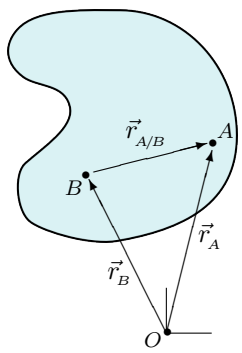
EM 311M - Dynamics
Exam 3 - **Solutions**- April 19, 2012

Write your work and solutions only on the **FRONT** side of the sheets provided, including white sheets.

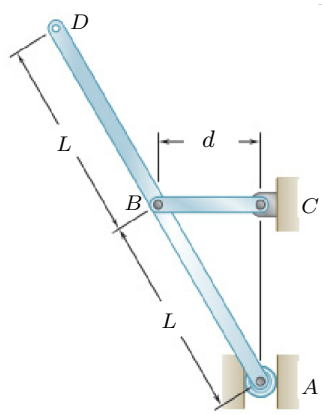
1. i. Given the positions of two points A and B in a rigid body with angular velocity $\vec{\omega}$, the rate of change of the relative position vector of A relative to B is given by

$$\frac{d\vec{r}_{A/B}}{dt} = \vec{\omega} \times \vec{r}_{A/B}.$$

(a) (1 point) Write the expression for the velocity of A as a function of the velocity of B and the angular velocity of the rigid body.
(b) (3 points) Derive the expression for the acceleration of A as a function of the acceleration of B and the angular velocity and acceleration of the rigid body.



ii. At the instant shown (see figure below) the crank BC has a given *constant* angular velocity ω_{BC} clockwise.
(a) (21 points) Determine the expressions for the acceleration of points A and D as functions of ω_{BC}, d and L.
(b) (2 bonus* points) Look at the velocities of A and B. What type of motion the rigid body AD undergoes instantaneously? Explain how to conclude that by using instantaneous centers.



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*Bonus points will be given only if exam grade is lower than 100 (maximum possible grade).

Solution:

i) (a) **(1 point)** The position vector of A can be written as

$$\vec{r}_A = \vec{r}_B + \vec{r}_{A/B},$$

and taking the time derivative we get

$$\frac{d\vec{r}_A}{dt} = \frac{d\vec{r}_B}{dt} + \frac{d\vec{r}_{A/B}}{dt} \Rightarrow \vec{\mathrm{v}}_A = \vec{\mathrm{v}}_B + \vec{\mathrm{v}}_{A/B}.$$

For A and B points in a rigid body, the velocity of A relative to B can be written as

$$\vec{\mathrm{v}}_{A/B} = \frac{d\vec{r}_{A/B}}{dt} = \vec{\omega} \times \vec{r}_{A/B},$$

with $\vec{\omega}$ the angular velocity of the rigid body. Then, we obtain

$$\boxed{\vec{\mathrm{v}}_A = \vec{\mathrm{v}}_B + \vec{\omega} \times \vec{r}_{A/B}}. \tag{1}$$

(b) **(3 points)** We now take the time derivative of (1):

$$\frac{d\vec{\mathrm{v}}_A}{dt} = \frac{d\vec{\mathrm{v}}_B}{dt} + \frac{d}{dt}\left(\vec{\omega} \times \vec{r}_{A/B}\right)$$

and get

$$\vec{a}_A = \vec{a}_B + \frac{d\vec{\omega}}{dt} \times \vec{r}_{A/B} + \vec{\omega} \times \frac{d\vec{r}_{A/B}}{dt},$$

from where the final result is obtained

$$\boxed{\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{A/B}\right)},$$

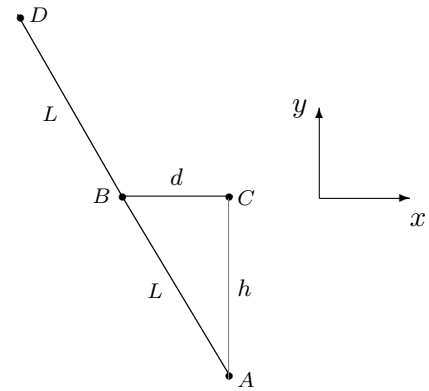
where $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$ is the angular acceleration. In a planar motion, the acceleration expression reduces to

$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \vec{r}_{A/B} - \omega^2\, \vec{r}_{A/B}.$$

ii) (a) In this type of questions, to obtain accelerations, we first need to find velocities. We have the following information:

- Point C is fixed: $\vec{v}_C = \vec{0}$ and $\vec{a}_C = \vec{0}$.
- Point A is constrained to move in the vertical direction: $\vec{v}_A = v_A\hat{j}$ and $\vec{a}_A = a_A\hat{j}$.
- Crank BC has constant clockwise angular velocity: $\vec{\omega}_{BC} = -\omega_{BC}\hat{k}$ and $\vec{\alpha}_{BC} = \vec{0}$.

Let's take a look at the following diagram:



where by the Pythagorean theorem

$$h = \sqrt{L^2 - d^2}.$$

(3 point) We compute first the velocity of B:

$$\begin{aligned}\vec{v}_B &= \vec{v}_C + \vec{\omega}_{BC} \times \vec{r}_{B/C} \\ &= \vec{0} - \omega_{BC}\hat{k} \times (-d\hat{i}) = d\,\omega_{BC}\hat{j} \\ &\boxed{\vec{v}_B = d\,\omega_{BC}\hat{j}}\end{aligned}$$

(6 point) We now can compute the velocity of A:

$$\begin{aligned}\vec{v}_A &= \vec{v}_B + \vec{\omega}_{AD} \times \vec{r}_{A/B} \\ v_A\hat{j} &= d\,\omega_{BC}\hat{j} + \omega_{AD}\hat{k} \times (d\hat{i} - h\hat{j}) \\ &= h\,\omega_{AD}\hat{i} + (d\,\omega_{BC} + d\,\omega_{AD})\hat{j}.\end{aligned}$$

Note that since A and B are points belonging to the rigid body AD, we used the angular velocity $\vec{\omega}_{AD}$. The above relation is a *vector* equation, thus we compare the components of the left-hand side and right-hand side:

$\hat{i}$ components: $0 = h\,\omega_{AD} \Rightarrow \boxed{\omega_{AD} = 0}$,

$\hat{j}$ components: $v_A = d\,\omega_{BC} + d\,\omega_{AD} = d\,\omega_{BC} \Rightarrow \boxed{\vec{v}_A = d\,\omega_{BC}\hat{j}}$.

(3 point) We compute now the acceleration of B:

$$\begin{aligned}\vec{a}_B &= \vec{a}_C + \vec{\alpha}_{BC} \times \vec{r}_{B/C} - \omega_{BC}^2\,\vec{r}_{B/C} \\ &= \vec{0} + \vec{0} + d\,\omega_{BC}^2\hat{i} = d\,\omega_{BC}^2\hat{i}\end{aligned}$$

$$\boxed{\vec{a}_B = d\,\omega_{BC}^2\hat{i}}.$$

(6 point) We now can compute the acceleration of A:

$$\begin{aligned}
\vec{a}_A &= \vec{a}_B + \vec{\alpha}_{AD} \times \vec{r}_{A/B} - \omega_{AD}^2\, \vec{r}_{A/B} \\
a_A \hat{j} &= d\,\omega_{BC}^2\, \hat{i} + \alpha_{AD} \hat{k} \times (d\hat{i} - h\hat{j}) + \vec{0} \\
&= (h\alpha_{AD} + d\,\omega_{BC}^2)\hat{i} + d\,\alpha_{AD}\, \hat{j}.
\end{aligned}$$

We compare the components of the left-hand side and right-hand side:

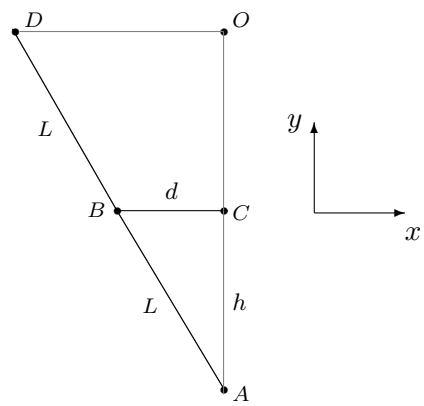
$\hat{i}$ components: $0 = h\alpha_{AD} + d\,\omega_{BC}^2 \Rightarrow \boxed{\alpha_{AD} = -\dfrac{d\,\omega_{BC}^2}{h}}$,

$\hat{j}$ components: $a_A = d\,\alpha_{AD} \Rightarrow \vec{a}_A = -\dfrac{d^2\,\omega_{BC}^2}{h}\hat{j}$.

The acceleration vector of A is

$$\boxed{\vec{a}_A = -\frac{d^2\,\omega_{BC}^2}{\sqrt{L^2 - d^2}}\hat{j}}.$$

(3 point) To compute the acceleration of D, we take a look at the diagram



where $\overline{DO} \parallel \overline{BC}$. We observe that the triangles $\triangle AOD$ and $\triangle ACB$ are similar. Then,

$$\frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AO}}{\overline{AC}} \Rightarrow \boxed{\overline{AO} = 2\overline{AC} = 2h} \qquad \text{and} \qquad \frac{\overline{AD}}{\overline{AB}} = \frac{\overline{DO}}{\overline{BC}} \Rightarrow \boxed{\overline{DO} = 2\overline{BC} = 2d}.$$

The acceleration of D is then

$$\begin{aligned}
\vec{a}_D &= \vec{a}_A + \vec{\alpha}_{AD} \times \vec{r}_{D/A} - \omega_{AD}^2\, \vec{r}_{D/A}, \\
\vec{a}_D &= -\frac{d^2\,\omega_{BC}^2}{h}\hat{j} + \alpha_{AD}\hat{k} \times (-2d\hat{i} + 2h\hat{j}) + \vec{0} \\
&= -2h\alpha_{AD}\hat{i} - \left(\frac{d^2\,\omega_{BC}^2}{h} + 2d\alpha_{AD}\right)\hat{j} = 2d\,\omega_{BC}^2\,\hat{i} + \frac{d^2\,\omega_{BC}^2}{h}\hat{j}.
\end{aligned}$$

The acceleration vector of D is

$$\boxed{\vec{a}_D = 2d\,\omega_{BC}^2\,\hat{i} + \frac{d^2\,\omega_{BC}^2}{\sqrt{L^2 - d^2}}\hat{j}}.$$

(b) **(2 bonus points)** We observe that $\vec{v}_A = \vec{v}_B$, i.e., they have the same direction and magnitude; thus, the body is instantaneously in *translation*. If we now draw lines through A and B perpendicular to their directions of motion, we observe that these lines are parallel and do not intersect. Then, there is no instantaneous center for bar AD at that moment, and the body is not rotating, thus it is in translation.

2. i. (5 points) Given a general system of N particles with masses m_i and velocities $\vec{v}_i$, $i = 1, \ldots, N$, derive the *Moment-Angular Momentum Principle.*

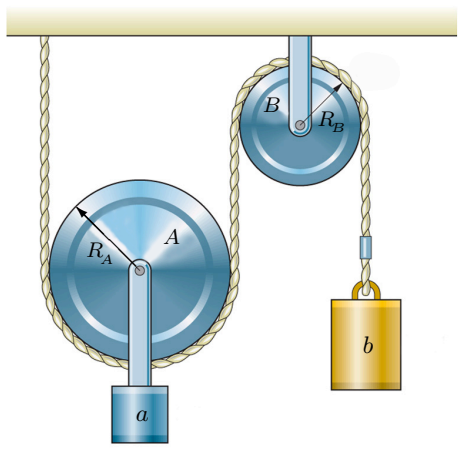
ii. In the system shown below, the following information is provided:

- The weight W_A of the pulley A, its radius R_A and its moment of inertia I_A about its center.
- The moment of inertia I_B of pulley B about its center and its radius R_B.
- The weights of the loads a and b: w_a and w_b.

Assume the system is released from rest.

(a) (18 points) What is the expression for the acceleration of the load b as a function of the given data?

(b) (2 points) Take the result of (a) and assume you do not know the weight w_b. What is the condition on the magnitude of w_b so that the velocity of the load b is downward?



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Solution:

i)($\frac{1}{2}$ points) By Newton's 2nd law for particle i

$$\sum_j \vec{f}_{ij} + \vec{f}_i^{\,\mathrm{E}} = \frac{d}{dt}(m_i \vec{\mathrm{v}}_i),$$

with $\vec{f}_{ij}$ the force that particle j exerts on particle i and $\vec{f}_i^{\,\mathrm{E}}$ an external force exerted on particle i. Taking the cross product with $\vec{r}_i$ and summing over all particles $i = 1$ to N we obtain

$$\sum_i \sum_j \vec{r}_i \times \vec{f}_{ij} + \sum_i \vec{r}_i \times \vec{f}_i^{\,\mathrm{E}} = \sum_i \vec{r}_i \times \frac{d}{dt}(m_i \vec{\mathrm{v}}_i). \tag{2}$$

(2 point) We now show that the total moment corresponding to internal forces in (2) is zero:

$$\begin{aligned}
\sum_i \sum_j \vec{r}_i \times \vec{f}_{ij} &= \frac{1}{2}\sum_i \sum_j \vec{r}_i \times \vec{f}_{ij} + \frac{1}{2}\sum_j \sum_i \vec{r}_j \times \vec{f}_{ji} && \leftarrow \quad \text{Change dummy indexes names in second term} \\
&= \frac{1}{2}\sum_i \sum_j \vec{r}_i \times \vec{f}_{ij} - \frac{1}{2}\sum_j \sum_i \vec{r}_j \times \vec{f}_{ij} && \leftarrow \quad \text{Newton's 3rd law : } \vec{f}_{ji} = -\vec{f}_{ij} \\
&= \frac{1}{2}\sum_i \sum_j (\vec{r}_i - \vec{r}_j) \times \vec{f}_{ij} && \leftarrow \quad \text{Change order of summation in second term} \\
&= \vec{0}. && \leftarrow \quad \text{Assume } \vec{f}_{ij} \parallel (\vec{r}_i - \vec{r}_j)
\end{aligned}$$

(2 point) Furthermore, we observe that

$$\frac{d}{dt}(\vec{r}_i \times m_i \vec{\mathrm{v}}_i) = \frac{d\vec{r}_i}{dt} \times m_i \vec{\mathrm{v}}_i + \vec{r}_i \times \frac{d}{dt}(m_i \vec{\mathrm{v}}_i) = \cancelto{0}{\vec{\mathrm{v}}_i \times m_i \vec{\mathrm{v}}_i} + \vec{r}_i \times \frac{d}{dt}(m_i \vec{\mathrm{v}}_i)$$

and thus

$$\sum_i \vec{r}_i \times \frac{d}{dt}(m_i \vec{\mathrm{v}}_i) = \frac{d}{dt}\left(\sum_i \vec{r}_i \times m_i \vec{\mathrm{v}}_i\right) = \frac{dH_O}{dt}$$

with H_O the total angular momentum about the origin O.

($\frac{1}{2}$ point) Let M_O denote the total moment exerted on the system by external forces:

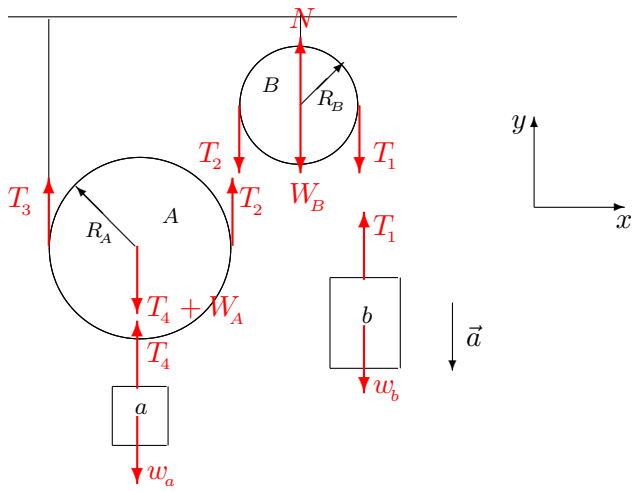
$$M_O = \sum_i \vec{r}_i \times \vec{f}_i^{\,\mathrm{E}}.$$

Then, we can write (2) as

$$\boxed{M_O = \frac{dH_O}{dt}},$$

the *Moment-Angular Momentum Principle* for a system of N particles.

ii) (a) **(2 points)** We draw the free-body diagram:



We now write the equations of motion:

- **(2 points)** Load b:

$$\sum F_y = \boxed{T_1 - w_b = m_b a}. \tag{1}$$

- **(2 points)** Pulley B:

$$\sum F_y = N - W_B = 0.$$

Note: We do not have the information about W_B, so we will not use this equation.

$$\sum M = \left[R_B \hat{i} \times (-T_1)\hat{j} + (-R_B)\hat{i} \times (-T_2)\hat{j}\right] \cdot \hat{k} = I_B \alpha_B$$

then,

$$\boxed{R_B(T_2 - T_1) = I_B \alpha_B}. \tag{2}$$

- **(2 points)** Pulley A:

$$\sum F_y = \boxed{T_2 + T_3 - (T_4 + W_A) = m_A a_A}. \tag{3}$$

$$\sum M = \left[R_A \hat{i} \times (T_2)\hat{j} + (-R_A)\hat{i} \times (T_3)\hat{j}\right] \cdot \hat{k} = I_A \alpha_A$$

then,

$$\boxed{R_A(T_2 - T_3) = I_A \alpha_A}. \tag{4}$$

- **(2 points)** Load a:

$$\sum F_y = \boxed{T_4 - w_a = m_a a_A}. \tag{5}$$

Note: Pulley A and load a move together; thus they have the *same* acceleration.

Unknowns: $T_1, T_2, T_3, T_4, \alpha_A, \alpha_B, a_A, a$. We have 8 unknowns, thus we need 8 equations. We will look for 3 additional equations using kinematics.

(1 points) Before that, let's combine some equations to get a reduced system of equations. Using (1) and (2):

$$T_2 = T_1 + \frac{I_B}{R_B}\alpha_B = w_b + m_b a + \frac{I_B}{R_B}\alpha_B,$$

then

$$\boxed{T_2 = w_b + m_b a + \frac{I_B}{R_B}\alpha_B}. \qquad (*)$$

Using (3) and (4):

$$2T_2 = T_4 + W_A + m_A a_A + \frac{I_A}{R_A}\alpha_A$$

and using (5) we get

$$2T_2 = w_a + m_a a_A + W_A + m_A a_A + \frac{I_A}{R_A}\alpha_A$$

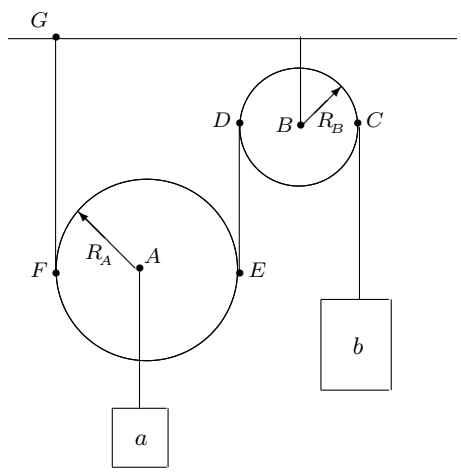
then

$$\boxed{2T_2 = (w_a + W_A) + (m_a + m_A)a_A + \frac{I_A}{R_A}\alpha_A}. \qquad (**)$$

Combine $(*)$ and $(**)$ to get

$$\boxed{2\left[w_b + m_b a + \frac{I_B}{R_B}\alpha_B\right] = (w_a + W_A) + (m_a + m_A)a_A + \frac{I_A}{R_A}\alpha_A}. \qquad (***)$$

The unknowns are: $a, a_A, \alpha_A, \alpha_B$. Thus, we need now three additional equations, to express a_A, α_A, α_B as functions of a. We now look for special points that will provide kinematic constraints:



<u>Kinematic constraints</u>

(2 points) Kinematic constraint 1:

$$(i) \qquad a = (\vec{a}_C)_y,$$

$$(ii) \qquad \vec{a}_C = \vec{a}_B + \vec{\alpha}_B \times \vec{r}_{C/B} - \omega_B^2\,\vec{r}_{A/B} = \vec{0} + \alpha_B\hat{k} \times R_B\hat{\imath} - \omega_B^2 R_B\hat{\imath}$$
$$= -R_B\omega_B^2\hat{\imath} + R_B\alpha_B\hat{\jmath}.$$

Then, we obtain

$$\boxed{a = R_B \alpha_B}. \tag{6}$$

(2 points) Kinematic constraint 2:

$$\begin{aligned}
(iii) \qquad \vec{a}_D &= \vec{a}_B + \vec{\alpha}_B \times \vec{r}_{D/B} - \omega_B^2 \vec{r}_{D/B} = \vec{0} + \alpha_B \hat{k} \times (-R_B)\hat{i} - \omega_B^2(-R_B)\hat{i} \\
&= R_B \omega_B^2 \hat{i} - R_B \alpha_B \hat{j} \\
(iv) \qquad \vec{a}_E &= \vec{a}_A + \vec{\alpha}_A \times \vec{r}_{E/A} - \omega_A^2 \vec{r}_{E/A} = a_A \hat{j} + \alpha_A \hat{k} \times R_A \hat{i} - \omega_A^2 R_A \hat{i} \\
&= -R_A \omega_A^2 \hat{i} + (a_A + R_A \alpha_A)\hat{j}
\end{aligned}$$

We have that $(\vec{a}_D)_y = (\vec{a}_E)_y$, then

$$\boxed{-R_B \alpha_B = a_A + R_A \alpha_A}. \tag{7}$$

(2 points) Kinematic constraint 3:

$$\begin{aligned}
(v) \qquad \vec{a}_F &= \vec{a}_A + \vec{\alpha}_A \times \vec{r}_{F/A} - \omega_A^2 \vec{r}_{F/A} = a_A \hat{j} + \alpha_A \hat{k} \times (-R_A)\hat{i} - \omega_A^2(-R_A)\hat{i} \\
&= R_A \omega_A^2 \hat{i} + \left(a_A \hat{j} - \alpha_A R_A\right)\hat{j} \\
(v) \qquad \vec{a}_G &= \vec{0}
\end{aligned}$$

We have that $(\vec{a}_G)_y = (\vec{a}_F)_y$, then

$$\boxed{a_A = \alpha_A R_A}. \tag{8}$$

We now have from (6)

$$\boxed{\alpha_B = \frac{a}{R_B}}.$$

Using this result in (7) together with (8) we have

$$-R_B\left(\frac{a}{R_B}\right) = \alpha_A R_A + R_A \alpha_A \Rightarrow \boxed{\alpha_A = -\frac{a}{2R_A}}.$$

And finally, from (8) we obtain

$$\boxed{a_A = -\frac{a}{2}}.$$

(1 points) Substituting a_A, α_A, α_B in $(***)$, we get

$$2\left[w_b + m_b a + \frac{I_B}{R_B^2} a\right] = (w_a + W_A) - (m_a + m_A)\frac{a}{2} - \frac{I_A}{2R_A^2} a,$$

from where we obtain

$$w_b - \frac{(w_a + W_A)}{2} = -\left[m_b + \frac{I_B}{R_B^2} + \frac{1}{4}(m_a + m_A) + \frac{I_A}{4R_A^2}\right] a,$$

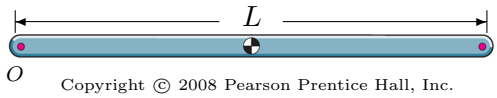
and the acceleration of the load b is

$$\boxed{a = -\frac{w_b - \dfrac{(w_a + W_A)}{2}}{\left[\dfrac{w_b}{g} + \dfrac{I_B}{R_B^2} + \dfrac{w_a + W_A}{4g} + \dfrac{I_A}{4R_A^2}\right]}}$$

(b) **(2 points)** For the velocity of load b to be downward, we need $a < 0$. Then, the condition is

$$\boxed{w_b > \frac{(w_a + W_A)}{2}}.$$

3. i. In the figure below, a uniform slender bar of length L and mass m is presented.
(a) (2 points) Use direct integration to compute the moment of inertia of the bar about an axis through its center of mass.
(b) (3 points) Use the parallel-axis theorem to compute the moment of inertia of the bar about an axis through the point O (see figure).



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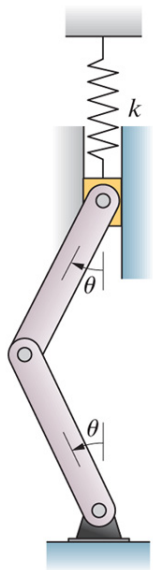
ii. (20 points) In the system below, each bar is of mass m and length L. The spring constant k is given, and the spring is unstretched when $\theta = 0$. If the system is released from rest with the bars vertical, what is the magnitude of the angular velocity of the bars for a given angle $\theta = \theta_0 > 0$, as a function of the given data.



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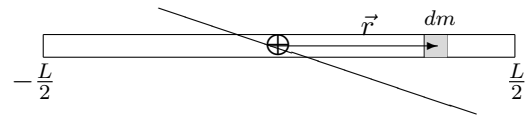
Solution:

i) (a) **(2 point)** To compute the moment of inertia of the slender bar, we divide the bar into infinitesimal pieces of material, each of mass dm as shown below:



The moment of inertia is

$$I = \int_m r^2 dm.$$

We need to find the differential of mass dm. For a uniform slender bar, the mass density is $\rho = \frac{m}{AL}$, where A is the cross-sectional area. Then, $dm = \rho A dr$. Now, we compute

$$I = \int_m r^2 dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} r^2 \rho A dr = \rho A \frac{r^3}{3}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \rho A \frac{L^3}{12},$$

where we assume A to be constant. We then obtain

$$\boxed{I = \tfrac{1}{12} m L^2}.$$

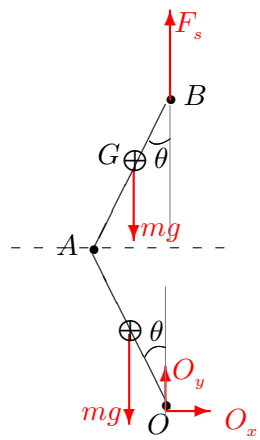
(b) **(3 point)** Since the center of mass is at a distance $L/2$ from O, by the parallel-axis theorem

$$I_O = m\left(\frac{L}{2}\right)^2 + I = m\frac{L^2}{4} + m\frac{L^2}{12}.$$

We then get

$$\boxed{I_O = \tfrac{1}{3} m L^2}.$$

ii) (2 point) We draw the free-body diagram:



The only forces *that do work* are the force of the spring F_s and the weights of the two bars; these forces are conservative.

By conservation of energy:

$$\underbrace{T_1 + V_1}_{\substack{\text{Total energy when}\\ \text{system is released}}} = \underbrace{T_2 + V_2}_{\substack{\text{Total energy for}\\ \text{given angle } \theta_0}} .$$

(3 point) The kinetic energy of the system for any instance is

$$T = \underbrace{\frac{1}{2} I_O \omega_{OA}^2}_{\substack{\text{Kinetic energy of bar } \overline{OA}\\ \text{(rotating about fixed axis though } O)}} + \underbrace{\frac{1}{2} m v_G^2 + \frac{1}{2} I \omega_{AB}^2}_{\text{Kinetic energy of bar } \overline{AB}} .$$

(3 point) The potential energy of the system for a given angle θ is

$$V = \underbrace{\frac{L}{2}\cos(\theta) mg}_{\text{Potential energy of bar } \overline{OA}} + \underbrace{\frac{3L}{2}\cos(\theta) mg}_{\text{Potential energy of bar } \overline{AB}} + \underbrace{\frac{1}{2} k \left(2L - 2L\cos(\theta)\right)^2}_{\text{Potential energy of spring}} .$$

Then, we have from the conservation of energy equation (with $T_1 = 0$)

$$\frac{L}{2} mg + \frac{3L}{2} mg = \frac{1}{2} I_O \omega_{OA}^2 + \frac{1}{2} m v_G^2 + \frac{1}{2} I \omega_{AB}^2 + \frac{L}{2}\cos(\theta_0) mg + \frac{3L}{2}\cos(\theta_0) mg + \frac{1}{2} k \left(2L - 2L\cos(\theta_0)\right)^2$$

which gives

$$\boxed{4Lmg(1 - \cos(\theta_0)) - 4kL^2 \left(1 - \cos(\theta_0)\right)^2 = I_O \omega_{OA}^2 + m v_G^2 + I \omega_{AB}^2} \tag{1}$$

The unknowns are : $\omega_{OA}, \omega_{AB}, v_G$.

By kinematics:

(2 point)

$$\begin{aligned}\vec{v}_A &= \vec{v}_O + \vec{\omega}_{OA} \times \vec{r}_{A/O} = \vec{0} + \omega_{OA}\hat{k} \times (-L\sin(\theta_0)\hat{\imath} + L\cos(\theta_0)\hat{\jmath}) \\ &= -L\,\omega_{OA}\cos(\theta_0)\hat{\imath} - L\,\omega_{OA}\sin(\theta_0)\hat{\jmath}\end{aligned}$$

(2 point)

$$\begin{aligned}\vec{v}_A &= \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{A/B} = v_B\hat{\jmath} + \omega_{AB}\hat{k} \times (-L\sin(\theta_0)\hat{\imath} - L\cos(\theta_0)\hat{\jmath}) \\ &= v_B\hat{\jmath} + L\,\omega_{AB}\cos(\theta_0)\hat{\imath} - L\,\omega_{AB}\sin(\theta_0)\hat{\jmath}\end{aligned}$$

(2 point) Equating both expressions for $\vec{v}_A$, we obtain

$$-L\,\omega_{OA}\cos(\theta_0)\hat{\imath} - L\,\omega_{OA}\sin(\theta_0)\hat{\jmath} = L\,\omega_{AB}\cos(\theta_0)\hat{\imath} + (v_B - L\,\omega_{AB}\sin(\theta_0))\,\hat{\jmath}$$

and we compare the components of the vector equation:

$$\begin{aligned}&\hat{\imath}\text{ components:} \quad -L\,\omega_{OA}\cos(\theta_0) = L\,\omega_{AB}\cos(\theta_0) \Rightarrow \boxed{\omega_{AB} = -\omega_{OA}}, \\ &\hat{\jmath}\text{ components:} \quad -L\,\omega_{OA}\sin(\theta_0) = v_B - L\,\omega_{AB}\sin(\theta_0) \Rightarrow \boxed{v_B = -2L\,\omega_{OA}\sin(\theta_0)}.\end{aligned}$$

(2 point) We now find the expression of the velocity of the center of mass of the bar $\overline{AB}$:

$$\begin{aligned}\vec{v}_G &= \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{G/B} = v_B\hat{\jmath} + \omega_{AB}\hat{k} \times \left(-\frac{L}{2}\sin(\theta_0)\hat{\imath} - \frac{L}{2}\cos(\theta_0)\hat{\jmath}\right) \\ &= \frac{L}{2}\,\omega_{AB}\cos(\theta_0)\hat{\imath} + \left(v_B - \frac{L}{2}\,\omega_{AB}\sin(\theta_0)\right)\hat{\jmath} \\ &= \frac{L}{2}\,\omega_{AB}\cos(\theta_0)\hat{\imath} + \frac{3L}{2}\,\omega_{AB}\sin(\theta_0)\hat{\jmath}\end{aligned}$$

Then,

$$\begin{aligned}v_G^2 &= \left(\frac{L}{2}\,\omega_{AB}\cos(\theta_0)\right)^2 + \left(\frac{3L}{2}\,\omega_{AB}\sin(\theta_0)\right)^2 \\ &= \frac{1}{4}L^2\omega_{AB}^2\left(\cos^2(\theta_0) + 9\sin^2(\theta_0)\right) = \frac{1}{4}L^2\omega_{AB}^2\left(1 - \sin^2(\theta_0) + 9\sin^2(\theta_0)\right)\end{aligned}$$

$$\boxed{v_G^2 = \frac{1}{4}L^2\omega_{AB}^2\left(1 + 8\sin^2(\theta_0)\right)}.$$

(2 point) We substitute the above into (1) to get

$$4Lmg(1-\cos(\theta_0)) - 4kL^2\left(1-\cos(\theta_0)\right)^2 = I_O\omega_{OA}^2 + m\frac{1}{4}L^2\omega_{OA}^2\left(1 + 8\sin^2(\theta_0)\right) + I\omega_{OA}^2.$$

(2 point) For a slender bar we have

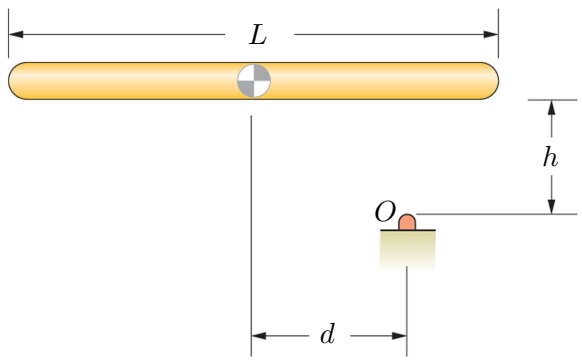
$$I = \frac{1}{12}mL^2 \qquad ; \qquad I_O = \frac{1}{3}mL^2$$

$$4Lmg(1-\cos(\theta_0)) - 4kL^2\left(1-\cos(\theta_0)\right)^2 = mL^2\left[\frac{5}{12} + \frac{1}{4}\left(1 + 8\sin^2(\theta_0)\right)\right]\omega_{OA}^2$$

then

$$\boxed{|\omega_{OA}| = \sqrt{\frac{4(1-\cos(\theta_0))\left[mg - kL\left(1-\cos(\theta_0)\right)\right]}{mL\left[\frac{5}{12} + \frac{1}{4}\left(1 + 8\sin^2(\theta_0)\right)\right]}}}$$

4. A slender bar of mass m is released from rest in the horizontal position from a height h above the fixed projection at O. The horizontal distance from the bar's center of mass to the fixed point O is d.



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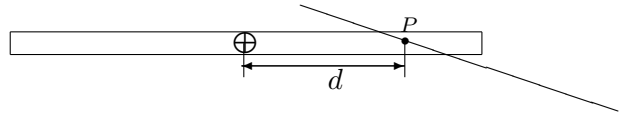
i. (a) (2 points) What is the moment of inertia of the bar about the point of impact?
(b) (5 points) Assume the impact of the bar with the projection at O is *perfectly plastic* (i.e., the bar adheres to the fixed point O). Find the expression of the bar's angular velocity immediately after the impact as a function of the given data.

ii. Assume now that an *arbitrary* value of the coefficient of restitution e is given.
(a) (10 points)
- What is the expression for the bar's angular velocity immediately after the impact as a function of the given data?
- Show that for $e = 0$, you recover the result obtained in "i.(b)".

(b) (8 points)
- Assuming a general value for e, what is the expression for the distance d that would cause the velocity of the bar's center of mass to be zero immediately after the impact?
- What is the bar's kinetic energy, in this case, immediately after the impact?

Solution:

i) (a) **(2 point)** We compute the moment of inertia of the slender bar about an axis through the point P, which is the point where the bar will impact the fixed point O:



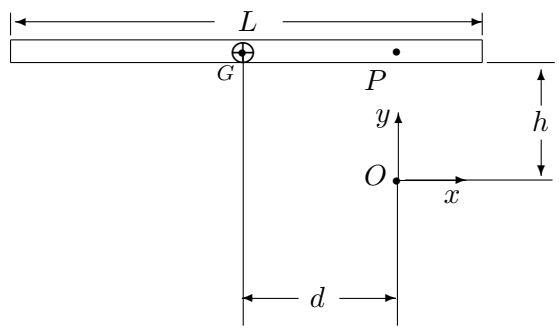
For this purpose, we use the parallel-axis theorem

$$I_P = I + md^2$$

with I the moment of inertia of the slender bar about an axis through its center of mass. Then, we obtain

$$\boxed{I_P = \tfrac{1}{12}mL^2 + md^2}.$$

(b) **(2 point)** The system is diagram below:



We first compute the velocity of the center of mass just before the impact. The only force acting on the bar is its weight, which is conservative. By conservation of energy

$$T_1 + V_1 = T_2 + V_2$$

then,

$$\underbrace{0}_{\substack{\text{Kinetic energy}\\ \text{when bar is released}}} + \underbrace{mgh}_{\substack{\text{Potential energy}\\ \text{when bar is released}}} = \underbrace{\frac{1}{2}mv_G^2 + \frac{1}{2}I\omega^2}_{\substack{\text{Kinetic energy}\\ \text{just before impact}}} + \underbrace{0}_{\substack{\text{Potential energy}\\ \text{just before impact}}}.$$

Note: At the release moment, the bar has no velocity and no angular velocity, i.e., the angular momentum is zero. Particularly, the angular momentum about the center of mass $H = 0$. The weight exerts a force at the center of mass, thus the moment about the center $\sum M = 0$. Then, we have

$$\sum M = \frac{dH}{dt} \Rightarrow H = \text{constant} = 0$$

until the impact; furthermore, $H = I\omega$, then $\omega = 0$.

From the conservation of energy equation we obtain

$$v_G = \sqrt{2gh} \tag{1}$$

so that the velocity of the center of mass just before the impact is

$$\boxed{\vec{\mathrm{v}}_G = -\sqrt{2gh}\,\hat{j}}.$$

(3 point) During the impact, we have conservation of angular momentum about O. Particularly, since the bar adheres to the fixed point O, after the impact it rotates about a fixed axis through O. Then, by conservation of angular momentum about O

$$\begin{aligned} \left[\vec{r}_{G/O} \times m\vec{\mathrm{v}}_G\right] \cdot \hat{k} + I\cancelto{0}{\omega} &= I_P\omega' \\ \Rightarrow \qquad \left[-d\hat{i} \times m(-\sqrt{2gh})\hat{j}\right] \cdot \hat{k} &= \left(\frac{1}{12}mL^2 + md^2\right)\omega'. \end{aligned}$$

The angular velocity just after the impact is

$$\boxed{\omega' = \frac{d\sqrt{2gh}}{d^2 + \dfrac{L^2}{12}}} \tag{2}$$

ii) (a) **(2 point)** We now assume an arbitrary value for the coefficient of restitution e. Then, we can relate the velocities (in the vertical direction) just before and after the impact at the point of impact P (which coincides with O at impact):

$$e = \frac{(\vec{\mathrm{v}}'_P)_y - \cancelto{\text{negligible}}{(\vec{\mathrm{v}}'_O)_y}}{\cancelto{0}{(\vec{\mathrm{v}}_O)_y} - (\vec{\mathrm{v}}_P)_y}.$$

Then,

$$\boxed{-(\vec{\mathrm{v}}_P)_y e = (\vec{\mathrm{v}}'_P)_y}. \tag{3}$$

(2 point) By kinematics:

$$\begin{aligned} \vec{\mathrm{v}}_P &= \vec{\mathrm{v}}_G + \cancelto{0}{\vec{\omega}} \times \vec{\mathrm{r}}_{P/G} = -v_G\hat{j}, \\ \vec{\mathrm{v}}'_P &= \vec{\mathrm{v}}'_G + \vec{\omega}' \times \vec{\mathrm{r}}_{P/G} = -v'_G\hat{j} + \omega'\hat{k} \times (d\hat{i}) \\ &= (-v'_G + d\omega')\hat{j}. \end{aligned}$$

Taking the y-components and using (3), we have

$$\boxed{v_G e = -v'_G + d\omega'}. \tag{4}$$

(2 point) We now use conservation of angular momentum about O during the impact:

$$\left(\vec{H}_O\right)_{\text{before}} = \left(\vec{H}_O\right)_{\text{after}}$$

then,

$$\vec{r}_{G/P} \times m\vec{v}_G + I\omega\hat{k} = \vec{r}_{G/P} \times m\vec{v}_G' + I\omega'\hat{k}$$

$$\boxed{dmv_G + I\cancelto{0}{\omega} = dmv_G' + I\omega'}. \tag{5}$$

(2 point) For a slender bar $I = \frac{1}{12}mL^2$, then by (5) we have

$$dmv_G = dmv_G' + \left(\frac{1}{12}mL^2\right)\omega'.$$

or

$$dv_G = dv_G' + \left(\frac{1}{12}L^2\right)\omega'. \tag{6}$$

Using (1) and (4), we have

$$\boxed{v_G' = d\omega' - \sqrt{2gh}\,e}. \tag{7}$$

Using (6)

$$d\sqrt{2gh} = d(d\omega' - \sqrt{2gh}\,e) + \left(\frac{1}{12}L^2\right)\omega',$$

from where we conclude

$$\boxed{\omega' = \frac{d\sqrt{2gh}(1+e)}{d^2 + \dfrac{L^2}{12}}}. \tag{8}$$

(2 point) If $e = 0$

$$\boxed{\omega' = \frac{d\sqrt{2gh}}{d^2 + \dfrac{L^2}{12}}},$$

and we recover the result of (2).

(b) We now assume a general value for e.

(2 point) Using (7) and (8)

$$\begin{aligned} v_G' &= \frac{d^2\sqrt{2gh}(1+e)}{d^2 + \dfrac{L^2}{12}} - \sqrt{2gh}\,e \\ &= \sqrt{2gh}\frac{d^2(1+\cancel{e}) - (\cancel{d^2} + \frac{L^2}{12})e}{d^2 + \dfrac{L^2}{12}}, \end{aligned}$$

then we get

$$v'_G = \left(\frac{d^2 - \frac{L^2}{12}e}{d^2 + \frac{L^2}{12}} \right) \sqrt{2gh}.$$

(2 point) We now observe that to get $v'_G = 0$ we need

$$\boxed{d = \sqrt{\frac{e}{12}}L}. \tag{9}$$

(2 point) In this case, the velocity of the center of mass of the bar is zero ($v'_G = 0$). However, the kinetic energy is composed of translational and rotational terms:

$$T = \frac{1}{2}m(v'_G)^2 + \frac{1}{2}I(\omega')^2 = \frac{1}{2}I(\omega')^2.$$

Then, we compute the angular velocity after the impact using (8) and (9):

$$\omega' = \frac{\sqrt{\frac{e}{12}}L\sqrt{2gh}(1+e)}{\frac{e}{12}L^2 + \frac{L^2}{12}} = \frac{\sqrt{\frac{e}{12}}L\sqrt{2gh}\cancel{(1+e)}}{\frac{L^2}{12}\cancel{(1+e)}}$$

and

$$\boxed{\omega' = \frac{\sqrt{24egh}}{L}}. \tag{10}$$

(2 point) Using (10), we compute the kinetic energy

$$T = \frac{1}{2}I(\omega')^2 = \frac{1}{2}\frac{mL^2}{12}\frac{24egh}{L^2} = megh$$

giving the kinetic energy

$$\boxed{T = emgh}.$$