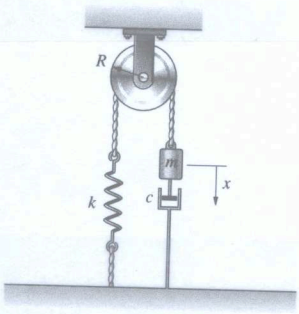
EX 21.46

Problem 21.46 The radius of the pulley is $R = 100$ mm and its moment of inertia is $I = 0.1$ kg-m^2. The mass $m = 5$ kg, and the spring constant is $k = 135$ N/m. The cable does not slip relative to the pulley. The coordinate x measures the displacement of the mass relative to the position in which the spring is unstretched. Determine x as a function of time if $c = 60$ N-s/m and the system is released from rest with $x = 0$.



Solution

Data:

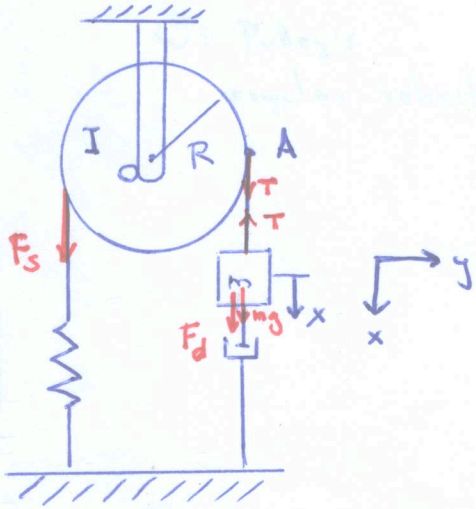
$R = 100\,mm = 0.1\,m$

$m = 5\,kg$

$I = 0.1\,kg\,m^2$

$k = 135\,N/m$

$c = 60\,Ns/m$

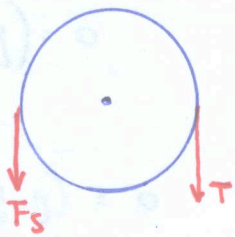


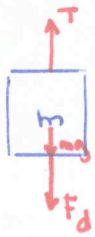
initial conditions $\begin{cases} x(0) = 0 \\ \dot{x}(0) = 0 \end{cases}$

Load:

$$\sum F_x = \boxed{mg - c\dot{x} - T = m\ddot{x}} \quad (1)$$

(damping force acts in opposition to velocity)





Pulley:

$$\sum M = (R\hat{j} \times T\hat{i})\cdot\hat{k} + (-R\hat{j} \times kx\hat{j})\cdot\hat{k} = -I\alpha \qquad (\alpha > 0)$$

$$-R(T - kx) = -I\alpha$$

$$\boxed{R(T - kx) = I\alpha} \quad (2)$$

$$(2) \Rightarrow \quad T = \frac{I\alpha}{R} + kx$$

$$(1) \Rightarrow \quad mg - c\dot{x} - \left(\frac{I\alpha}{R} + kx\right) = m\ddot{x}$$

$$m\ddot{x} + \frac{I}{R}\alpha + c\dot{x} + kx - mg = 0 \qquad (3)$$

<u>Kinematics</u> : Relate α to $\ddot{x}$

$$\vec{a}_A = \vec{a}_O + \vec{\alpha} \times \vec{r}_{A/O} - \omega^2 \vec{r}_{A/O}$$

ω: Pulley's angular velocity

$$= \vec{0} + -\alpha \hat{k} \times R\hat{j} - \omega^2 R\hat{j}$$

$$= R\alpha\hat{i} - R\omega^2\hat{j}$$

$$(\vec{a}_A)_x = R\alpha = \ddot{x} \quad \Rightarrow \quad \boxed{\alpha = \frac{\ddot{x}}{R}}$$

$$(3) \Rightarrow \quad \left(m + \frac{I}{R^2}\right)\ddot{x} + c\dot{x} + k\left(x - \frac{m}{k}g\right) = 0$$

$$\ddot{x} + \underbrace{\frac{2c}{2\left(m + \frac{I}{R^2}\right)}}_{2d}\dot{x} + \underbrace{\frac{k}{m + \frac{I}{R^2}}}_{\omega^2}\underbrace{\left(x - \frac{m}{k}g\right)}_{\tilde{x}} = 0$$

$$\boxed{d = \frac{c}{2\left(m + \frac{I}{R^2}\right)} \qquad \omega = \sqrt{\frac{k}{m + \frac{I}{R^2}}}}$$

ω: angular frequency

$$\ddot{\tilde{x}} + 2d\dot{\tilde{x}} + \omega^2\tilde{x} = 0$$

$d = 2 \qquad \omega = 3 \Rightarrow d < \omega \Rightarrow$ subcritical damping

$$\omega_d = \sqrt{\omega^2 - d^2} = \sqrt{5}$$

The general solution is

$$\tilde{x}(t) = e^{-dt}\left[A\sin(\omega_d t) + B\cos(\omega_d t)\right]$$

$$\Longrightarrow \boxed{x(t) = \frac{mg}{k} + e^{-dt}\left[A\sin(\omega_d t) + B\cos(\omega_d t)\right]}$$

Use initial conditions to find A and B

$$x(0) = \frac{mg}{k} + B = 0 \Longrightarrow \boxed{B = -\frac{mg}{k}}$$

$$\dot{x}(t) = -de^{-dt}\left[A\sin(\omega_d t) + B\cos(\omega_d t)\right]$$
$$+ e^{-dt}\left[\omega_d A\cos(\omega_d t) - B\omega_d\sin(\omega_d t)\right]$$

$$\dot{x}(0) = -dB + \omega_d A = 0 \Longrightarrow \boxed{A = \frac{dB}{\omega_d}}$$

$$\boxed{x(t) = \frac{mg}{k}\left[1 + e^{-dt}\left(\frac{d}{\omega_d}\sin(\omega_d t) - \cos(\omega_d t)\right)\right]}$$

$$x(t) = 0.363\left[1 + e^{-2t}\left(\frac{2}{\sqrt{5}}\sin(\sqrt{5}\,t) - \cos(\sqrt{5}\,t)\right)\right]$$