APPENDIX A

THE DISPLACEMENT OPERATOR

A useful construct in the analysis of the quantum-mechanical harmonic oscillator is the displacement operator

\[ D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} , \] (A.1)

where \( \alpha \) is a dimensionless complex number.\(^1\) The ladder operator \( a \) is defined as

\[ a = \left( \frac{\mu \omega}{2} \right)^{1/2} \left( x + i \frac{p}{\mu \omega} \right) , \] (A.2)

and satisfies the commutation relation

\[ [a, a^\dagger] = 1 . \] (A.3)

The harmonic oscillator Hamiltonian

\[ H = \frac{p^2}{2\mu} + \frac{\mu \omega^2 x^2}{2} \] (A.4)

can be expressed in terms of the ladder operator as

\[ H = \omega \left( a^\dagger a + \frac{1}{2} \right) . \] (A.5)

The eigenstates of Eq. (A.5) are defined by

\[ H |n\rangle = E(n) |n\rangle \] (A.6)

with \( n = 0, 1, 2... \) the quantum of excitation and
\[ E(n) = \omega(n + 1/2) \]  
the corresponding eigenenergy.

The displacement operator can be factored as

\[ D(\alpha) = e^{-\alpha^2/2} e^{\alpha a} e^{-\alpha^* a} , \]  

owing to the commutation relation of Eq. (A.3) and the Baker-Campbell-Hausdorff Theorem (a.k.a. Glauber’s Formula). Then, the action of the displacement operator on the ground state of Eq. (A.5), i.e. \( n = 0 \), yields

\[ D(\alpha)|0\rangle = e^{-\alpha^2/2} e^{\alpha a} e^{-\alpha^* a} |0\rangle \]

\[ = e^{-\alpha^2/2} e^{\alpha a} |0\rangle \]

\[ = e^{-\alpha^2/2} \sum_j \frac{\alpha^j}{\sqrt{j!}} |j\rangle . \]

The final result of Eq. (A.9) is the energy eigenstate representation of the harmonic oscillator coherent state

\[ |\alpha\rangle = e^{-\alpha^2/2} \sum_j \frac{\alpha^j}{\sqrt{j!}} |j\rangle , \]

which is also as an eigenstate of the ladder operator,

\[ a|\alpha\rangle = \alpha |\alpha\rangle . \]  

Consequently, the displacement operator transforms the ground state of the harmonic oscillator into the coherent state, i.e.

\[ |\alpha\rangle = D(\alpha)|0\rangle . \]

More generally, the unitary transformation
serves to translates the ladder operator $a$ by an amount $\alpha$. Since,

$$D(\alpha) a D(\alpha)^\dagger |0\rangle = (a - \alpha) |0\rangle,$$  \hspace{1cm} (A.14)

it follows from Eq. (A.13) that

$$a' = a - \alpha.$$  \hspace{1cm} (A.15)

In particular, for real values of $\alpha$, the displacement operator spatially translates the Hamiltonian. For example, the displaced harmonic Hamiltonian

$$H' = \frac{p^2}{2\mu} + \frac{\mu \omega^2}{2} (x - x_0)^2,$$  \hspace{1cm} (A.16)

can be expressed in terms of the undisplaced Hamiltonian, given by Eq. (A.5), as

$$H' = D(\alpha_0) H D(\alpha_0)^\dagger,$$  \hspace{1cm} (A.17)

for $\alpha_0 = (\mu \omega / 2)^{1/2} x_0$. The eigenstates of $H'$ have identical eigenenergies, see Eq. (A.7), but are translated by the amount $\alpha_0$

$$|\tilde{n}\rangle = D(\alpha_0) |n\rangle.$$  \hspace{1cm} (A.18)

Inserting the definition of the ladder operator into $D(\alpha_0)$, we also see that

$$D(\alpha_0) = e^{i\phi_0},$$  \hspace{1cm} (A.19)

i.e. $D(\alpha)$ is the spatial translation operator for real values of $\alpha$. The latter result is particularly useful for translating the wave functions of Eq. (A.18);

$$\langle x | \tilde{n} \rangle = \langle x | D(\alpha_0) | n \rangle$$

$$= \langle x - x_0 | n \rangle.$$  \hspace{1cm} (A.20)
Other useful properties of the displacement operator include the composition property

\[ D(\alpha_1)D(\alpha_2) = D(\alpha_1 + \alpha_2)e^{(\alpha_1^2-\alpha_2^2)/2} \]  

(A.21)

and the unitary transformation

\[ e^{-iHt}D(\alpha)e^{iHt} = D(\alpha e^{-i\omega t}) , \]  

(A.22)

both of which follow from Eq. (A.9). See Ref. [1] for additional details.

The overlap between a displaced and undisplaced eigenstate,

\[ \langle \tilde{m} | n \rangle = \langle m | D^\dagger(\alpha) | n \rangle , \]

is calculated using Eq. (A.8). Expanding the exponential operators, we have

\[ \langle \tilde{m} | n \rangle = e^{-|\alpha|^2/2} \langle m | e^{-aa^\dagger} e^{\alpha^*a} | n \rangle \]  

(A.23)

\[ = e^{-|\alpha|^2/2} \sum_{j=0}^{\infty} \sum_{k=0}^{n} \frac{(-\alpha)^j (\alpha^*)^k}{j! k!} \langle m | (a^\dagger)^j (a)^k | n \rangle \]

\[ = e^{-|\alpha|^2/2} \sum_{j=0}^{\infty} \sum_{k=0}^{n} \frac{(-\alpha)^j (\alpha^*)^k}{j! k!} \left( \frac{m!}{(m-j)! (n-k)!} \right)^{1/2} \langle m-j | n-k \rangle , \]

where in the last line finite upper bounds on the summations are a result of \( a|0\rangle = 0 \). Due to the orthonormality of the harmonic oscillator eigenstates, we must have

\[ m - j = n - k , \]

which includes two cases of the summation in Eq. (A.23). In the first, \( m \geq n \), we substitute \( j = k + m - n \) and find

\[ \langle \tilde{m} | n \rangle = e^{-|\alpha|^2/2} (-\alpha)^{m-n} \left( \frac{n!}{m!} \right)^{1/2} \sum_{k=0}^{n} \frac{(-|\alpha|^2)^k}{k! (n-k)! (k+m-n)!} \]

(A.24)
\begin{align*}
&= e^{-\frac{\alpha^2}{2}} (-\alpha)^{m-n} \left( \frac{n!}{m!} \right) \frac{1}{n-m} L_{n-m} (|\alpha|^2),
\end{align*}

where in going to the last line we have used the following identity for the associated Laguerre polynomial:

\begin{equation}
L_n^\beta (x) = \sum_{k=0}^{n} \frac{(-x)^k}{k!} \frac{(n+\beta)!}{(n-k)!(k+\beta)!}. \tag{A.25}
\end{equation}

Similarly, for the second case, \( m \leq n \), we let \( k = j + n - m \) to find

\begin{equation}
\langle \bar{m} | n \rangle = e^{-\frac{\alpha^2}{2}} (\alpha^*)^{n-m} \left( \frac{m!}{n!} \right) \frac{1}{n-m} L_{n-m} (|\alpha|^2). \tag{A.26}
\end{equation}

When \( m = n \) both cases coincide and the overlap is given by

\begin{equation}
\langle \bar{n} | n \rangle = e^{-\frac{\alpha^2}{2}} L_n^0 (|\alpha|^2), \tag{A.27}
\end{equation}

where \( L_n^0 (x) = L_n (x) \) is the usual Laguerre polynomial.
Notes
