Review session for exam 2

(I) Principle of work and energy

By Newton's 2nd law

$$\sum \vec{F} = m \frac{d\vec{v}}{dt}$$

sum of external forces

Total mass

acceleration of center of mass

Take dot product with $\vec{v}$

$$\sum \vec{F} \cdot \frac{d\vec{r}}{dt} = \sum \vec{F} \cdot \vec{v} = m \frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{1}{2} m \frac{d}{dt}(\vec{v} \cdot \vec{v}) = \frac{1}{2} m \frac{d}{dt}(v^2)$$

$$\Longrightarrow \sum \vec{F} \cdot d\vec{r} = \frac{1}{2} m \, d(v^2)$$

Integrate

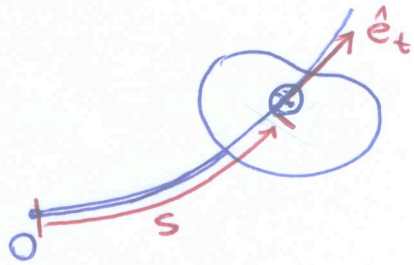
$$\int_{\vec{r}_1}^{\vec{r}_2} \sum \vec{F} \cdot d\vec{r} = \int_{v(\vec{r}_1)}^{v(\vec{r}_2)} \frac{1}{2} m \, d(v^2) = \frac{1}{2} m v^2(\vec{r}_2) - \frac{1}{2} m v^2(\vec{r}_1)$$

work $\longleftarrow$ $$\boxed{\int_{\vec{r}_1}^{\vec{r}_2} \sum \vec{F} \cdot d\vec{r} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2}$$ $\longrightarrow$ change in kinetic energy

with $\left.\begin{array}{l} v_1 = v(\vec{r}_1) \\ v_2 = v(\vec{r}_2) \end{array}\right\}$ magnitudes of velocities.

(II) Evaluating the work

$$U_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \sum \vec{F} \cdot d\vec{r}$$



$$\vec{v} = \frac{d\vec{r}}{dt} = v\,\hat{e}_t = \frac{ds}{dt}\,\hat{e}_t$$

in tangential/normal

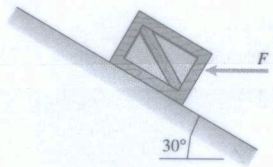
s: coordinate measured along the path

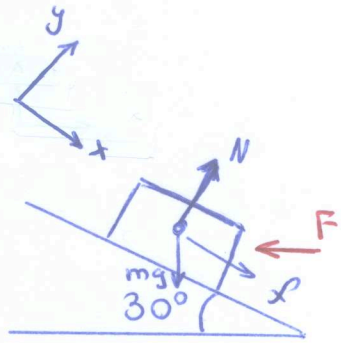
$$\Rightarrow d\vec{r} = ds\,\hat{e}_t \quad \Rightarrow \quad \sum \vec{F} \cdot d\vec{r} = \left(\sum \vec{F} \cdot \hat{e}_t\right) ds = \sum F_t\, ds$$

$$U_{12} = \int_{s_1}^{s_2} \sum F_t\, ds$$

EX 15.122

Problem 15.121 The coefficients of friction between the 20-kg crate and the inclined surface are $\mu_s = 0.24$ and $\mu_k = 0.22$. If the crate starts from rest and the horizontal force $F = 200$ N, what is the magnitude of the velocity of the crate when it has moved 2 m?





$m = 20\,kg$; $\mu_s = 0.24$; $\mu_k = 0.22$; $F = 200\,N$

(1) Choose directions

(2) Use Newton's 2nd law on each direction

$$\sum F_y = N - mg\cos 30° - F\sin 30° = m a_y = 0$$

$$\Rightarrow \boxed{N = F\sin 30° + mg\cos 30° = 270\,N}$$

$$\sum F_x = f - F\cos 30° + mg\sin 30° = m a_x$$

(3) Check if crate moves:

Friction force necesary for equilibrium ($a_x = 0$)

$$f = F\cos 30^\circ - mg\sin 30^\circ = 75.1\,N$$

$$\mu_s N = 64.8\,N < 75.1\,N \Rightarrow \text{crate moves!}$$

(4) Use work and energy principle

The force in direction of motion is

$$\mu_k N - F\cos 30^\circ + mg\sin 30^\circ$$

$$\int_{0\,m}^{-2\,m} (\mu_k N - F\cos 30^\circ + mg\sin 30^\circ)\,dx = \frac{1}{2}m v_2^2 - \underbrace{\frac{1}{2}m v_1^2}_{=0 \rightarrow \text{start from rest}}$$

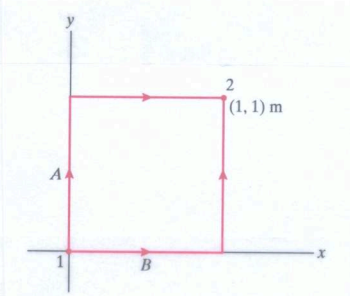
$$(F\cos 30^\circ - mg\sin 30^\circ - \mu_k N)(2\,m) = \frac{1}{2}m v_2^2$$

$$\boxed{v_2 = 1.77\,m/s}$$

EX 15.111 (a)

Problem 15.111 An object is subjected to the force $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ (N), where x and y are in meters.

(a) Show that $\mathbf{F}$ is *not* conservative.
(b) Suppose the object moves from point 1 to point 2 along the paths A and B shown in Problem 15.110. Determine the work done by $\mathbf{F}$ along each path.



$$\vec{F} = y\hat{i} - x\hat{j}\ N$$

$$U_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}$$

Along path B:

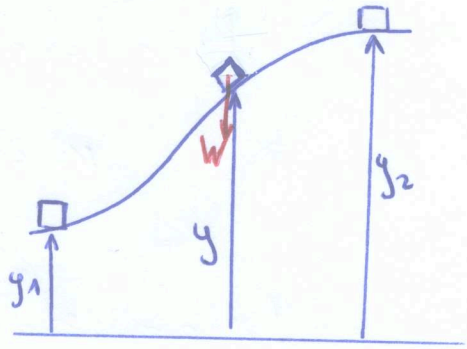
$$U_{12}^{(B)} = \int_0^1 (y\hat{i} - x\hat{j})\Big|_{y=0}\cdot\hat{i}\,dx + \int_0^1 (y\hat{i} - x\hat{j})\Big|_{x=1}\cdot\hat{j}\,dy$$

$$= 0 - 1 = -1\ N\cdot m$$

Similarly: $U_{12}^{(A)} = \int_0^1 (y\hat{i} - x\hat{j})\Big|_{x=0}\cdot\hat{j}\,dy + \int_0^1 (y\hat{j} - x\hat{i})\Big|_{y=1}\cdot\hat{i}\,dx = 0 + 1 = +1\ N\cdot m \neq U_{12}^{(B)}$

(III) Work done by particular forces

1) Work done by weight



$$\vec{F} = -mg\,\hat{j}$$

In cartesian coordinates

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \longrightarrow \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

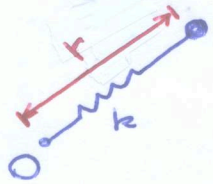
$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{F}\cdot d\vec{r} = -mg\,dy$$

$$U_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{y_1}^{y_2} -mg\,dy = -mg\,(y_2 - y_1)$$

$$\boxed{U_{12} = mg\,(y_1 - y_2)}$$

2) Work done by spring



$$\vec{F} = -k\,S\,\hat{e}_r$$

$S = r - r_0$

stretch

unstretched length

In polar coordinates

$$\vec{r} = r\hat{e}_r \longrightarrow \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\theta}{dt}\hat{e}_\theta$$

$$d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$$

$$dS = dr$$

$$\vec{F}\cdot d\vec{r} = -k\,S\,dS$$

$$U_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{S_1}^{S_2} -k\,S\,dS = -\frac{1}{2}kS_2^2 + \frac{1}{2}kS_1^2$$

$$\boxed{U_{12} = \frac{1}{2}kS_1^2 - \frac{1}{2}kS_2^2}$$

(IV) Potential energy and conservation of energy

By principle of work and energy

$$U_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \Sigma \vec{F} \cdot d\vec{r} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

Suppose a scalar function of position V satisfies

$$dV = -\Sigma \vec{F} \cdot d\vec{r}$$

$$U_{12} = \int_{V_1}^{V_2} -dV = -(V_2 - V_1)$$

$V_1 = V(\vec{r}_1)$

$V_2 = V(\vec{r}_2)$

$$\Rightarrow \boxed{\frac{1}{2} m v_1^2 + V_1 = \frac{1}{2} m v_2^2 + V_2} \rightarrow \text{conservation of energy}$$

kinetic energy: $\frac{1}{2} m v^2$

Potential energy: V

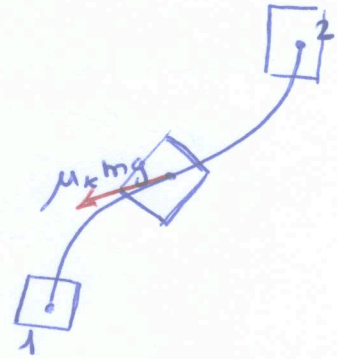
Conservative forces: if exists V s.t. $dV = -\vec{F} \cdot d\vec{r}$ then $\vec{F}$ is a <u>conservative force</u>

If $\vec{F}$ conservative $\Rightarrow U_{12} = V_1 - V_2 \rightarrow$ work does not depend on path.

If all forces are conservative $\Rightarrow$ system is conservative

Frictional forces are not conservative

$$\vec{f} = -\mu_k m g \hat{e}_t$$



$$d\vec{r} = ds\,\hat{e}_t \qquad \vec{v} = v\,\hat{e}_t = \frac{ds}{dt}\hat{e}_t$$

$$U_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{f}\cdot d\vec{r} = \int_0^L -\mu_k m g\, ds = -\mu_k m g L \longrightarrow \text{depend on path!}$$

Potential energy of weight

$$\vec{F} = -mg\hat{j} \qquad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$dV = -\vec{F}\cdot d\vec{r} = mg\,dy$$

$$\int_{V_0}^{V} dV = \int_0^y mg\,dy \implies V = mgy + \underset{\substack{\parallel \\ 0}}{V_0} \text{ arbitrary}$$

$$\boxed{V(y) = mgy}$$

Potential energy of spring

$$\vec{F} = -k(r-r_0)\hat{e}_r \qquad d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$$

$$dV = -\vec{F}\cdot d\vec{r} = k(r-r_0)\,dr$$

$$\int_{V_0}^{V} dV = \int_{r_0}^{r} k(r-r_0)\,dr \implies V = \frac{1}{2}k(r-r_0)^2 + \underset{\substack{\parallel \\ 0}}{V_0} \text{ arbitrary}$$

$$\boxed{V(r) = \frac{1}{2}k(r-r_0)^2} \quad \text{or} \quad \boxed{V(s) = \frac{1}{2}k s^2} \qquad s = r - r_0$$

EX

(1) Let $\vec{F} = -[k(r-r_0) + q(r-r_0)^3]\,\hat{e}_r$ ← Nonlinear spring

Find $V(r)$.

Solution: $\vec{F} = -[k(r-r_0) + q(r-r_0)^3]\,\hat{e}_r$

$$d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$$

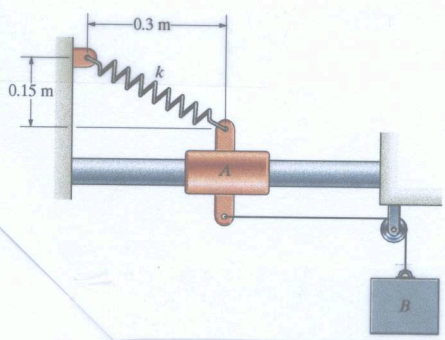
$$dV = -\vec{F}\cdot d\vec{r} = -[k(r-r_0) + q(r-r_0)^3]\,dr$$

$$\int_{V_0}^{V} dV = \int_{r_0}^{r} [k(r-r_0) + q(r-r_0)^3]\,dr$$

$$\boxed{V = \frac{1}{2}k(r-r_0)^2 + \frac{1}{4}q(r-r_0)^4} \qquad V_0 = 0$$

EX 15.92:

Problem 15.92 The spring constant $k = 700$ N/m. The masses $m_A = 14$ kg and $m_B = 18$ kg. The horizontal bar is smooth. At the instant shown, the spring is unstretched and the mass B is moving downward at 1 m/s. How fast is B moving when it has moved downward 0.2 m from its present position?



(1) Compute unstretched length of spring (state 1)

$$r_0 = \sqrt{(0.3\,m)^2 + (0.15\,m)^2} = 0.335\,m$$

(2) Compute spring length at state 2

$$r = \sqrt{(0.5\,m)^2 + (0.15\,m)^2} = 0.522\,m$$

Conservation of energy

$$\underbrace{\frac{1}{2} m_A v_1^2 + \frac{1}{2} m_B v_1^2}_{\text{only kinetic energy}} = \underbrace{\frac{1}{2} m_A v_2^2 + \frac{1}{2} m_B v_2^2}_{\text{kinetic energy}} \underbrace{- m_B g h + \frac{1}{2} k (r - r_0)^2}_{\text{potential energy}}$$

<u>Note</u>: Mass A remains at the same height so there is no change in its gravitational potential energy

$$\Longrightarrow \quad (m_A + m_B) v_1^2 = (m_A + m_B) v_2^2 - 2 m_B g h + k (r - r_0)^2$$

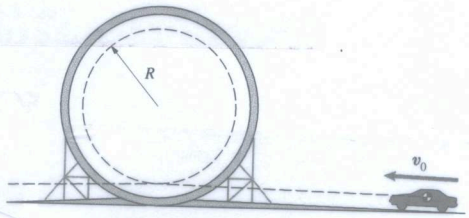
$$v_2 = \sqrt{v_1^2 + \frac{2 m_B g h - k (r - r_0)^2}{m_A + m_B}}$$

$$\boxed{v_2 = 1.56 \text{ m/s}}$$

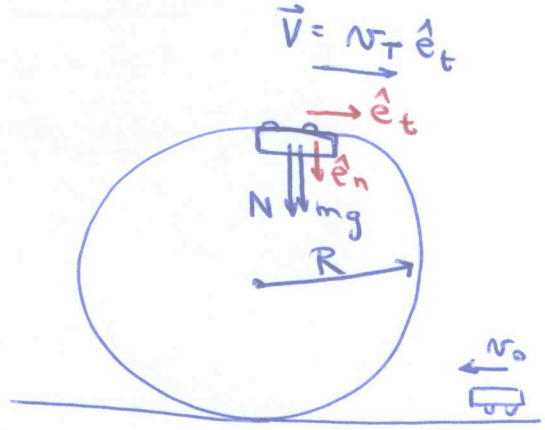
Other scenarios

EX 15.40

Problem 15.40 A stunt driver wants to drive a car through the circular loop of radius $R = 5$ m. Determine the minimum velocity v_0 at which the car can enter the loop and coast through without losing contact with the track. What is the car's velocity at the top of the loop?



(a) Find the velocity at the top



$$\sum F_n = N + mg = m a_n = m \frac{v_T^2}{R}$$

For minimum velocity $N = 0$

$$\cancel{m} g = \cancel{m} \frac{v_T^2}{R}$$

$$\boxed{v_T = \sqrt{Rg} = 7.00 \text{ m/s}}$$

(b) Use conservation of energy to find v_0

$$\frac{1}{2} \cancel{m} v_0^2 = \frac{1}{2} \cancel{m} v_T^2 + \cancel{m} g \, 2R$$

$$v_0 = \sqrt{v_T^2 + 4Rg}$$

$$\boxed{v_0 = 15.68 \text{ m/s}}$$

(V) Relation between force and potential energy

(I) In Cartesian coordinates

$$V = V(x,y,z) \qquad \vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$dV = -\vec{F}\cdot d\vec{r} \qquad d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = -\left(F_x\,dx + F_y\,dy + F_z\,dz\right)$$

$$\Rightarrow \quad F_x = -\frac{\partial V}{\partial x} \; ; \quad F_y = -\frac{\partial V}{\partial y} \; ; \quad F_z = -\frac{\partial V}{\partial z}$$

$$\vec{F} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) = -\vec{\nabla}V$$

$$\boxed{\vec{F} = -\vec{\nabla}V}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \underbrace{\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)}_{\frac{\partial}{\partial y}\left(-\frac{\partial V}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial V}{\partial y}\right) = 0}\hat{i} - \underbrace{\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)}_{=0}\hat{j} + \underbrace{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)}_{=0}\hat{k}$$

$$\Rightarrow \quad \boxed{\vec{\nabla}\times\vec{F} = \vec{0}}$$

EX. 15.117

Problem 15.117 Determine which of the following are conservative.

(a) $\mathbf{F} = (3x^2 - 2xy)\mathbf{i} - x^2\mathbf{j}$;
(b) $\mathbf{F} = (x - xy^2)\mathbf{i} + x^2y\mathbf{j}$;

(a) $\vec{F} = (3x^2 - 2xy)\hat{i} - x^2\hat{j}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 2xy & -x^2 & 0 \end{vmatrix} = 0\hat{i} - 0\hat{j} + (-2x - (-2x))\hat{k}$$

$$= \vec{0}$$

Force is conservative

(b) $\vec{F} = (x - xy^2)\hat{i} + x^2y\hat{j}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - xy^2 & x^2y & 0 \end{vmatrix} = 0\hat{i} - 0\hat{j} + (2xy + 2xy)\hat{k}$$

$= 4xy\hat{k} \longrightarrow$ Not conservative.

(VI) Principle of impulse and momentum

By Newton's 2nd law

$$\sum \vec{F} = m \frac{d\vec{v}}{dt}$$

Integrate in time

$$\boxed{\int_{t_1}^{t_2} \sum \vec{F}\, dt = m\vec{v}_2 - m\vec{v}_1}$$

Principle of (linear) impulse and momentum

Define average of total force on time

$$\sum \vec{F}_{av} = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \sum \vec{F}\, dt$$

$$\Longrightarrow \quad (t_2 - t_1) \sum \vec{F}_{av} = m\vec{v}_2 - m\vec{v}_1$$

Impulsive force: large force over short interval

In tangential component

$$\int_{t_1}^{t_2} \Sigma F_t \, dt = m v_2 - m v_1$$

velocities along path

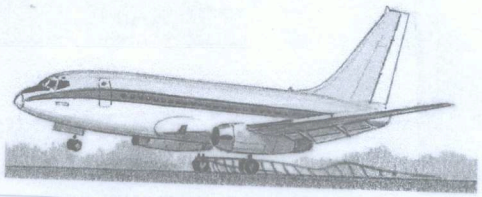
$$(t_2 - t_1) \Sigma F_{t\,av} = m v_2 - m v_1 .$$

EX 16.123

Problem 16.123 An aircraft arresting system is used to stop airplanes whose braking systems fail. The system stops a 47.5-Mg airplane moving at 80 m/s in 9.15 s.

(a) What impulse is applied to the airplane during the 9.15 s?

(b) What is the average deceleration to which the passengers are subjected?



(a) The impulse is

$$\int_{t_1}^{t_2} F \, dt = m v_2 - \underset{0}{\underset{\shortparallel}{m v_1}} = 47500 \text{ kg} \cdot 80 \text{ m/s} = 3.8 \times 10^6 \text{ N} \cdot \text{s}$$

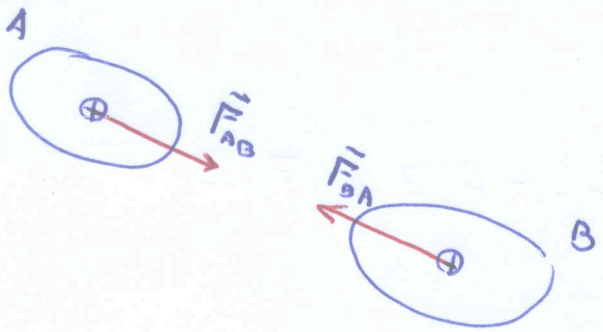
(b) The average force is

$$F_{av} = \frac{\int_{t_1}^{t_2} F \, dt}{t_2 - t_1} = \frac{3.8 \times 10^6 \text{ N} \cdot \text{s}}{9.15 \text{ s}} = \boxed{4.153 \times 10^5 \text{ N}}$$

From Newton's 2nd law

$$\boxed{a_{av} = \frac{F_{av}}{m} = 8.743 \text{ m/s}^2}$$

(VII) Conservation of linear momentum



By Newton's 3rd law $\vec{F}_{AB} + \vec{F}_{BA} = \vec{0}$

Assume no external forces;

By impulse and momentum on A <u>and</u> on B

$$\int_{t_1}^{t_2} \vec{F}_{AB}\, dt = m_A \vec{V}_A(t_2) - m_A \vec{V}_A(t_1)$$

$$+ \quad \int_{t_1}^{t_2} \vec{F}_{BA}\, dt = m_B \vec{V}_B(t_2) - m_B \vec{V}_B(t_1)$$

$$\boxed{m_A \vec{V}_A(t_1) + m_B \vec{V}_B(t_1) = m_A \vec{V}_A(t_2) + m_B \vec{V}_B(t_2)}$$

$\Rightarrow$ conservation of linear momentum.

center of mass of combined system

$$\vec{r} = \frac{m_A \vec{r}_A + m_B \vec{r}_B}{m_A + m_B}$$

$$\Rightarrow \quad \vec{V} = \frac{m_A \vec{V}_A + m_B \vec{V}_B}{m_A + m_B} = \text{constant}$$

$\leftarrow$ velocity of combined center of mass remains constant

Impacts

Perfectly plastic: objects remain together after impact

$$\vec{V}_A(t_e) = \vec{V}_B(t_e) = \vec{V}'$$

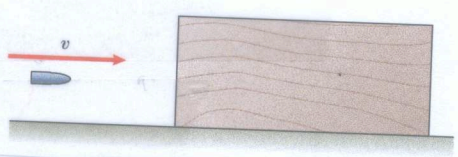
$$m_A \vec{V}_A + m_B \vec{V}_B = m_A \vec{V}_A' + m_B \vec{V}_B' = (m_A + m_B)\vec{V}'$$

velocity of combined center of mass ←

$$\boxed{\vec{V}' = \frac{m_A \vec{V}_A + m_B \vec{V}_B}{m_A + m_B}}$$

EX 16.52

Problem 16.52 A bullet with a mass of 3.6 grams is moving horizontally with velocity v and strikes a 5-kg block of wood, becoming embedded in it. After the impact, the bullet and block slide 24 mm across the floor. The coefficient of kinetic friction between the block and the floor is $\mu_k = 0.4$. Determine the velocity v.



Solution: * Momentum is conserved through collision

* Use work-energy to finish problem

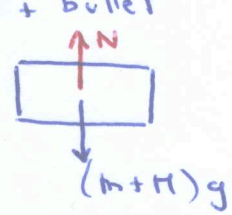
(a) $m = 0.0036\,kg$; $M = 5\,kg$;

$\mu_k = 0.4$; $d = 0.024\,m$

By conservation of linear momentum

(*) $m v = (m + M) v_2$

Block + bullet



(b) By work-energy principle

$$\int_0^d -\mu_k \underset{(m+M)g}{\underset{\parallel}{N}}\, ds = \underset{\text{Final kinetic energy}}{\underset{\uparrow}{0}} - \frac{1}{2}(m+M)v^2$$

$$\frac{1}{2}\cancel{(m+M)}v_2^2 - \mu_k \cancel{(m+M)} g d = 0$$

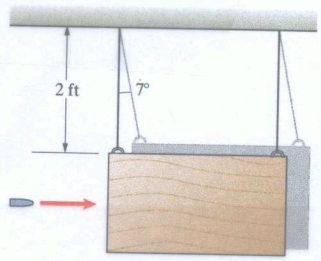
$$v_2^2 = 2\mu_k g d$$

Use (*) $\Rightarrow \left(\left(\frac{m}{m+M}\right)v\right)^2 = 2\mu_k g d$

$$\boxed{v = \frac{m+M}{m}\sqrt{2\mu_k g d}} \Rightarrow \boxed{v = 603 \text{ m/s}}$$

EX 16.53

Problem 16.53 A 0.12-ounce bullet hits a suspended 15-lb block of wood and becomes embedded in it. The angle through which the wires supporting the block rotate as a result of the impact is measured and determined to be 7°. What was the bullet's velocity?

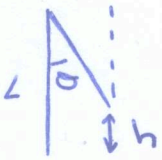


Solution : * Momentum is conserved through collision

* Energy is conserved after collision

(a) Momentum is conserved

(*) $m v = (m+M) v_2$



$h = L - L\cos\theta$

(b) Conservation of energy

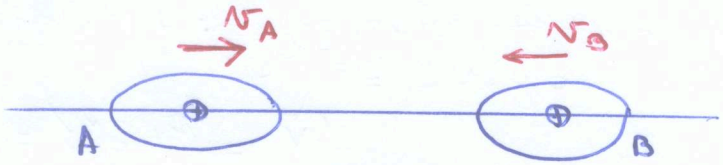
$$\frac{1}{2}\cancel{(M+m)} v_2^2 = \cancel{(M+m)} g L (1-\cos\theta)$$

$$v_2^2 = 2gL(1-\cos\theta)$$

Use (*) $\left(\frac{m}{m+M} v\right)^2 = 2gL(1-\cos\theta)$

$$\boxed{v = \left(\frac{m+M}{m}\right)\sqrt{2gL(1-\cos\theta)}} \quad \boxed{v = 1960 \text{ ft/s}}$$

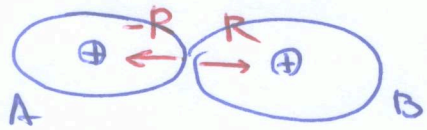
Direct central impact : objects traveling along straight line



Neglect external forces.

Analyze impact

t_c



← nearest proximity

$v_A(t_c) = v_B(t_c)$

Object A:

$$\int_{t_1}^{t_c} -R\,dt = m_A v_A(t_c) - m_A v_A(t_1)$$

$$\int_{t_c}^{t_2} -R\,dt = m_A v_A(t_2) - m_A v_A(t_c)$$

$$e = \frac{\int_{t_c}^{t_2} R\,dt}{\int_{t_1}^{t_c} R\,dt} = \frac{\cancel{m_A} v_A(t_2) - \cancel{m_A} v_A(t_c)}{\cancel{m_A} v_A(t_c) - \cancel{m_A} v_A(t_1)}$$

$v_A = v_A(t_1)$

$v_A' = v_A(t_2)$

$v_c = v_A(t_c)$

$$\boxed{(v_c - v_A)\,e = v_A' - v_c} \quad (*)$$

Object B :

$$\int_{t_1}^{t_c} R\,dt = m_B v_B(t_c) - m_B v_B(t_1)$$

$$\int_{t_c}^{t_2} R\,dt = m_B v_B(t_2) - m_B v_B(t_c)$$

$$e = \frac{m_B v_B(t_2) - m_B v_B(t_c)}{m_B v_B(t_c) - m_B v_B(t_1)}$$

$v_B(t_c) = v_A(t_c) = v_c$

$v_B = v_B(t_1)$

$v_B' = v_B(t_2)$

$$\boxed{(v_c - v_B)\,e = v_B' - v_c} \quad (**)$$

$$(**) - (*) \Rightarrow (\cancel{v_c} - v_B - \cancel{v_c} + v_A)\,e = v_B' - \cancel{v_c} - v_A' + \cancel{v_c}$$

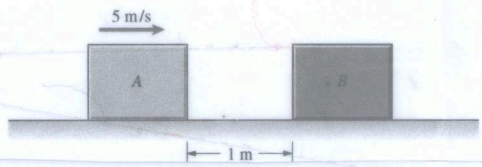
$$\boxed{e = \frac{v_B' - v_A'}{v_A - v_B}}$$

Impact analysis

$$\begin{cases} m_A v_A + m_B v_B = m_A v_A' + m_B v_B' & \longrightarrow \text{conservation of linear momentum} \\ e = \dfrac{v_B' - v_A'}{v_A - v_B} & \longrightarrow \text{coefficient of restitution} \end{cases}$$

EX 16.133

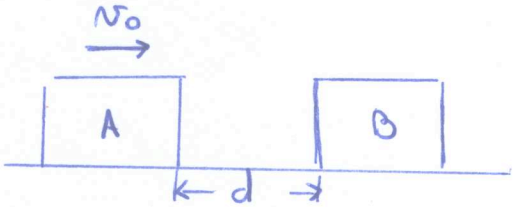
Problem 16.133 The 10-kg mass A is moving at 5 m/s when it is 1 m from the stationary 10-kg mass B. The coefficient of kinetic friction between the floor and the two masses is $\mu_k = 0.6$, and the coefficient of restitution of the impact is $e = 0.5$. Determine how far B moves from its initial position as a result of the impact.



Solution

(1) Use work-energy to get v_A (before impact)

$$\int_0^d -\mu_k N\, ds = \int_0^d -\mu_k m_A g\, ds = \frac{1}{2} m_A v_A^2 - \frac{1}{2} m_A v_0^2$$



$$-\mu_k \cancel{m_A} g d = \frac{1}{2} \cancel{m_A} v_A^2 - \frac{1}{2} \cancel{m_A} v_0^2$$

$$\boxed{v_A = \sqrt{v_0^2 - 2\mu_k g d}}$$

(2) Analyze impact

$$\begin{cases} m_A v_A + m_B \underset{0}{v_B} = m_A v_A' + m_B v_B' \\ e = \dfrac{v_B' - v_A'}{v_A - v_B} \end{cases}$$

$$\Rightarrow \quad m_A v_A = m_A v_A' + m_B v_B'$$

$$\Rightarrow \quad v_A e = v_B' - v_A'$$

$$\downarrow$$

$$v_A' = v_B' - v_A e$$

We want v_B':

$$\Rightarrow \quad m_A v_A = m_A (v_B' - v_A e) + m_B v_B'$$

$$m_A (1+e) v_A = (m_A + m_B) v_B' \quad \Rightarrow \quad v_B' = \frac{m_A}{m_A + m_B} (1+e) v_A$$

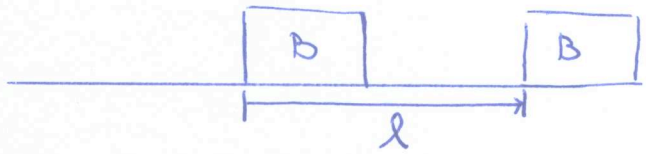
Substitute v_A

$$\boxed{v_B' = \frac{m_A}{m_A + m_B} (1+e) \sqrt{v_0^2 - 2\mu_k g d}}$$

(3) Find how for B slides

Use work-energy:

$$\int_0^{l} -\mu_k m_B g \, ds = \frac{1}{2} m_B \underbrace{v_B^2(l_{final})}_{=0} - \frac{1}{2} m_B (v_B')^2$$



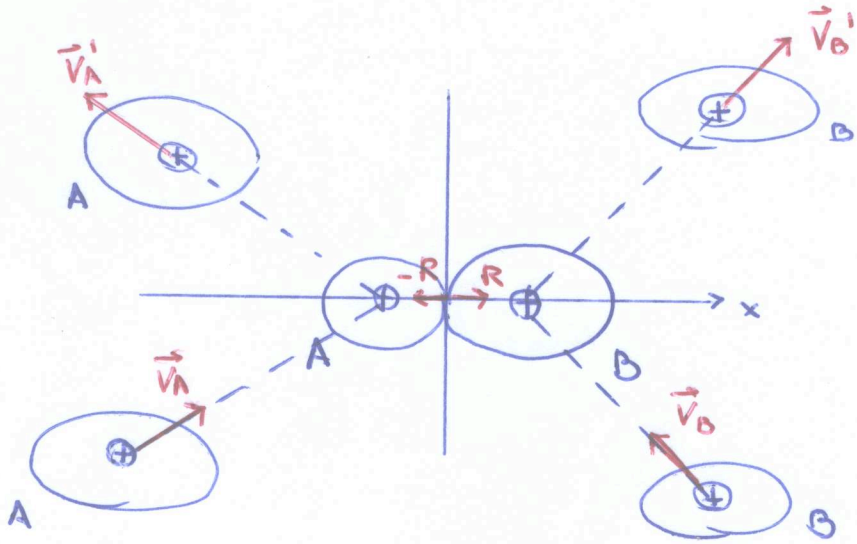
$$\cancel{-}\mu_k \cancel{m_B} g l = \cancel{-}\frac{1}{2} \cancel{m_B} (v_B')^2$$

$$l = \frac{1}{2\mu_k g}(v_B')^2 = \frac{1}{2\mu_k g}\left(\frac{m_A}{m_A+m_B}\right)^2 (1+e)^2 \left(v_0^2 - 2\mu_k g d\right)$$

$$\boxed{l = \left(\frac{m_A}{m_A+m_B}\right)^2 (1+e)^2 \left(\frac{v_0^2}{2\mu_k g} - d\right)}$$

$$\boxed{l = 0.632 \text{ m}}$$

Oblique central impact



Change in velocity only in direction of impact

$(\vec{V}_A')_y = (\vec{V}_A)_y$ $\qquad$ $(\vec{V}_B')_y = (\vec{V}_B)_y$

$(\vec{V}_A')_z = (\vec{V}_A)_z$ $\qquad$ $(\vec{V}_B')_z = (\vec{V}_B)_z$

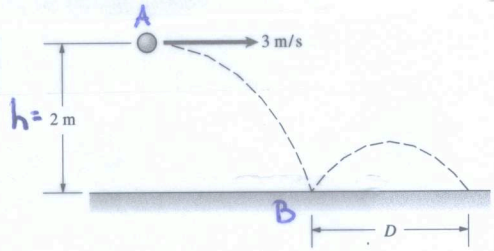
In x-direction

$$m_A (\vec{V}_A)_x + m_B (\vec{V}_B)_x = m_A (\vec{V}_A')_x + m_B (\vec{V}_B')_x$$

$$e = \frac{(\vec{V}_B')_x - (\vec{V}_A')_x}{(\vec{V}_A)_x - (\vec{V}_B)_x}$$

EX 16.140

Problem 16.140 A ball is given a horizontal velocity of 3 m/s at 2 m above the smooth floor. Determine the distance D between the ball's first and second bounces if the coefficient of restitution is $e = 0.6$.



Solution:

(1) Find velocity of ball before impact

By kinematics

$v_x = v_{x0} \longrightarrow$ constant $\qquad v_x = (\bar{v}_A)_x \qquad v_y = (\bar{v}_A)_y$

$v_y = \underset{\overset{\shortparallel}{0}}{v_{y0}} - gt$ and $y = h + \underset{\overset{\shortparallel}{0}}{v_{y0}}\, t - \frac{1}{2} g t^2$

Let $y = 0 \Rightarrow 0 = h - \frac{1}{2} g t^2 \Rightarrow h = \frac{1}{2} g t^2$

$$t = \sqrt{\frac{2h}{g}}$$

Use result to find $v_y \Rightarrow v_y = -g\sqrt{\frac{2h}{g}} = -\sqrt{2gh}$

$$\boxed{v_y = -\sqrt{2gh}}$$

Can we find v_y by energy conservation?

$$mgh + \frac{1}{2} m v_x^2 = \frac{1}{2} m v^2$$

$$mgh = \frac{1}{2} m (v^2 - v_x^2) = \frac{1}{2} m v_y^2$$

$$\boxed{v_y = -\sqrt{2gh}}$$

ball moves downward!

(2) By the coefficient of restitution

$$e = \frac{\cancel{(\vec{v}_B')_y}^{\text{negligible}} - (\vec{v}_A')_y}{(\vec{v}_A)_y - \cancel{(\vec{v}_B)_y}^{\,0}} \Rightarrow e(\vec{v}_A)_y = -(\vec{v}_A')_y$$

$$\Rightarrow \quad v_y' = -e v_y$$

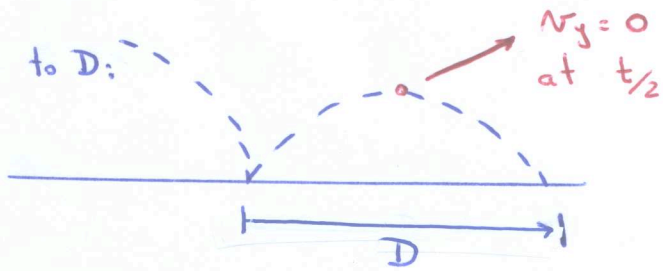
$$\boxed{v_y' = e\sqrt{2gh}}$$

(3) From kinematics:

Find time t to travel from impact to D:

$$v_y = v_{oy} - g\left(\frac{t}{2}\right)$$

$$\overset{\shortparallel}{0}$$



$$t = 2\frac{v_{oy}}{g} = 2\frac{v_y'}{g} = \frac{2e\sqrt{2gh}}{g} = 2e\sqrt{\frac{2h}{g}}$$

Use motion in x-direction to find D:

$$D = v_{ox} t = v_{ox}\, 2e\sqrt{\frac{2h}{g}}$$

$$\boxed{D = 2e v_{ox}\sqrt{\frac{2h}{g}}} \qquad \boxed{D = 2.30\,m}$$

(VIII) **Angular momentum**

By Newton's 2nd law

$$\sum \vec{F} = m \frac{d\vec{v}}{dt}$$

Take cross product with $\vec{r}$

$$\vec{r} \times \sum \vec{F} = \vec{r} \times m \frac{d\vec{v}}{dt}$$

Notice that $\frac{d}{dt}(\vec{r} \times m\vec{v}) = \underbrace{\left(\frac{d\vec{r}}{dt} \times m\vec{v}\right)}_{\vec{v} \times m\vec{v} = \vec{0}} + \left(\vec{r} \times m \frac{d\vec{v}}{dt}\right)$

$$\Rightarrow \quad \vec{r} \times \sum \vec{F} = \frac{d}{dt}(\vec{r} \times m\vec{v})$$

$\vec{H}_o = \vec{r} \times m\vec{v}$ = angular momentum about **O**.

$$\vec{r} \times \sum \vec{F} = \frac{d\vec{H}_o}{dt}$$

Integrate in time

$$\boxed{\int_{t_1}^{t_2} \vec{r} \times \sum \vec{F} \, dt = \vec{H}_o(t_2) - \vec{H}_o(t_1)}$$

"Principle of angular impulse and momentum"

If $\sum \vec{F} \parallel \vec{r} \Rightarrow \vec{r} \times \sum \vec{F} = \vec{0} \Rightarrow \vec{H}_o = \text{constant}$

parallel

Assume central force ($\vec{F} \parallel \vec{r}$) on a plane

In polar coordinates

$$\vec{r} = r\,\hat{e}_r$$

$$\vec{v} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta$$

$$\vec{H}_o = \vec{r} \times m\vec{v} = m r v_\theta\,(\hat{e}_r \times \hat{e}_\theta) = m r v_\theta\,\hat{e}_z$$

unit vector perpendicular to plane

$$\vec{H}_o = \text{constant} \Rightarrow \boxed{r v_\theta = \text{constant}}$$