Review

position vector $\vec{r}$

velocity vector $\vec{v} = \frac{d\vec{r}}{dt}$

acceleration vector $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

Angular motion: $\omega = \frac{d\theta}{dt}$

$\alpha = \frac{d^2\theta}{dt^2}$

Rotating vector

$$\frac{d\hat{e}}{dt} = \omega \hat{n}$$

$$\frac{d\hat{n}}{dt} = -\omega \hat{e}$$

In one-dimension:

Given $a(t)$:

$$\frac{dv}{dt} = a(t) \rightarrow v(t) = v(0) + \int_0^t a(t)\,dt$$

$$\frac{ds}{dt} = v(t) \rightarrow s(t) = s(0) + \int_0^t v(t)\,dt$$

Given $a(s)$

$$a(s) = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$$

$$\int_{s_0}^{s} a(s)\,ds = \int_{v(s_0)}^{v(s)} v\,dv$$

Given $a(v)$ $\quad \frac{dv}{dt} = a(v) \Rightarrow \int_{v(0)}^{v(t)} \frac{dv}{a(v)} = \int_0^t dt$

Example 1 Let $a(t) = c(t-1)$ with $c>0$ constant.

At $t_0 = 1 \quad s(t_0) = s_0$, $v(t_0) = v_0$

Find $x(t)$ and $v(t)$.

Solution:

$$\frac{dv}{dt} = a(t) = c(t-1)$$

$$\int_{v(t_0)}^{v(t)} dv = \int_{t_0}^{t} c(t-1)\,dt$$

$$v(t) - v_0 = c\frac{(t-1)^2}{2}\Big|_{t_0}^{t} = c\frac{(t-1)^2}{2} - c\frac{(t_0-1)^2}{2}$$

($t_0 = 1$, so $c\frac{(t_0-1)^2}{2} = 0$)

$$\boxed{v(t) = v_0 + \frac{c}{2}(t-1)^2}$$

$$\frac{ds}{dt} = v(t)$$

$$\int_{s(t_0)}^{s(t)} ds = \int_{t_0}^{t} v(t)\,dt = \int_{t_0}^{t} v_0 + c\frac{(t-1)^2}{2}\,dt$$

$$s(t) - s(t_0) = v_0(t-t_0) + c\frac{(t-1)^3}{6}\Big|_{t_0}^{t}$$

$$\boxed{s(t) = s_0 + v_0(t-1) + \frac{c}{6}(t-1)}$$

Example 2

1. A particle moves in a straight path with an acceleration $a = -4s$ m/sec^2 where s is distance. When $s = 0$ the particle has a velocity of 6 m/sec. Determine the velocity of the particle as a function of s, and find the distance the particle travels before coming to a stop.

Solution

(a)

$$a = \frac{dv}{dt} = \frac{dv}{ds} v$$

$$\int_{s_0}^{s} a(s)\, ds = \int_{v(s_0)}^{v(s)} v\, dv$$

$$\int_{0}^{s} -4s\, ds = \int_{v_0}^{v} v\, dv$$

$$-4\frac{s^2}{2} = \frac{v^2}{2} - \frac{v_0^2}{2} \quad / \cdot 2$$

$$v^2 = v_0^2 - 4s^2 \qquad v_0 = 6 \text{ m/s}$$

$$\boxed{v(s) = \sqrt{36 - 4s^2} \text{ m/s}}$$

(b)

$$v(s) = 0 = \sqrt{36 - 4s^2} \text{ m/s} \Rightarrow 36 - 4s^2 = 0$$

$$\boxed{s = 3 \text{ m}}$$

Particular case: $a = a_0 = \text{constant}$

$$v = v_0 + a_0 (t - t_0)$$

integration ↓

$$s = s_0 + v_0 + \frac{a_0}{2} (t - t_0)$$

Also $$a_0 = \frac{dv}{ds} v$$

$$\Rightarrow \quad v^2 = v_0^2 + 2 a_0 (s - s_0)$$

Curvilinear motion

Cartesian:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

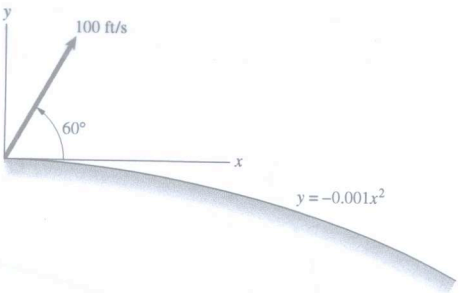
$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

Note: We can decompose the motion on each direction!

Example 3:

Problem 13.85 A projectile is launched at 100 ft/s at 60° above the horizontal. The surface on which it lands is described by the equation shown. Determine the point of impact.



$v_0 = 100\ ft/s$

$\theta = 60°$

Solution:

x direction: $x = x_0 + v_0 \cos\theta_0 t$ $\quad t_0 = 0$

y direction: $y = y_0 + v_0 \sin\theta_0 t - \frac{g}{2} t^2$

At impact $y = -0.001x^2$

$$\Rightarrow \quad v_0 \sin\theta_0 t - \frac{g}{2} t^2 = -0.001 (v_0 \cos\theta_0 t)^2$$

$$= -0.001\, v_0^2 \cos^2\theta_0 t^2 \quad / t \quad (t \neq 0)$$

$$v_0 \sin\theta_0 = \left(\frac{g}{2} - 0.001 v_0^2 \cos^2\theta_0\right) t$$

$$t = \frac{v_0 \sin\theta_0}{\frac{g}{2} - 0.001 v_0^2 \cos^2\theta_0}$$

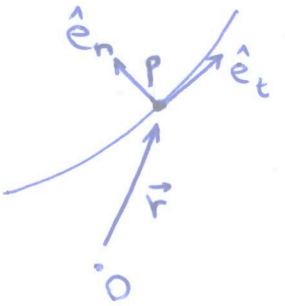
Note: $g = 32.2\ ft/s^2$

$$t = 6.37\ s$$

$$x = v_0 \cos\theta_0 t = \boxed{318.4\ ft}$$

Curvilinear motion

Unit vectors $\hat{e}_t$ and $\hat{e}_n$



$$\frac{d\hat{e}_t}{dt} = \omega \hat{e}_n$$

$$\frac{d\hat{e}_n}{dt} = \omega \hat{e}_t$$

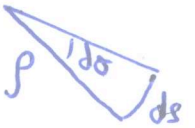
Given $\vec{r}$

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = v \hat{e}_t}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{dt} = \frac{dv}{dt}\hat{e}_t + v\omega\hat{e}_n$$

Using the radius of curvature ρ

$$ds = \rho\, d\theta$$



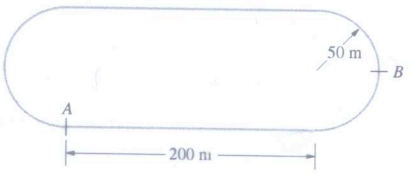
$$v = \frac{ds}{dt} = \rho\frac{d\theta}{dt} = \rho\omega$$

$$\omega = \frac{v}{\rho}$$

$$\boxed{\vec{a} = \frac{dv}{dt}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n}$$

EXAMPLE 4

Problem 13.118 Suppose that the tangential component of acceleration of the car described in Problem 13.117 is given in terms of the car's position by $a_t = 0.4 - 0.001s$ m/s^2, where s is the distance the car travels along the track from point A. What are the car's velocity and acceleration in terms of normal and tangential components at point B?



Assume the car starts from rest.

Solution:

$a_t(s) = 0.4 - 0.001s \quad m/s^2$ Let s_A: s at A, s_B: s at B

$a_t(s) = \frac{dv}{dt} = \frac{dv}{ds} v$

$$\int_{s_0=0}^{s} 0.4 - 0.001s \, ds = \int_{v_0=0}^{v} v \, dv =$$

$$0.4s - 0.001\frac{s^2}{2} = \frac{v^2}{2}$$

$$\boxed{v(s) = \sqrt{0.8s - 0.001s^2}}$$

Let s_B denote s

$s_B = 200\,m + \frac{2\pi}{4}\rho \qquad \rho = 50\,m$

$$\boxed{s_B = 25\pi + 200 \; m}$$

$$\boxed{v_B = 12.05 \; m/s}$$

$a_t = 0.4 - 0.001 \cdot (25\pi + 200) \quad m/s^2$

$$\boxed{a_t|_B = 0.122 \; m/s^2}$$

$$\boxed{a_r = \frac{v^2}{\rho} = 2.905 \; m/s^2}$$

Example 5

The position of the motion of a given particle is described by the position vector

$$\vec{r}(t) = 3t + 2t^2 \hat{j} + 3\hat{k} \quad m$$

(i) Find the tangential and normal unit vectors at $t = 1\,s$.

(ii) Find the instantaneous radius of curvature at $t = 1s$

Solution

$$\vec{r}(t) = 3t\,\hat{i} + 2t^2\,\hat{j} + 3\hat{k} \quad m$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 3\hat{i} + 4t\,\hat{j} \quad m/s \quad \Rightarrow \quad \vec{v}(1) = 3\hat{i} + 4\hat{j} \quad m/s$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 4\hat{j} \quad m/s^2 \quad \Rightarrow \quad \vec{a}(1) = 4\hat{j} \quad m/s^2$$

$$\vec{v} = v\,\hat{e}_t \qquad v = \|\vec{v}\| = \sqrt{3^2 + 4^2} = 5\ m$$

$$\hat{e}_t = \frac{\vec{v}}{v} = \frac{1}{5}\left(3\hat{i} + 4\hat{j}\right) = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

$$\boxed{\hat{e}_t = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}}$$

$$a_t = \vec{a} \cdot \hat{e}_t = \frac{16}{5}$$

$$\vec{a} = a_t \hat{e}_t + a_n \hat{e}_n$$

$$a_n \hat{e}_n = \vec{a} - a_t \hat{e}_t = 4\hat{j} - \frac{16}{5}\left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right)$$

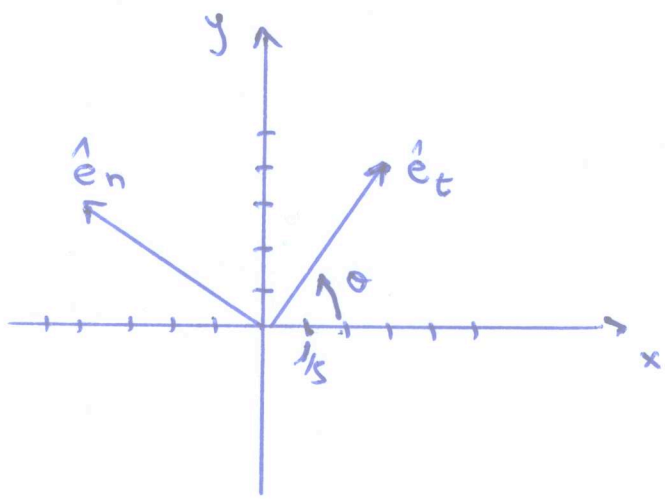
$$= 4\hat{j} - \frac{48}{25}\hat{i} - \frac{64}{25}\hat{j}$$

$$a_n \hat{e}_n = -\frac{48}{25}\hat{i} + \frac{36}{25}\hat{j} = \frac{6}{25}\left(-8\hat{i} + 6\hat{j}\right)$$

$$a_n \hat{e}_n = \frac{12}{25}\left(-4\hat{i} + 3\hat{j}\right)$$

$$a_n = \frac{12}{25}\sqrt{4^2 + 3^2} = \frac{12 \cdot 5}{25_5} = \frac{12}{5}$$

$$\boxed{\hat{e}_n = \frac{a_n \hat{e}_n}{a_n} = -\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}}$$



check:

$$\hat{e}_t \cdot \hat{e}_n = 0$$

(ii) $a_n = \frac{v^2}{\rho}$

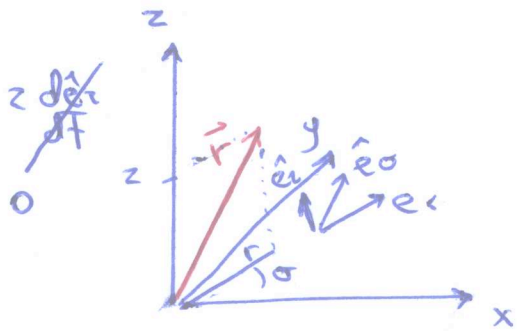
$$\rho = \frac{v^2}{a_n} = \frac{5^2 \text{ m}^2/\text{s}^2}{\frac{12}{5} \text{ m}/\text{s}^2} = \frac{75}{12} \text{ m}$$

$$\boxed{\rho = 6.25 \text{ m}}$$

Polar / Cylindrical coordinates

$$\vec{r} = r\hat{e}_r + z\hat{e}_z$$

$$\vec{v} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} + \frac{dz}{dt}\hat{e}_z + z\frac{d\hat{e}_z}{dt}$$



$$\frac{d\hat{e}_r}{dt} = \omega\hat{e}_\theta$$

$$\frac{d\hat{e}_\theta}{dt} = -\omega\hat{e}_r$$

$$\frac{d\hat{e}_z}{dt} = 0$$

$$\boxed{\vec{v} = \frac{dr}{dt}\hat{e}_r + r\omega\hat{e}_\theta + \frac{dz}{dt}\hat{e}_z}$$

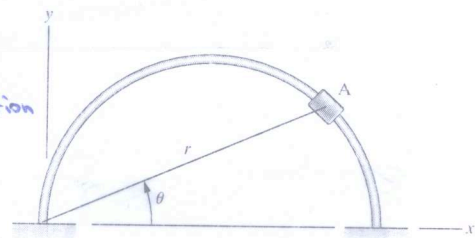
$$\vec{a} = \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\omega\hat{e}_\theta + \frac{dr}{dt}\omega\hat{e}_\theta + r\frac{d\omega}{dt}\hat{e}_\theta + r\omega\frac{d\hat{e}_\theta}{dt} + \frac{d^2z}{dt}\hat{e}$$

($\frac{d\omega}{dt} = \alpha$; $\frac{d\hat{e}_\theta}{dt} = -\omega\hat{e}_r$)

$$\boxed{\vec{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{e}_r + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{e}_\theta + \frac{d^2z}{dt^2}\hat{e}_z}$$

Example 6

Problem 13.145 The collar A slides on the circular bar. The radial position of A (in meters) is given as a function of θ by $r = 2\cos\theta$. ~~At the instant shown, $\theta = 25°$ and $d\theta/dt = 4$ rad/s~~. Determine the ~~velocity~~ acceleration of A in terms of polar coordinates.



$$r = 2\cos\theta$$

$$\frac{dr}{dt} = -2\sin\theta \frac{d\theta}{dt} = -2\omega_0 \sin\theta$$

$$\frac{d^2r}{dt^2} = -2\omega_0 \cos\theta \frac{d\theta}{dt} = -2\omega_0^2 \cos\theta$$

$$a_r = \frac{d^2r}{dt^2} - r\omega^2 = -2\omega_0^2\cos\theta - 2\cos\theta\,\omega_0^2$$

$$\boxed{a_r = -4\omega_0^2\cos\theta}$$

$$a_\theta = r\alpha + 2\frac{dr}{dt}\omega \underset{\uparrow}{=} 2(-2\omega_0\sin\theta)\omega_0 = -4\omega_0^2\sin\theta$$

$$\alpha = \frac{d\omega}{dt} = 0$$

$$\boxed{a_\theta = -4\omega_0^2\sin\theta}$$

Newton's 2nd law

$$\vec{f} = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$$

if m = constant

$$\vec{f} = m\frac{d\vec{v}}{dt} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$$

in a system of particles

$$\sum_{j(\neq i)} \vec{f}_{ij} + \vec{f}_i^{\,E} = m_i \frac{d^2\vec{r}_i}{dt^2}$$

$$\underbrace{\sum_i \sum_{j(\neq i)} \vec{f}_{ij}}_{= 0 \;\; (\vec{f}_{ij} = -\vec{f}_{ji})} + \underbrace{\sum_i \vec{f}_i^{\,E}}_{\sum \vec{F}} = \underbrace{\sum_i m_i \frac{d^2\vec{r}_i}{dt^2}}_{m \frac{d^2\vec{r}}{dt^2}}$$

$\sum \vec{F}$: sum of external forces.

m: Total mass

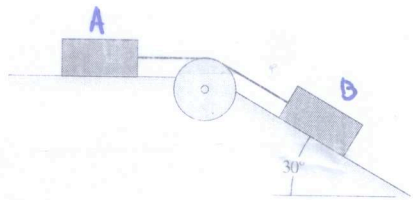
$\frac{d^2\vec{r}}{dt^2}$: acceleration of center of mass

Center of mass

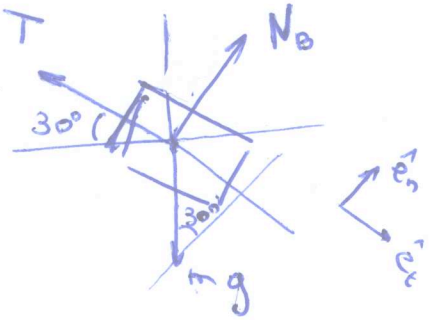
$$\vec{r} = \frac{\sum_i m_i \vec{r}_i}{m}$$

Example 7

Problem 14.31 The mass of each box is 14 kg. One second after they are released from rest, they have moved 0.3 m from their initial positions. What is the coefficient of kinetic friction between the boxes and the surface?



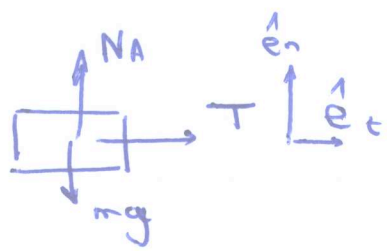
B



$$\textcircled{1}\begin{cases}\sum F_t = mg\sin\theta - T - \mu_k N_B = m a_t \\ \sum F_n = N_B - mg\cos\theta = 0\end{cases}$$

$$\textcircled{2}\begin{cases}\sum F_t = T - \mu_k N_A = m a_t \\ \sum F = N_A - mg = 0\end{cases}$$

A



$\textcircled{2} \Longrightarrow \quad N_A = mg \longrightarrow T - \mu_k mg = m a_t$

$\textcircled{1} \Longrightarrow \quad N_B = mg\cos\theta \longrightarrow mg\sin\theta - T - \mu_k mg\cos\theta = m a_t$

$\textcircled{1}+\textcircled{2} \Rightarrow mg\sin\theta - \mu_k mg(1+\cos\theta) = 2m a_t$

$$\frac{mg\sin\theta - 2m a_t}{mg(1+\cos\theta)} = \mu_k$$

What is a_t ?

For constant acceleration

$$s = s_0 + v_0(t - t_0) + \frac{a}{2}(t - t_0)^2$$

$$s = \frac{a}{2} t^2 \qquad s_0 = 0 \quad v_0 = 0 \quad t_0 = 0$$

$$a = \frac{2s}{t^2} = \boxed{0.6 \text{ m/s}^2}$$

$$\mu_k = \frac{mg\sin\theta - 2ma_t}{mg(1+\cos\theta)}$$

$$\boxed{\mu_k = 0.202}$$

Things to remember

Position vector $\vec{r}$

Velocity vector $\vec{v} = \frac{d\vec{r}}{dt}$

Acceleration vector $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

Angular velocity $\omega = \frac{d\theta}{dt}$

Angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$

Rotating vector

$\frac{d\hat{e}}{dt} = \omega \hat{n}$

$\frac{d\hat{n}}{dt} = -\omega \hat{e}$

Chain rule : $a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}$

Separation of variables: $a = \frac{dv}{dt} \Rightarrow a\,dt = dv$

Curvilinear motion:

Tangential / normal description:

Given $\vec{r} \Rightarrow \vec{v} = v\hat{e}_t$

$\vec{a} = a_t \hat{e}_t + a_n \hat{e}_n$

Useful relation: $v = \rho\omega \Rightarrow a_n = \frac{v^2}{\rho}$

Polar / Cylindrical coordinates

$\vec{r} = r\hat{e}_r + z\hat{e}_z$

Newton's 2nd law:

$$\vec{f} = \frac{d}{dt}(m\vec{v})$$

if m constant : $\vec{f} = m\frac{d\vec{v}}{dt} = m\vec{a}$

Newton's 3rd law : $\vec{f}_{ij} = -\vec{f}_{ji}$

Center of mass : $\vec{r} = \frac{\sum_i m_i \vec{r}_i}{m}$