Q} - \langle Q \rangle)\Psi \rangle \\ &= \langle u, u \rangle = ||u||^2,\end{aligned}$$

   where we used the following:

   i. $\langle \langle Q \rangle \Psi, \tilde{Q}\Psi \rangle = \langle Q \rangle \langle \Psi, \tilde{Q}\Psi \rangle = \langle Q \rangle \langle Q \rangle = \langle \langle Q \rangle \Psi, \langle Q \rangle \Psi \rangle$,
   ii. $\tilde{Q}$ is Hermitian, then: $\langle \langle Q \rangle \Psi, (\tilde{Q} - \langle Q \rangle)\Psi \rangle = \langle (\tilde{Q} - \langle Q \rangle)\Psi, \langle Q \rangle \Psi \rangle$.
   iii. $\langle Q \rangle$ is real.

   By replacing Q by M we obtain

$$\sigma_M^2 = \langle v, v \rangle = ||v||^2.$$

   Then, by the Cauchy-Schwarz inequality we obtain

$$\boxed{\sigma_Q^2 \sigma_M^2 = ||u||^2 \, ||v||^2 = \langle u, u \rangle \langle v, v \rangle \geq |\langle u, v \rangle|^2}.$$

(b)

$$\begin{aligned}
\langle u, v\rangle &= \langle(\tilde{Q} - \langle Q\rangle)\Psi, (\tilde{M} - \langle M\rangle)\Psi\rangle \\
&= \langle\tilde{Q}\Psi, (\tilde{M} - \langle M\rangle)\Psi\rangle - \langle\langle Q\rangle\Psi, (\tilde{M} - \langle M\rangle)\Psi\rangle \\
&= \langle\tilde{Q}\Psi, \tilde{M}\Psi\rangle - \langle\tilde{Q}\Psi, \langle M\rangle\Psi\rangle - \langle\langle Q\rangle\Psi, \tilde{M}\Psi\rangle + \langle\langle Q\rangle\Psi, \langle M\rangle\Psi\rangle \\
&= \langle\Psi, \tilde{Q}\tilde{M}\Psi\rangle - \langle\tilde{Q}\Psi, \Psi\rangle\langle M\rangle - \langle Q\rangle\langle\Psi, \tilde{M}\Psi\rangle + \langle Q\rangle\langle\Psi, \Psi\rangle\langle M\rangle \\
&= \langle\Psi, \tilde{Q}\tilde{M}\Psi\rangle - \langle Q\rangle\langle M\rangle - \langle Q\rangle\langle M\rangle + \langle Q\rangle\langle M\rangle \\
&= \langle QM\rangle - \langle Q\rangle\langle M\rangle.
\end{aligned}$$

We then obtain

$$\boxed{\langle u, v\rangle = \langle QM\rangle - \langle Q\rangle\langle M\rangle}.$$

(c) For any complex number $z = a + ib$ we have $|z|^2 = z^* z = (a - ib)(a + ib) = a^2 + b^2$. In addition

$$\begin{aligned}
\mathrm{Re}(z) &= \frac{1}{2}(z + z^*) = \frac{1}{2}((a + ib) + (a - ib)) = a, \\
\mathrm{Im}(z) &= \frac{1}{2i}(z - z^*) = \frac{1}{2i}((a + ib) - (a - ib)) = b,
\end{aligned}$$

so that $|z|^2 \geq b^2 = \left(\frac{1}{2i}(z - z^*)\right)^2$. Then,

$$|z|^2 \geq \left(\frac{1}{2i}(z - z^*)\right)^2.$$

Following (a) above, and noting that $\langle u, v\rangle^* = \langle v, u\rangle$ we have

$$\boxed{\sigma_Q^2\sigma_M^2 \geq |\langle u, v\rangle|^2 \geq \left(\frac{1}{2i}(\langle u, v\rangle - \langle v, u\rangle)\right)^2}.$$

(d) From (b) we have

$$\begin{aligned}
\langle u, v\rangle &= \langle QM\rangle - \langle Q\rangle\langle M\rangle \\
\langle v, u\rangle &= \langle MQ\rangle - \langle M\rangle\langle Q\rangle \\
\langle u, v\rangle - \langle v, u\rangle &= \langle QM\rangle - \langle MQ\rangle = \langle(QM - MQ)\rangle = \langle[Q, M]\rangle.
\end{aligned}$$

Then, from (c) we have

$$\boxed{\sigma_Q^2\sigma_M^2 \geq \left(\frac{1}{2i}\langle[Q, M]\rangle\right)^2},$$

i.e., Equation (11.27) in the notes.

2. (a) The Schödinger's equation is

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) + \frac{\hbar}{i}\frac{\partial}{\partial t}\Psi(x,t) = 0.$$

<u>Inside the well:</u> Inside the well no forces act on the particle, because $V(x) = 0$ for $0 \leq x \leq a$, so we expect a solution similar to a free particle inside the well.
<u>Outside the well:</u> Outside the well, the probability of finding the particle is zero, so that $\rho(x,t) = \Psi(x,t)^*\Psi(x,t) = 0$, then $\Psi(x,t) = 0$ outside the well.
Assume a solution using separation of variables $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$, so that we obtain inside the well the time-independent Schödinger's equation

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) - E\psi(x) = 0.$$

We can rewrite this equation, together with the boundary conditions as

$$\boxed{\frac{d^2}{dx^2}\psi(x) + k^2\psi(x) = 0, \qquad \psi(0) = \psi(a) = 0},$$

with $k^2 = 2mE/\hbar^2$.

(b) We assume a solution of the form $\psi(x) = A\sin(kx) + B\cos(kx)$. By using the boundary conditions, we find

$$0 = \psi(0) = A\sin(k0) + B\cos(k0) = B \Rightarrow B = 0.$$
$$0 = \psi(a) = A\sin(ka) \Rightarrow ka = n\pi,$$

with n an integer. We assume $A \neq 0$, otherwise $\psi^*(x)\psi(x) \equiv 0$ and thus it cannot represent a density function. Then,

$$k^2 = \frac{2mE}{\hbar^2} = \left(\frac{n\pi}{a}\right)^2 \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2ma^2},$$

so we get

$$\boxed{E = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, 3, \ldots}.$$

(c) We now find the normalization of the wave function

$$1 = \int_{-\infty}^{\infty}\psi(x)^*\psi(x)dx = \int_0^a \psi(x)^*\psi(x)dx = \int_0^a A^2\sin^2(kx)dx = \int_0^a A^2\left(\frac{1-\cos(2kx)}{2}\right)dx$$
$$= A^2\left(\frac{x}{2} - \frac{\sin(2kx)}{4k}\right)\Bigg|_0^a = A^2\frac{a}{2} \Rightarrow A = \sqrt{\frac{2}{a}}.$$

The wave function for a given n is

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right), \quad 0 \leq x \leq a}.$$

We now show that the functions $\{\psi_n(x)\}$ are orthonormal in $L^2(0,a)$. For that we use the following:

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \Rightarrow \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right) = \sin(x)\sin(y).$$

Then,

$$\begin{aligned}\int_0^a \psi_m(x)\psi_n(x)dx &= \frac{2}{a}\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx \\ &= \frac{2}{a}\int_0^a \frac{1}{2}\left(\cos\left(\frac{(m-n)\pi x}{a}\right) - \cos\left(\frac{(m+n)\pi x}{a}\right)\right)dx.\end{aligned}$$

If $m = n$ then

$$\begin{aligned}\int_0^a \psi_m(x)\psi_n(x)dx &= \frac{2}{a}\int_0^a \frac{1}{2}\left(1 - \cos\left(\frac{(m+n)\pi x}{a}\right)\right)dx \\ &= \frac{2}{a}\left(\frac{a}{2} - \frac{1}{2}\frac{a}{(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{a}\right)\Bigg|_0^a\right) = 1.\end{aligned}$$

Similarly, if $m \neq n$

$$\int_0^a \psi_m(x)\psi_n(x)dx = 0.$$

Therefore, we obtain

$$\boxed{\int_0^a \psi_m(x)\psi_n(x)dx = \delta_{m,n}},$$

with $\delta_{m,n}$ the Kronecker delta. Therefore the functions are orthonormal in $L^2(0,a)$. The solution $\psi(x)$ is then the superposition of the above functions:

$$\psi(x) = \sum_n \psi_n(x).$$

This can be shown as follows:

$$\frac{d^2}{dx^2}\psi(x) + k^2\psi(x) = \frac{d^2}{dx^2}\sum_n \psi_n(x) + k^2\sum_n \psi_n(x) = \sum_n\left(\frac{d^2}{dx^2}\psi_n(x) + k^2\psi_n(x)\right) = 0,$$

because each function $\psi_n(x)$ satisfies the equation.

(d) We now use the functions $\{\psi_n(x)\}$ as a basis in $L^2(0,a)$. We calculate the Fourier series representation of the function $f(x) = x$ in $(0,a)$ as follows.

$$f(x) = \sum_{n=1}^{\infty} c_n\psi_n(x).$$

To find the coefficients c_n, we use the orthogonality of the basis functions so that

$$\int_0^a f(x)\psi_m(x)dx = \int_0^a \sum_{n=1}^{\infty} c_n\psi_n(x)\psi_m(x)dx = \sum_{n=1}^{\infty} c_n\int_0^a \psi_n(x)\psi_m(x)dx = \sum_{n=1}^{\infty} c_n\delta_{n,m} = c_m.$$

Then,

$$c_m = \int_0^a f(x)\psi_m(x)dx.$$

For $f(x) = x$

$$c_m = \int_0^a x\psi_m(x)dx = \int_0^a x\sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right)dx.$$

We use the change of variables $\xi = \frac{m\pi x}{a}$ and get

$$c_m = \sqrt{\frac{2}{a}}\int_0^{m\pi}\frac{a}{m\pi}\xi\sin(\xi)\frac{a}{m\pi}d\xi = \sqrt{\frac{2}{a}}\left(\frac{a}{m\pi}\right)^2\int_0^{m\pi}\xi\sin(\xi)d\xi = \sqrt{\frac{2}{a}}\left(\frac{a}{m\pi}\right)^2(\sin(\xi)-\xi\cos(\xi))\Big|_0^{m\pi}$$

$$= \sqrt{\frac{2}{a}}\left(\frac{a}{m\pi}\right)^2(-m\pi\cos(m\pi)) = \frac{a^2}{m\pi}\sqrt{\frac{2}{a}}(-1)^{m+1}.$$

Then,

$$\boxed{c_m = \frac{a^2}{m\pi}\sqrt{\frac{2}{a}}(-1)^{m+1}},$$

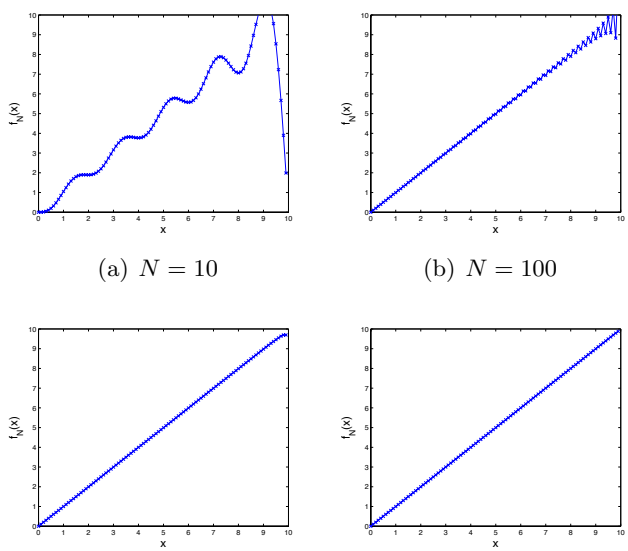
and

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2a}{n\pi}\sin\left(\frac{n\pi x}{a}\right).$$

In Figure 1, we plot partial Fourier series, i.e., we choose N and plot

$$f_N(x) = \sum_{n=1}^{N} c_n\psi_n(x)$$

in the range $(0, a) = (0, 10)$.



(a) $N = 10$ (b) $N = 100$

(c) $N = 1,000$ (d) $N = 10,000$

Figure 1: Partial Fourier Series for $f(x) = x$ in the interval $(0, 10)$, for $N = 10, 100, 1000, 10000$.

(e) If Q is an Hermitian operator where its eigenfunctions represent a complete set, then Q is an observable. We already found that the eigenfunctions of Q are a complete set, then we check whether Q satisfies $\langle \Psi, Q\Psi \rangle = \langle Q\Psi, \Psi \rangle$. We first observe that for $0 \leq x \leq a$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right),$$
$$\frac{d^2\psi_n}{dx^2}(x) = \left(\frac{n\pi x}{a}\right)^2 \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) = \left(\frac{n\pi x}{a}\right)^2 \psi_n(x).$$

$$\begin{aligned}\langle \Psi, Q\Psi \rangle &= \int_0^a \Psi^*(x) \frac{d^2\Psi}{dx^2}(x)dx = \sum_m \sum_n \int_0^a \Psi_m^*(x) \frac{d^2\Psi_n}{dx^2}(x)dx = \sum_m \sum_n \int_0^a \Psi_m^*(x) \left(\frac{n\pi x}{a}\right)^2 \Psi_n(x)dx \\ &= \sum_m \sum_n \left(\frac{n\pi x}{a}\right)^2 \int_0^a \Psi_m^*(x)\Psi_n(x)dx = \sum_m \sum_n \left(\frac{n\pi x}{a}\right)^2 \delta_{m,n} = \sum_m \left(\frac{m\pi x}{a}\right)^2.\end{aligned}$$

On the other hand

$$\begin{aligned}\langle Q\Psi, \Psi \rangle &= \int_0^a \frac{d^2\Psi^*}{dx^2}(x)\Psi(x)dx = \sum_m \sum_n \int_0^a \frac{d^2\Psi_m^*}{dx^2}(x)\Psi_n(x)dx = \sum_m \sum_n \int_0^a \left(\frac{m\pi x}{a}\right)^2 \Psi_m^*(x)\Psi_n(x)dx \\ &= \sum_m \sum_n \left(\frac{m\pi x}{a}\right)^2 \int_0^a \Psi_m^*(x)\Psi_n(x)dx = \sum_m \sum_n \left(\frac{m\pi x}{a}\right)^2 \delta_{m,n} = \sum_m \left(\frac{m\pi x}{a}\right)^2.\end{aligned}$$

We then get

$$\boxed{\langle \Psi, Q\Psi \rangle = \langle Q\Psi, \Psi \rangle}.$$

Then, Q is an observable.

3. We check whether the operators A and B commute:

$$[A, B]\varphi = (AB - BA)\varphi = AB\varphi - BA\varphi = x\frac{\partial\varphi}{\partial x} - \frac{\partial}{\partial x}(x\varphi) = x\frac{\partial\varphi}{\partial x} - \varphi - x\frac{\partial\varphi}{\partial x} = -\varphi.$$

Then, $[A, B]\varphi \neq 0$ for $\varphi \neq 0$, thus the operators do not commute.

<u>Note:</u> The momentum operator is $p = \frac{\hbar}{i}\frac{\partial}{\partial x} = \frac{\hbar}{i}B$, and we know that x and p do not commute because $[x, p]\varphi = i\hbar\varphi$.

4. (a) We show that the operators Q and M commute:

$$\begin{aligned}QM\psi(x) &= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\left(-i\hbar\frac{\partial\psi}{\partial x}\right) = \left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial^2\psi}{\partial x^2}\right) \\ &= \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\right) = MQ\psi(x).\end{aligned}$$

Then we have

$$[Q, M]\psi(x) = (QM - MQ)\psi(x) = 0.$$

(b) Let $\psi(x) = e^{ikx}$ then

$$Q\psi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}e^{ikx} = \frac{\hbar^2 k^2}{2m}e^{ikx} = \lambda_Q \psi(x),$$
$$M\psi(x) = -i\hbar\frac{\partial}{\partial x}e^{ikx} = \hbar k e^{ikx} = \lambda_M \psi(x),$$

with $\lambda_Q = \frac{\hbar^2 k^2}{2m}$ and $\lambda_M = \hbar k$. Therefore, $\psi(x)$ is a simultaneous eigenfunction of Q and M with eigenvalues λ_Q and λ_M respectively.

The Hamiltonian for a free particle is $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} = Q$. Then $H\psi(x) = E\psi(x)$ with $E = \frac{\hbar^2 k^2}{2m}$. Furthermore, the momentum operator $\tilde{p} = -i\hbar\frac{\partial}{\partial x} = M$, then $\tilde{p}\psi(x) = p\psi(x)$ with $p = \hbar k$. Consequently

$$\boxed{E = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}}.$$

Proof of (10.27) The Shrödinger equation is

$$i\hbar\frac{\partial\Psi}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}-V\Psi=0, \tag{1}$$

then

$$\Psi^*\frac{\partial\Psi}{\partial t}=-\frac{\hbar}{2mi}\Psi^*\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{i\hbar}V\Psi^*\Psi.$$

If we now apply the complex conjugate operation to (1), we get

$$-i\hbar\frac{\partial\Psi^*}{\partial t}+\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2}-V\Psi^*=0,$$

then

$$\frac{\partial\Psi^*}{\partial t}\Psi=\frac{\hbar}{2mi}\frac{\partial^2\Psi^*}{\partial x^2}\Psi-\frac{1}{i\hbar}V\Psi^*\Psi.$$

Using the above results, we have

$$\begin{aligned}\frac{\partial\Psi^*}{\partial t}\Psi+\Psi^*\frac{\partial\Psi}{\partial t}&=\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2}-\frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)=\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}-\frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)\\&=\frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right).\end{aligned}$$

We now look at

$$\begin{aligned}\frac{d\langle x\rangle}{dt}&=\frac{d}{dt}\int_{-\infty}^{\infty}x\Psi^*(x,t)\Psi(x,t)dx=\int_{-\infty}^{\infty}x\frac{\partial}{\partial t}\left(\Psi^*\Psi\right)dx=\int_{-\infty}^{\infty}x\left(\frac{\partial\Psi^*}{\partial t}\Psi+\Psi^*\frac{\partial\Psi}{\partial t}\right)dx\\&=\frac{i\hbar}{2m}\int_{-\infty}^{\infty}x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)dx.\end{aligned}$$

We now assume that the wave function decays sufficiently fast so that $\lim_{x\to\pm\infty}x\Psi(x,t)=0$, and, in addition, the derivative $\frac{\partial\Psi}{\partial x}$ is bounded for $x\to\pm\infty$. Then, by performing integration by parts we obtain

$$\begin{aligned}\frac{d\langle x\rangle}{dt}&=\frac{i\hbar}{2m}x\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty}-\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)dx\\&=-\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x}-\frac{\partial\Psi^*}{\partial x}\Psi\right)dx=-\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx+\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial x}\Psi dx\\&=-\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx+\frac{i\hbar}{2m}\Psi^*\Psi\Bigg|_{-\infty}^{\infty}-\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx\\&=-\frac{i\hbar}{m}\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx=\frac{1}{m}\int_{-\infty}^{\infty}\Psi^*\frac{\hbar}{i}\frac{\partial\Psi}{\partial x}dx=\frac{1}{m}\int_{-\infty}^{\infty}\Psi^*\tilde{p}\Psi dx=\frac{1}{m}\langle p\rangle.\end{aligned}$$

Then,

$$\boxed{\langle p\rangle=m\frac{d\langle x\rangle}{dt}}.$$

Proof of (10.28)

$$\frac{d\langle p\rangle}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \Psi^* \tilde{p} \Psi dx = \int_{-\infty}^{\infty} \frac{\partial \Psi^*}{\partial t} \tilde{p} \Psi dx + \int_{-\infty}^{\infty} \Psi^* \tilde{p} \frac{\partial \Psi}{\partial t} dx.$$

We now observe that

$$\begin{aligned} \frac{\hbar}{i}\frac{\partial \Psi}{\partial t} &= -H\Psi \\ -\frac{\hbar}{i}\frac{\partial \Psi^*}{\partial t} &= -(H\Psi)^* \end{aligned}$$

with H the Hamiltonian. Because H is Hermitian, we can write:

$$\begin{aligned} \int_{-\infty}^{\infty} \Psi^* \tilde{p} \frac{\partial \Psi}{\partial t} dx &= \frac{1}{i\hbar}\int_{-\infty}^{\infty} \Psi^* \tilde{p} H \Psi dx \\ \int_{-\infty}^{\infty} \frac{\partial \Psi^*}{\partial t} \tilde{p} \Psi dx &= -\frac{1}{i\hbar}\int_{-\infty}^{\infty} (H\Psi)^* \tilde{p} \Psi dx = -\frac{1}{i\hbar}\int_{-\infty}^{\infty} \Psi^* H \tilde{p} \Psi dx. \end{aligned}$$

We then can write

$$\frac{d\langle p\rangle}{dt} = \int_{-\infty}^{\infty} -\frac{1}{i\hbar}\left(\Psi^* H \tilde{p} \Psi - \Psi^* \tilde{p} H \Psi\right) dx = \int_{-\infty}^{\infty} -\frac{1}{i\hbar}\Psi^*\left(H\tilde{p} - \tilde{p}H\right)\Psi dx.$$

We now observe that because $H = \frac{p^2}{2m} + V$ we have

$$\begin{aligned} (Hp - pH)\Psi &= \left(\frac{p^3}{2m} + Vp - \frac{p^3}{2m} + pV\right)\Psi = V\frac{\hbar}{i}\frac{\partial \Psi}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(V\Psi) \\ &= V\frac{\hbar}{i}\frac{\partial \Psi}{\partial x} - \frac{\hbar}{i}\left(\frac{\partial V}{\partial x}\right)\Psi - V\frac{\hbar}{i}\frac{\partial \Psi}{\partial x} \end{aligned}$$

then,

$$\frac{d\langle p\rangle}{dt} = \int_{-\infty}^{\infty} -\frac{1}{i\hbar}\Psi^*\left(-\frac{\hbar}{i}\left(\frac{\partial V}{\partial x}\right)\Psi\right) dx = \int_{-\infty}^{\infty} \Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi dx = \left\langle -\frac{\partial V}{\partial x}\right\rangle.$$

Then,

$$\boxed{\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle}.$$