Post-Processing of Discontinuous Galerkin Methods for Hyperbolic Equations

Jennifer K. Ryan

Computer Science and Mathematics Division
Oak Ridge National Laboratory

Research supported by NASA grant NGT-1-01037 and Householder Fellowship in Scientific Computing sponsored by the Department of Energy Applied Mathematical Sciences program. Program of Oak Ridge National Laboratory (ORNL), managed by UT-Battelle, LLC for the U.S. Department of Energy under Contract No. DE-AC05-00OR22725.
Outline

• Discontinuous Galerkin Method
• Post-processing for linear hyperbolic equations
  \( (C.-W. \ Shu, \ H.L. \ Atkins) \)
• One-sided post-processing
• Post-processing stencil choosing techniques
  \( (R. \ Archibald, \ A. \ Gelb, \ S. \ Gottlieb) \)
• Post-processing for smoothly varying mesh
• Summary & Future Work
Discontinuous Galerkin Method

Properties:

• Can handle complicated geometries.
• Simple treatment of boundary conditions.
• High order accuracy.
  – Proven: \( k + \frac{1}{2} \) order accuracy for general case.
  – \( k + 1 \) order accuracy in special cases.
  – Numerically: \( k + 1 \) order accuracy
• Flexibility for adaptivity.
• Highly parallelizable.
Discontinuous Galerkin Method

Consider \( u_t + f(u)_x = 0 \).

Find \( u_h(x, t) \in V_h \) such that

\[
\int_{I_i} (u_h)_t v \, dx = \int_{I_i} f(u_h)v_x \, dx - f((u_h)_{i+1/2}) v_{i+1/2} + f((u_h)_{i-1/2}) v_{i-1/2}
\]

for all \( v \in V_h \).

\( V_h = \text{span}\{1, \xi_i, \xi_i^2, \ldots, \xi_i^k, i = 1, \ldots, N\} \), where \( \xi_i = \frac{x-x_i}{\Delta x_i} \) on

\( I_i = (x_i - \frac{\Delta x_i}{2}, x_i + \frac{\Delta x_i}{2}) \), and \( u_h(x, t) = \sum_{l=0}^{k} u_{i(l)}^{(l)}(t) \xi_i^l \) if \( x \in I_i \).

Use upwind monotone flux

Numerical Scheme:

\[
\int_{I_i} (u_h)_t v \, dx = \int_{I_i} f(u_h)v_x \, dx - \hat{f}_{i+1/2} v_{i+1/2}^- + \hat{f}_{i-1/2} v_{i-1/2}^+
\]

\( \forall v \in V_h \).
Post-Processor

Cockburn, Luskin, Shu, & Süli (2003)

• Discontinuous Galerkin approximation allows us to use negative order error estimates:

\[ ||u_h - u||_{-l} = \mathcal{O}(h^{2k+1}). \]

• Post-processor extracts this information.

• Works for a locally uniform mesh:
  → Translation invariant
  → Post-Processor is local
Negative Order Sobolev Norm

\[ \|u\|_{-\ell,\Omega} = \sup_{\phi \in C_0^\infty} \frac{\int_\Omega u(x)\phi(x)dx}{\|\phi\|_{\ell,\Omega}}, \quad \ell \geq 1 \]

Example:

\[ u_N = \sin(2\pi N x), \quad \Omega = (-1, 1), \quad \ell \geq 1 \]

\[ \Rightarrow \quad \|u_N\|_{-\ell,\Omega} = \frac{1}{(2\pi N)^\ell} \]
Post-Processor Kernel

- Independent of the partial differential equation.
- Applied only at the final time.
- Filters out oscillations in the error.

Kernel Properties

Bramble & Schatz (1977)
Mock & Lax (1978)

- Compact Support.
- Reproduces polynomials of degree $2k + 1$ by convolution.
- Linear combination of $B$-splines.
Post-Processed Solution

Post-processed solution: \( u^*(x) = K_{h}^{2(k+1),k+1} * u_h. \)

\[
K_{h}^{2(k+1),k+1}(x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c_{\gamma}^{2(k+1),k+1} \psi^{(k+1)} \left( \frac{x}{h} - \gamma \right)
\]

\( h = \Delta x_i \) for all \( i \), and \( c_{\gamma}^{2(k+1),k+1} \in \mathbb{R}. \)

\( \psi^{(0)} = \delta_0, \psi^{(n)} = \psi^{(n-1)} * \chi \) for \( n \geq 2 \), where

\[
\chi(x) = \begin{cases} 
1, & x \in (-\frac{1}{2}, \frac{1}{2}), \\
0, & \text{else}.
\end{cases}
\]
Example

Second Order Approximation

\[ u^*(y) = \frac{1}{h} \int_{-\infty}^{\infty} K_{4,2}^{4,2} \left( \frac{x - y}{h} \right) u_h(x) dx \]

where

\[ K_{h}^{4,2}(y) = \frac{1}{h} \left( c_{-1}^{4,2} \psi^{(2)} \left( \frac{y}{h} - 1 \right) + c_{0}^{4,2} \psi^{(2)} \left( \frac{y}{h} \right) + c_{1}^{4,2} \psi^{(2)} \left( \frac{y}{h} + 1 \right) \right) \]

and \( \psi^{(2)}(x) = \begin{cases} 
1 - |x| & |x| \leq 1 \\
0 & \text{else}
\end{cases} \)

where

\[ c_{-1}^{4,2}, c_{0}^{4,2}, c_{1}^{4,2} \]

and \( \psi^{(2)}(x) \) is defined as

\[ \psi^{(2)}(x) = \begin{cases} 
1 - |x| & |x| \leq 1 \\
0 & \text{else}
\end{cases} \]
Example

Second Order Approximation

Find $c_\gamma$, $\gamma = -1, 0, 1$:

Use $K_h^{4,2} \ast p = p$ for $p = 1, x, x^2$

$$
\begin{bmatrix}
1 & 1 & 1 \\
x + 1 & x & x - 1 \\
x^2 + 2x + \frac{7}{6} & x^2 + \frac{1}{6} & x^2 - 2x + \frac{7}{6}
\end{bmatrix}
\begin{bmatrix}
c_{-1} \\
c_0 \\
c_1
\end{bmatrix} =
\begin{bmatrix}
1 \\
x \\
x^2
\end{bmatrix}
$$

$$
K_h^{4,2}(y) = \frac{1}{h} \left( \frac{-1}{12} \psi^{(2)}(\frac{y}{h}) - 1 + \frac{7}{6} \psi^{(2)}(\frac{y}{h}) - \frac{1}{12} \psi^{(2)}(\frac{y}{h} + 1) \right)
$$
Implementation

Know form of approximation and kernel ⇒

\[ u^*(x) = \frac{1}{h} \int_{-\infty}^{\infty} K^{2(k+1),k+1} \left( \frac{y-x}{h} \right) u_h(y) \, dy \]

\[ = \frac{1}{h} \sum_{j=-2k}^{2k} \int_{I_{i+j}} K^{2(k+1),k+1} \left( \frac{y-x}{h} \right) \sum_{l=0}^{k} u_{i+j}^{(l)} \left( \frac{y-x_{i+j}}{h} \right)^l \, dy \]

\[ = \sum_{j=-2k}^{2k} \sum_{l=0}^{k} u_{i+j}^{(l)} C(j, l, k, x) \]

\[ C(j, l, k, x) = \frac{1}{h} \sum_{\gamma=-k}^{k} C_{\gamma} 2^{2(k+1),k+1} \int_{I_{i+j}} \psi^{(k+1)} \left( \frac{y-x}{h} - \gamma \right) \left( \frac{y-x_{i+j}}{h} \right)^l \, dy \in \mathbb{P}^{2k+1} \]

\[ k' = \left\lceil \frac{(3k + 1)}{2} \right\rceil \leq 2k \]
## Derivatives

<table>
<thead>
<tr>
<th>N</th>
<th>Approximation</th>
<th>Post-Processed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$ error</td>
<td>order</td>
</tr>
<tr>
<td>----</td>
<td>-------------</td>
<td>-------</td>
</tr>
<tr>
<td><strong>Errors in First Derivative for $\mathbb{P}^2$</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>3.48E-03</td>
<td>—</td>
</tr>
<tr>
<td>40</td>
<td>8.72E-04</td>
<td>2.00</td>
</tr>
<tr>
<td>80</td>
<td>2.18E-04</td>
<td>2.00</td>
</tr>
<tr>
<td>160</td>
<td>5.45E-05</td>
<td>2.00</td>
</tr>
<tr>
<td><strong>Errors in Second Derivative for $\mathbb{P}^2$</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>6.78E-02</td>
<td>—</td>
</tr>
<tr>
<td>40</td>
<td>3.39E-02</td>
<td>1.00</td>
</tr>
<tr>
<td>80</td>
<td>1.70E-02</td>
<td>1.00</td>
</tr>
<tr>
<td>160</td>
<td>8.48E-03</td>
<td>1.00</td>
</tr>
</tbody>
</table>

\[
\frac{du^*(x)}{dx} \in \mathbb{P}^{2k}
\]

\[
u_t + u_x = 0
\]

\[
\begin{align*}
  u(x, 0) &= \sin(x) \\
  x &\in (0, 2\pi) \\
  T &= 12.5
\end{align*}
\]
Derivatives of Post-Processed Solution

$P^2$: $|d/dx(u-u_h)|$, Before Post-Processing

$P^2$: $|d/dx(u-u_h)|$, After Post-Processing

$P^2$: $|d^2/dx^2(u-u_h)|$, Before Post-Processing

$P^2$: $|d^2/dx^2(u-u_h)|$, After Post-Processing
2-D Approximation & Kernel

\[ K_h^{2(k+1),k+1}(x, y) = \]
\[ \frac{1}{h^2} \sum_{-\gamma_x - \gamma_y}^{\gamma_x} \sum_{-\gamma_x - \gamma_y}^{\gamma_y} c_{\gamma_x + \gamma_y} 2^{(k+1),(k+1,k+1)} \psi^{(k+1)} \left( \frac{x}{h_x} - \gamma_x \right) \psi^{(k+1)} \left( \frac{y}{h_y} - \gamma_y \right) \]

Solving: \( u_t + f(u)_x + g(u)_y = 0 \)

\[ V_h = \text{span}\{1, \xi_i, \eta_j, \xi_i^2, \xi_i \eta_j, \eta_j^2, \cdots, \xi_i^k, \cdots, \eta_j^k, \]
\[ i = 1, \cdots, N_x, j = 1, \cdots, N_y \}\]

on \( I_{i,j} = (x_i - \frac{h_x}{2}, x_i + \frac{h_x}{2}) \times (y_j - \frac{h_y}{2}, y_j + \frac{h_y}{2}) \) for \( \xi_i = \frac{x-x_i}{h_x}, \eta_j = \frac{y-y_j}{h_y} \).

Approximation form: \( u_h(x, y, t) = \sum_{l=0}^{k} \sum_{m=0}^{k-l} u_{i,j}^{(l,m)}(t) \xi_i^l \eta_j^m \) for \( x, y \in I_{i,j} \).
### 2 – D System

#### Errors in $u$

\[
  u_t - u_x - v_y = 0 \\
  v_t + v_x - u_y = 0
\]

\[
  u(0, t) = u(2\pi, t) \\
  v(0, t) = v(2\pi, t) \\
  T = 12.5
\]

<table>
<thead>
<tr>
<th>$N$</th>
<th>$L^2$ Error</th>
<th>Order</th>
<th>$L^2$ Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^2$</td>
<td>1.22E-01</td>
<td></td>
<td>1.22E-01</td>
<td></td>
</tr>
<tr>
<td>$20^2$</td>
<td>1.96E-02</td>
<td>2.63</td>
<td>1.90E-02</td>
<td>2.68</td>
</tr>
<tr>
<td>$40^2$</td>
<td>2.85E-03</td>
<td>2.78</td>
<td>2.48E-03</td>
<td>2.93</td>
</tr>
<tr>
<td>$80^2$</td>
<td>4.71E-04</td>
<td>2.59</td>
<td>3.14E-04</td>
<td>2.98</td>
</tr>
<tr>
<td>$P^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$10^2$</td>
<td>2.66E-03</td>
<td></td>
<td>1.97E-03</td>
<td></td>
</tr>
<tr>
<td>$20^2$</td>
<td>2.52E-04</td>
<td>3.40</td>
<td>5.66E-05</td>
<td>5.12</td>
</tr>
<tr>
<td>$40^2$</td>
<td>3.10E-05</td>
<td>3.02</td>
<td>1.67E-06</td>
<td>5.08</td>
</tr>
<tr>
<td>$80^2$</td>
<td>3.88E-06</td>
<td>3.00</td>
<td>5.06E-08</td>
<td>5.05</td>
</tr>
</tbody>
</table>
1 – D Variable Coefficient Equation

\[ u_t + (au)_x = f \]

\[ a(x) = 2 + \sin(x) \]
\[ u(x, 0) = \sin(3x) \]
\[ u(0, t) = u(2\pi, t) \]
\[ T = 12.5 \]

<table>
<thead>
<tr>
<th>mesh</th>
<th>( L^2 ) error ( P^1 )</th>
<th>order</th>
<th>( L^2 ) error ( P^2 )</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.83E-02</td>
<td>—</td>
<td>7.82E-02</td>
<td>—</td>
</tr>
<tr>
<td>20</td>
<td>4.35E-03</td>
<td>2.07</td>
<td>1.08E-03</td>
<td>2.86</td>
</tr>
<tr>
<td>40</td>
<td>1.07E-03</td>
<td>2.03</td>
<td>1.39E-04</td>
<td>2.96</td>
</tr>
<tr>
<td>80</td>
<td>2.66E-04</td>
<td>2.01</td>
<td>1.75E-05</td>
<td>2.99</td>
</tr>
<tr>
<td>10</td>
<td>8.61E-04</td>
<td>—</td>
<td>1.34E-04</td>
<td>—</td>
</tr>
<tr>
<td>20</td>
<td>1.07E-04</td>
<td>3.01</td>
<td>2.34E-06</td>
<td>5.84</td>
</tr>
<tr>
<td>40</td>
<td>1.34E-05</td>
<td>3.00</td>
<td>4.55E-08</td>
<td>5.69</td>
</tr>
<tr>
<td>80</td>
<td>1.67E-06</td>
<td>3.00</td>
<td>1.09E-09</td>
<td>5.38</td>
</tr>
</tbody>
</table>
1-D Variable Coefficient

\[ u_t + (a(x)u)_x = f(x,t) \]
\[ a(x) = 2 + \sin(x) \]
\[ u(x,0) = \sin(3x) \]
\[ x \text{ in } (0,2\pi), \quad T = 12.5 \]
Aeroacoustic Example

Scatter of a plane wave off of a cylinder: wavelength $\lambda = 2.5r$
Aeroacoustic Example

Scatter of a plane wave off of a cylinder: Fine Mesh

Without Post-Processing  With Post-Processing
Aeroacoustic Test Problem

- $p=2$
- $p=2$, post-processed
- $p=4$

Graph showing the dependence of $\varepsilon$ on $h$. The graph includes data points and lines for different $p$ values.
Symmetric Post-Processor

\[ u^*(x) = \sum_{j=-k'}^{k'} \sum_{l=0}^k u_{i+j}^{(l)} C(j, l, k, x) \]

\[ C(j, l, k, x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c^{2(k+1), k+1}_{\gamma} \int_{I_{i+j}} \psi^{(k+1)} \left( \frac{y-x}{h} - \gamma \right) \left( \frac{y-x_{i+j}}{h} \right)^l dy \]
\[ u^*(x) = \sum_{j=-2k'}^{0} \sum_{l=0}^{k} u_{i+j}^{(l)} C(j, l, k, x) \]

\[ C(j, l, k, x) = \frac{1}{h} \sum_{\gamma=-2k-1}^{-1} c_{\gamma}^{2(k+1), k+1} \int_{-\frac{1}{2}-(\xi_i+\gamma)}^{\frac{1}{2}-(\xi_i+\gamma)} \psi^{(k+1)}(\eta) (\xi_i + \eta + \gamma - j)^l \, d\eta \]

For \( k = 1 \):

\[ K(x) = \frac{11}{12} \psi^{(2)}(x + 3) - \frac{17}{6} \psi^{(2)}(x + 2) + \frac{35}{12} \psi^{(2)}(x + 1) \]
Domains with Different Mesh Sizes

- \( N = \) total number of elements for the 2 domains combined.
- \([0, \pi)\), has a more refined mesh.
- Solving a hyperbolic equation with smooth initial conditions over \([0, 2\pi]\).
- Calculating approximation for 2 periods in time.
1-D Multi-Domain

\textbf{P}^2: Without Post-Processing

\textbf{P}^2: With Post-processing

\begin{align*}
u_t + u_x &= 0 \\
u(x,0) &= \sin(3x) \\
x &\text{ in } (0,2\pi), \ T=12.5
\end{align*}
1-D Multi-Domain
Using One-Sided Post-Processor

\[ \begin{align*}
P^2: & \text{ Without Post-Processing} \\
\text{error} & \text{vs } x \\
N=20 & \text{ (red)} \\
N=40 & \text{ (blue)} \\
N=80 & \text{ (green)} \\
N=160 & \text{ (orange)} \\
\end{align*} \]

\[ \begin{align*}
P^2: & \text{ With Post-processing} \\
\text{error} & \text{vs } x \\
N=20 & \text{ (red)} \\
N=40 & \text{ (blue)} \\
N=80 & \text{ (green)} \\
N=160 & \text{ (orange)} \\
\end{align*} \]

\[ \begin{align*}
u_x & = 0 \\
u(x,0) & = \sin(3x) \\
x & \in (0, 2\pi), \ T=12.5
\end{align*} \]
## Multi-Domain Problem

<table>
<thead>
<tr>
<th>mesh</th>
<th>$L^2$ error</th>
<th>order</th>
<th>$L^2$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_h(x, 12.5)$</td>
<td></td>
<td>$u^*(x, 12.5)$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{P}^1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>3.48E-01</td>
<td>—</td>
<td>3.76E-01</td>
<td>—</td>
</tr>
<tr>
<td>40</td>
<td>6.20E-02</td>
<td>2.49</td>
<td>6.32E-02</td>
<td>2.57</td>
</tr>
<tr>
<td>80</td>
<td>9.55E-03</td>
<td>2.70</td>
<td>1.02E-02</td>
<td>2.63</td>
</tr>
<tr>
<td>160</td>
<td>1.46E-03</td>
<td>2.71</td>
<td>1.25E-03</td>
<td>3.03</td>
</tr>
<tr>
<td>$\mathbb{P}^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>9.77E-03</td>
<td>—</td>
<td>3.20E-01</td>
<td>—</td>
</tr>
<tr>
<td>40</td>
<td>7.95E-04</td>
<td>3.62</td>
<td>9.96E-03</td>
<td>5.01</td>
</tr>
<tr>
<td>80</td>
<td>1.05E-04</td>
<td>2.92</td>
<td>2.67E-04</td>
<td>5.22</td>
</tr>
<tr>
<td>160</td>
<td>1.31E-05</td>
<td>3.01</td>
<td>1.03E-05</td>
<td>4.69</td>
</tr>
</tbody>
</table>

\[ u_t + u_x = 0 \]

\[ u(x, 0) = \sin(3x) \]

\[ x \in (0, 2\pi) \]

\[ T = 12.5 \]
For $\mathbb{P}^1$, 5 candidate stencils:
Choosing the Post-Processing Stencil

Two Approaches

- Using Essentially Non-Oscillatory (ENO) type method: Smoothness of candidate post-processing stencils. (S. Gottlieb)
- Edge Detection method: Finds shock location based on numerical solution. (R. Archibald, A. Gelb)
ENO Type Stencil Choosing

- Calculate undivided differences at downwind point:

  \[ D^1_j = |u_{j+1}(x_{j-1/2}) - u_j(x_{j-1/2})|, \]
  \[ D^{r+1}_j = |D^r_{j+1} - D^r_j|, \quad r = 1, \cdots, 3 \]

- Find \( \max_{j-4 \leq i \leq j+3} D^1_i, \cdots, \max_{j-4 \leq i \leq j} D^4_i \).
ENO Type Stencil Choosing, $\mathbb{P}^1$

\[
    D_{j-4}^4 \quad D_{j-3}^4 \quad D_{j-2}^4 \quad D_{j-1}^4 \quad D_j^4 \\
    D_{j-4}^3 \quad D_{j-3}^3 \quad D_{j-2}^3 \quad D_{j-1}^3 \quad D_j^3 \\
    D_{j-4}^2 \quad D_{j-3}^2 \quad D_{j-2}^2 \quad D_{j-1}^2 \quad D_j^2 \\
    D_{j-4}^1 \quad D_{j-3}^1 \quad D_{j-2}^1 \quad D_{j-1}^1 \quad D_j^1 \\
    \text{Center} \quad \text{Center} \quad \text{Left+1} \quad \text{Left} \quad \text{Right} \quad \text{Right-1} \quad \text{Center} \quad \text{Center}
\]
1 − \(D\) Discontinuous Coefficient Equation

\[
\begin{align*}
    u(x, t)_t + (a(x)u(x, t))_x &= 0 \\
    a(x) &= \begin{cases} 
        1, & x \in [-1, 1] \setminus (-\frac{1}{2}, \frac{1}{2}), \\
        \frac{1}{2}, & x \in (-\frac{1}{2}, \frac{1}{2})
    \end{cases} \\
    u(x, 0) &= \begin{cases} 
        \cos(2\pi x), & x \in [-1, 1] \setminus (-\frac{1}{2}, \frac{1}{2}), \\
        -2\pi \cos(4\pi x), & x \in (-\frac{1}{2}, \frac{1}{2})
    \end{cases} \\
    u(-1, t) &= u(1, t), \quad T = 12.5
\end{align*}
\]

Case of two stationary shocks.
Two stationary shocks problem for $k = 1$. The $S_{32}$ stencil is on the left and the $S_{432}$ and $S_{5432}$ stencils on the right. The $S_{32}$ stencil has smeared the left shock location and biases unnecessarily, especially for 40 points. The $S_{432}$ and $S_{5432}$ are perfect.
Stencil Choices

Two stationary shocks problem for $k = 2$.

The $S_{32}$, $S_{432}$ and $S_{5432}$ stencils are all the same.
1-D Discontinuous Coefficient

\[ u_t + (a(x)u)_x = 0 \]

\[ x \text{ in } (-1, 1), \ T = 12.5 \]
1-D Discontinuous Coefficient Using One-Sided Post-Processor

\[ u_t + (au)_x = 0 \]

\( x \) in \((-1,1)\), \( T=12.5 \)
1 – $D$ Discontinuous Coefficient Equation

Case of two stationary shocks

<table>
<thead>
<tr>
<th>mesh</th>
<th>$L^2$ error $u_h(x, 12.5)$</th>
<th>order</th>
<th>$L^2$ error $u^*(x, 12.5)$</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\mathbb{P}^1$</td>
<td></td>
<td>$\mathbb{P}^2$</td>
</tr>
<tr>
<td>20</td>
<td>8.55E-01</td>
<td>—</td>
<td>8.17E-01</td>
<td>—</td>
</tr>
<tr>
<td>40</td>
<td>1.93E-01</td>
<td>2.15</td>
<td>1.80E-01</td>
<td>2.18</td>
</tr>
<tr>
<td>80</td>
<td>2.72E-02</td>
<td>2.83</td>
<td>2.58E-02</td>
<td>2.80</td>
</tr>
<tr>
<td>160</td>
<td>3.69E-03</td>
<td>2.88</td>
<td>3.34E-03</td>
<td>2.95</td>
</tr>
<tr>
<td>40</td>
<td>1.45E-03</td>
<td>—</td>
<td>2.04E-02</td>
<td>—</td>
</tr>
<tr>
<td>80</td>
<td>1.54E-04</td>
<td>3.24</td>
<td>4.48E-04</td>
<td>5.51</td>
</tr>
<tr>
<td>160</td>
<td>1.90E-05</td>
<td>3.02</td>
<td>5.87E-06</td>
<td>6.25</td>
</tr>
</tbody>
</table>
1 – $D$ Discontinuous Coefficient Equation

$$\frac{\partial u(x,t)}{\partial t} + (a(x)u(x,t))_x = 0,$$

$$a(x) = \begin{cases} 
1, & x \in [-2, 2] \setminus (-1,1), \\
\frac{1}{2}, & x \in (-1,1)
\end{cases}$$

$$u(x,0) = \begin{cases} 
\cos\left(\frac{\pi}{2} x\right), & x \in [-2, 2] \setminus (-1,1), \\
\frac{2}{3} \sin(\pi x), & x \in (-1,1).
\end{cases}$$

$$u(-2,t) = u(2,t), \quad T = 1.$$  

Case of two stationary and two moving shocks.
1-D Discontinuous Coefficient

\[ u_t + (a(x)u)_x = 0 \]
\[ x \text{ in } (-2, 2), \; T=1 \]
1-D Discontinuous Coefficient Using One-Sided Post-Processor

\[ u_t + (au)_x = 0 \]
\[ x \text{ in } (-2,2), \quad T=1 \]
1 - \( D \) Discontinuous Coefficient Equation

Case of two stationary and two moving shocks

<table>
<thead>
<tr>
<th>mesh</th>
<th>( L^2 ) error</th>
<th>order</th>
<th>( L^2 ) error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( u_h(x, 2) )</td>
<td></td>
<td>( u^*(x, 2) )</td>
<td></td>
</tr>
<tr>
<td>( P^1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>1.36E-03</td>
<td></td>
<td>2.05E-03</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>3.35E-04</td>
<td>2.02</td>
<td>9.57E-05</td>
<td>4.42</td>
</tr>
<tr>
<td>160</td>
<td>8.30E-05</td>
<td>2.01</td>
<td>4.61E-06</td>
<td>4.38</td>
</tr>
<tr>
<td>320</td>
<td>2.07E-05</td>
<td>2.01</td>
<td>3.40E-07</td>
<td>3.76</td>
</tr>
<tr>
<td>( P^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>3.79E-05</td>
<td></td>
<td>1.90E-04</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>4.77E-06</td>
<td>2.99</td>
<td>2.44E-06</td>
<td>6.28</td>
</tr>
<tr>
<td>160</td>
<td>5.98E-07</td>
<td>3.00</td>
<td>2.80E-08</td>
<td>6.45</td>
</tr>
</tbody>
</table>
Smoothly Varying Mesh

Mesh defined by $x = \xi + \frac{1}{2} \sin(\xi)$ where $\xi = ih$, $i = 1, \cdots, N$ is the uniform mesh variable:

$$u^*(x) = \frac{1}{\Delta x_i} \sum_{j=-k'}^{k'} \sum_{l=0}^{k} u_{i+j}^{(l)} \sum_{\gamma=-k}^{k} c_{\gamma}^{2(k+1),k+1}$$

$$\int_{I_{i+j}} \psi^{(k+1)} \left( \frac{y - x}{\Delta x_i} - \gamma \right) \left( \frac{y - x_{i+j}}{\Delta x_{i+j}} \right)^l \, dy$$

$$k' = \lceil (3k + 1)/2 \rceil$$
Smoothly Varying Mesh

- Create locally uniform mesh of mesh size: \( h = x_j \)
- Project \( u_h(x, T) \) to locally uniform mesh for all \( x_i \) in the post-processing region.
- Use \( u_n(x, T) \) to find post-processed solution on \( I_j \):
  \[
  u^*(x) = \sum_{j=-2k}^{2k} \sum_{l=0}^{k} u_{n_l}^{(i+j)} C(j, l, k, x)
  \]
Smoothly Varying Mesh

Approximation level errors for mesh type \( x = \xi + \frac{1}{2} \sin(\xi) \).
Smoothly Varying Mesh

<table>
<thead>
<tr>
<th>( N )</th>
<th>( P^1 )</th>
<th>( P^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L^2 ) error</td>
<td>order</td>
<td>( L^2 ) error</td>
</tr>
<tr>
<td>10</td>
<td>1.5358E-02</td>
<td>—</td>
</tr>
<tr>
<td>20</td>
<td>3.9029E-03</td>
<td>1.98</td>
</tr>
<tr>
<td>40</td>
<td>9.7975E-04</td>
<td>1.99</td>
</tr>
<tr>
<td>80</td>
<td>2.4519E-04</td>
<td>2.00</td>
</tr>
</tbody>
</table>

Approximation level errors.

\[ u(x) = \sin(x) \]

\[ x = \xi + \frac{1}{2} \sin(\xi) \]
Summary

• $(2k + 2 - d)$-th order accuracy in for the $d$-th derivative.
• $(2k + 1)$-th order accuracy for 2–D linear hyperbolic systems.
• One-Sided post-processing is able to handle computational boundaries, mesh interfaces or discontinuous coefficients.
• ENO stencil choosing is able to find discontinuities in numerical solution.
• Post-processor is able to improve accuracy for smoothly varying meshes.
Future Work

• Comparing different accuracy enhancement methods.
  – Adjerid, Devine, Flaherty, Krivodonova
  – Cockburn, Luskin, Shu, Süli
  – Zienkiewicz, Zhu

• Nonlinear hyperbolic equations.

• Applications ...
References

ryanjk@ornl.gov

www.csm.ornl.gov/~ryq/home.html


