Critical Damping

We have the equation

$$\ddot{x} + 2d\dot{x} + \omega^2 x = 0 \quad (*)$$

Let $x(t) = C e^{\lambda t}$, the we obtain from (*)

$$\lambda^2 + 2\lambda d + \omega^2 = 0$$

with solution

$$\lambda = -\frac{2d \pm \sqrt{4d^2 - 4\omega^2}}{2} = -d \pm \sqrt{d^2 - \omega^2}$$

For $d = \omega \Rightarrow \boxed{\lambda = -d}$. This gives only one solution of the form

$$x(t) = C e^{-dt}$$

Second solution: We now show that $x(t) = Dt e^{-dt}$ is also a solution to (*):

$$x(t) = Dt e^{-dt}$$

$$\dot{x}(t) = De^{-dt} - dt\, De^{-dt} = (1-dt) D e^{-dt}$$

$$\ddot{x}(t) = -dDe^{-dt} + (1-dt)(-d)De^{-dt} = (d^2t - 2d) D e^{-dt}$$

$$\Rightarrow \ddot{x} + 2d\dot{x} + \omega^2 x = (d^2t - 2d) De^{-dt} + 2d(1-dt) D e^{-dt} + \omega^2 Dt e^{-dt}$$

$$= (d^2t - 2d + 2d - 2d^2t + \omega^2 t) De^{-dt}$$

$$= (\omega^2 - d^2) t\, De^{-dt} = 0 \quad \checkmark$$

The general solution is:

$$\boxed{x(t) = Ce^{-dt} + Dte^{-dt}}$$

C and D are constants determined by the initial conditions.