Lecture 33 - April 16, 2012

Impacts

We would like to learn how to determine:

- Velocities of centers of mass
- angular velocities

of rigid bodies after they collide.

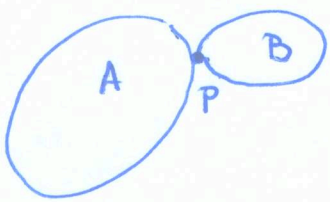
(I) Conservation of Momentum

Assume two rigid bodies A and B in two-dimensional motion in the same plane collide.

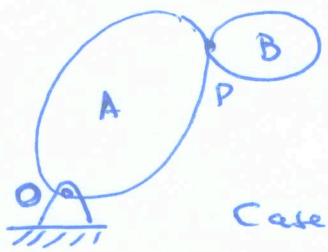
* Linear momentum:

If other forces are negligible in comparison to impact forces between A and B (Case A)

$\Rightarrow$ Total Linear Momentum is conserved during impact.



Case A



Case B

Note: If a rigid body has a pin support then linear momentum is not conserved (Case B).

* Angular Momentum

- If other forces and couples are negligible in comparison to impact forces and couples between A and B

  $\Rightarrow$ Total Angular Momentum about any fixed point O is conserved during the impact.

- If also A and B exert forces on each other only at their point of impact P and exert no couples on each other $\Rightarrow$ the angular momentum of each rigid body about P is conserved during the impact.

(II) Coefficient of restitution

Let

P = point of contact of A and B during impact

$\bar{V}_{Ap}$ = velocity of A at P just before impact

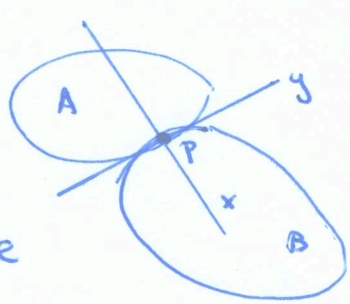
$\bar{V}_{BP}$ = velocity of B at P just before impact

$\bar{V}'_{AP}$ = velocity of A at P just after impact

$\bar{V}'_{BP}$ = velocity of B at P just after impact

If frictional forces resulting from impact are negligible

$\Rightarrow$ component of velocities normal to surfaces at P are related to e



coefficient of restitution $\leftarrow$

$$e = \frac{(\vec{v}'_{BP})_x - (\vec{v}'_{AP})_x}{(\vec{v}_{AP})_x - (\vec{v}_{BP})_x}$$

Analysis of impact

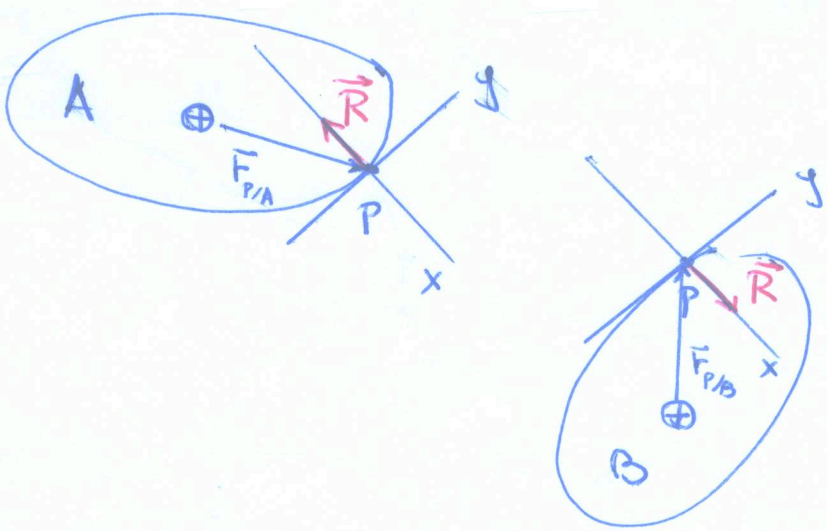
t_1 : time at which A and B first come into contact

t_c : time at which maximum deformation occurs and objects begin to "recover"

t_2 : time at which they separate.

R : magnitude of normal force exerted during impact

$\bar{v}_A, \bar{v}_{Ac}, \bar{v}_A'$
$\bar{v}_B, \bar{v}_{Bc}, \bar{v}_B'$ $\Rightarrow$ velocities of centers of mass of A and B at t_1, t_c, t_2.



By the Principle of Impulse and Momentum

A:
$$\int_{t_1}^{t_c} -R\,dt = m_A(\bar{V}_{Ac})_x - m_A(\bar{V}_A)_x \qquad (i)$$

$$\int_{t_c}^{t_2} -R\,dt = m_A(\bar{V}_A')_x - m_A(\bar{V}_{Ac})_x \qquad (ii)$$

B:
$$\int_{t_1}^{t_c} R\,dt = m_B(\bar{V}_{Bc})_x - m_B(\bar{V}_B)_x \qquad (iii)$$

$$\int_{t_c}^{t_2} R\,dt = m_B(\bar{V}_B')_x - m_B(\bar{V}_{Bc})_x \qquad (iv)$$

Recall
$$e = \frac{\int_{t_c}^{t_2} R\,dt}{\int_{t_1}^{t_c} R\,dt}$$

→ linear impulse during recovery phase

→ linear impulse during deformation phase

$$\frac{(ii)}{(i)} \Rightarrow e = \frac{(\bar{V}_A')_x - (\bar{V}_{Ac})_x}{(\bar{V}_{Ac})_x - (\bar{V}_A)_x} \Rightarrow (\bar{V}_A')_x = -(\bar{V}_A)_x e + (\bar{V}_{Ac})_x(1+e) \quad (*_1)$$

$$\frac{(iv)}{(iii)} \Rightarrow e = \frac{(\bar{V}_B')_x - (\bar{V}_{Bc})_x}{(\bar{V}_{Bc})_x - (\bar{V}_B)_x} \Rightarrow (V_B')_x = -(\bar{V}_B)_x e + (\bar{V}_{Bc})_x(1+e) \quad (*_2)$$

Now, apply Principle of Angular Impulse and Momentum:

let

$\omega_A, \omega_{AC}, \omega_{A'}$
$\omega_B, \omega_{BC}, \omega_{B'}$
} counterclockwise angular velocities of A and B at t_1, t_c, t_2

$$\bar{r}_{P/A} = x_A \hat{i} + y_A \hat{j}$$

$$\bar{r}_{P/B} = x_B \hat{i} + y_B \hat{j}$$

$$M_A = \bar{r}_{P/A} \times (-R\hat{i}) = y_A R \hat{k} \rightarrow \text{moment about center of mass of A due to } -\vec{R}$$

$$M_B = \bar{r}_{P/B} \times (R\hat{i}) = -y_B R \hat{k} \rightarrow \text{moment about center of mass of B due to } \vec{R}$$

By the Principle of Angular Impulse and Momentum

A:
$$\int_{t_1}^{t_c} y_A R \, dt = I_A \omega_{AC} - I_A \omega_A \quad (a)$$

$$\int_{t_c}^{t_2} y_A R \, dt = I_A \omega_{A'} - I_A \omega_{AC} \quad (b)$$

B:
$$\int_{t_1}^{t_c} -y_B R \, dt = I_B \omega_{BC} - I_B \omega_B \quad (c)$$

$$\int_{t_c}^{t_2} -y_B R \, dt = I_B \omega_{B'} - I_B \omega_{BC} \quad (d)$$

$\frac{(b)}{(a)}$ = $e = \frac{\omega_A' - \omega_{AC}}{\omega_{AC} - \omega_A} \Rightarrow \omega_A' = -\omega_A e + \omega_{AC}(1+e)$ $(*_3)$

$\downarrow$

y_A const.

$\frac{(d)}{(c)}$ = $e = \frac{\omega_B' - \omega_{BC}}{\omega_{BC} - \omega_B} \Rightarrow \omega_B' = -\omega_B e + \omega_{BC}(1+e)$ $(*_4)$

$\downarrow$

y_B const

By kinematics:

$(*_5)$
$$\begin{cases} (\vec{V}_{AP})_x = (\vec{V}_A)_x + (\vec{\omega}_A \times \vec{r}_{P/A})_x = (\vec{V}_A)_x - \omega_A y_A \\ (\vec{V}_{AP}')_x = (\vec{V}_A')_x - \omega_A' y_A \\ (\vec{V}_{BP})_x = (\vec{V}_B)_x - \omega_B y_B \\ (\vec{V}_{BP}')_x = (\vec{V}_B')_x - \omega_B' y_B \end{cases}$$

At tc the velocities of A and B equal at P:

$(*_6)$ $(\vec{V}_{AC})_x - \omega_{AC} y_A = (\vec{V}_{BC})_x - \omega_{BC} y_B$

$$(*_5) \Rightarrow \frac{(\vec{V}_{BP}')_x - (\vec{V}_{AP}')_x}{(\vec{V}_{AP})_x - (\vec{V}_{BP})_x} = \frac{(\vec{V}_B')_x - \omega_B' y_B - (\vec{V}_A')_x + \omega_A' y_A}{(\vec{V}_A)_x - \omega_A y_A - (\vec{V}_B)_x + \omega_B y_B}$$

$$= \frac{-(\vec{V}_B)_x e + (\vec{V}_{BC})_x (1+e) - \omega_B' y_B + (\vec{V}_A)_x e - (\vec{V}_{AC})_x (1+e) + \omega_A' y_A}{(\vec{V}_A)_x - \omega_A y_A - (\vec{V}_B)_x + \omega_B y_B}$$

$(*_1)$ and $(*_2)$

Use $(*_3)$ and $(*_4)$

$$= \Big[-(\bar{V}_B)_x e + (\bar{V}_{BC})_x (1+e) + \omega_B y_B e - \omega_{BC} y_B (1+e)$$

$$+ (\bar{V}_A)_x e - (\bar{V}_{AC})_x (1+e) - \omega_A y_A e + \omega_{AC} y_A (1+e) \Big] \frac{1}{(\bar{V}_A)_x - \omega_A y_A + (\bar{V}_B)_x + \omega_B y_B}$$

$$= e - \left[\frac{(\bar{V}_{AC})_x - \omega_{AC} y_A - (\bar{V}_{BC})_x + \omega_{BC} y_B}{(\bar{V}_A)_x - \omega_A y_A - (\bar{V}_B)_x + \omega_B y_B} \right] (1+e)$$

$$\underset{(*_6)}{=} e$$

$$\Longrightarrow \quad \boxed{e = \frac{(\bar{V}'_{BP})_x - (\bar{V}'_{AP})_x}{(\bar{V}_{AP})_x - (\bar{V}_{BP})_x}}$$