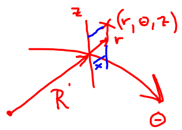
Transverse Motion and Stability

R radius of synchronous particle.



$$x \equiv r - R$$

$$y \equiv z$$

$$ds \equiv R\,d\theta \equiv \beta_s c\,dt \quad \text{synchronous particle}$$

r, θ, z coordinates (let $\dot{} \equiv \frac{d}{dt}$)

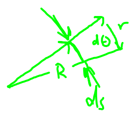
$$|v| = \beta c = \sqrt{\dot r^2 + r^2\dot\theta^2 + \dot z^2}$$

$$= r\dot\theta\sqrt{1 + \frac{\dot r^2}{r^2\dot\theta^2} + \frac{\dot z^2}{r^2\dot\theta^2}}$$

$$\simeq r\dot\theta \quad \text{because} \quad \frac{\dot r}{r\dot\theta} \text{ and } \frac{\dot z}{r\dot\theta} << 1$$

$$\frac{ds}{dt} = \frac{R\,d\theta}{dt} = r\,\frac{R}{r}\frac{d\theta}{dt}$$

$$= \frac{R}{r}\,r\dot\theta = \frac{R}{r}\beta c$$



$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = \beta c\,\frac{R}{r}\frac{d}{ds}$$

Unit vectors $\quad \hat r \ (\hat x)$

$\hat z \ (\hat y)$

$\hat\theta \ (\hat s)$

$$\frac{d\hat r}{ds} = \frac{d\hat r}{d\theta}\frac{d\theta}{ds} = \hat\theta / R$$

$$\frac{d\hat\theta}{ds} = \frac{d\hat\theta}{d\theta}\frac{d\theta}{ds} = -\hat r / R$$

$$\frac{d\hat z}{ds} = 0$$

Define $\boxed{\ ' \equiv \frac{d}{ds}\ }$

particle position: $\vec r = r\hat r + z\hat z$

$$\frac{d\vec r}{ds} = \hat r r' + r\hat r' + z'\hat z + \cancel{z\hat z'}^{\,0}$$

$$= \hat r r' + \hat\theta\frac{r}{R} + \hat z z'$$

$$\frac{d^2\vec r}{ds^2} = \overset{3}{\hat r' r'} + \overset{1}{\hat r r''} + \overset{2}{\hat\theta'\frac{r}{R}} + \overset{3}{\hat\theta\frac{r'}{R}} + \cancel{\hat z' z'}^{\,0} + \overset{4}{\hat z z''}$$

$$= \hat r\left(r'' - \frac{r}{R^2}\right) + 2\hat\theta\frac{r'}{R} + \hat z z''$$