Spin-Nernst effect in the paramagnetic regime of an antiferromagnetic insulator

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We theoretically investigate a pure spin Hall current driven by a longitudinal temperature gradient, i.e., the spin Nernst effect (SNE), in a paramagnetic state of a collinear antiferromagnetic insulator with the Dzyaloshinskii-Moriya interaction. The SNE in a magnetic ordered state in such an insulator was proposed by Cheng et al. [R. Cheng, S. Okamoto, and D. Xiao, Phys. Rev. Lett. 117, 217202 (2016)]. Here we show that the Dzyaloshinskii-Moriya interaction can generate a pure spin Hall current even without magnetic ordering. By using a Schwinger boson mean-field theory, we calculate the temperature dependence of SNE in a disordered phase. We also discuss the implication of our results to experimental realizations.

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I. INTRODUCTION

Recent years have seen a surge of interest in issues related to spin transport in magnetic insulators. For practical purposes, the ability to transfer spin information in the absence of charge flow holds great potential for energy-efficient applications [1–9]. On the fundamental side, spin transport measurements can also provide valuable information about the ground state and low-energy excitations of correlated electronic systems [10]. In particular, a thermal Hall effect (THE) of spin excitations has been predicted [11]. In this effect, a longitudinal temperature gradient can drive a transverse heat current carried by charge-neutral excitations such as magnons or spinons. Since its prediction, the THE has been observed in a number of magnetic insulators [12–15], accompanied by extensive theoretical efforts [16–24]. It is now recognized that, microscopically, the THE originates from nontrivial magnon dispersions due to either chiral spin textures or nonsymmetric spin-spin interactions, such as the Dzyaloshinskii-Moriya interaction (DMI).

However, in certain classes of magnetic insulators, the THE is symmetry-prohibited. Examples include magnetically disordered states at high temperatures and collinear antiferromagnets with combined time-reversal (T) and inversion (I) symmetry. For these systems, a spin Nernst effect (SNE) is symmetry-allowed nonetheless. In the SNE, spin currents with opposite polarization flow in the opposite transverse direction in response to a longitudinal temperature gradient. As a result, the heat current vanishes, and we are left with a pure transverse spin current. The relation between the THE and the SNE is akin to the relation between the anomalous Hall effect and the spin Hall effect. The SNE has been predicted for magnets on a honeycomb lattice, either in antiferromagnets (AFM) below the Néel temperature in which the SNE is realized by magnons [25–27], or ferromagnets (FM) above the Curie temperature in which the SNE is realized by spinons [28]. Possible experimental signature of the SNE has also been reported in the antiferromagnetic insulator MnPS3 in the ordered phase [29].

Actually, the honeycomb magnets can display either the THE or the SNE depending on their magnetic configurations, as summarized in Table I. The key ingredient here is a second nearest-neighbor DMI, which plays a similar role in spin transport as the spin-orbit interaction in electron transport. In the ordered phase of a honeycomb FM, the broken time-reversal symmetry together with the DMI leads to the THE [22,28]. On the other hand, in both the disordered phase of the FM and the ordered phase of the AFM, the vanishing magnetization forbids the THE, but the DMI still allows the SNE [25,26,28]. These results strongly hint that the SNE should also exist in the high-temperature disordered phase of the honeycomb AFM.

In this paper we present a detailed study of this effect using the Schwinger boson mean-field approach. We show that the SNE is indeed enabled by the DMI in the high-temperature disordered phase of a honeycomb AFM, and the transverse spin current is carried by the two pairs of conjugated spinon states connected by the combined T_I symmetry. Supplemented by a symmetry analysis, we calculate the reduced mean-field order parameters of the spinons, establish the disordered phase regime, and then identify the effect of a T_I conjugate pair on the pure SNE. Finally, we calculate the temperature dependence of the SNE coefficient in this disordered phase, and discuss its realization in real materials.

This paper is organized as follows. In Sec. II, we introduce the honeycomb AFM model with a second nearest-neighbor DMI, and present the mean-field solution to the Schwinger boson Hamiltonian. This is followed by a discussion of the SNE in Sec. III, including its dependence on the temperature, the staggered field, and the DMI strength. Finally, we comment on the limitations of our theoretical treatment and discuss possible material realizations of the SNE in Sec. IV.

II. MODEL AND METHOD

A. Honeycomb AFM

We begin with the following spin Hamiltonian on a honeycomb lattice:

$$H = J_1 \sum_{\langle i,j \rangle} S_i \cdot S_j + D_2 \sum_{\langle i,j \rangle} v_{ij} \hat{z} \cdot (S_i \times S_j)$$

$$- \hbar \sigma \sum_i \langle 1 \rangle S_i^z .$$

(1)
TABLE I. Summary of the thermal Hall effect (THE) and the spin Nernst effect (SNE) in honeycomb magnets with a second nearest-neighbor Dzyaloshinskii-Moriya interaction. Depending on the symmetry, the system exhibits either a THE or a SNE.

<table>
<thead>
<tr>
<th>Collinear order</th>
<th>Ordered</th>
<th>Disordered</th>
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<tbody>
<tr>
<td>FM</td>
<td>THE$^a$</td>
<td>SNE$^b$</td>
</tr>
<tr>
<td>AFM</td>
<td>SNE$^c$</td>
<td>SNE$^d$</td>
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</table>

$^a$References [22,28].
$^b$Reference [28].
$^c$References [25,26].
$^d$This work.

The first term describes the antiferromagnetic nearest-neighbor (NN) Heisenberg exchange with $J_1 > 0$. The second term is a second-NN DMI. Here $v_{ij} = 2\sqrt{3}(d_1 \times d_2)_z = \pm 1$ with $d_1$ and $d_2$ the vectors connecting site $i$ to its second NN site $j$, as shown in Fig. 1. This second-NN DMI is allowed by crystal symmetry [30,31]; it is intrinsic to the honeycomb lattice. The third term is a staggered Zeeman field along the $z$ direction that stabilizes the system in the collinear AFM ground state at low temperatures.$^1$ Throughout this paper, we will use $J_1$ as the energy and temperature unit.

In the high-temperature paramagnetic (PM) phase, the low-energy spin dynamics can be described by spinons. We introduce the Schwinger boson (SB) representation for the spin operator [33]

$$ S_i = \frac{1}{2} \sum_{s,s'} c_{i,s}^\dagger \sigma_{ss'} c_{i,s'}^\dagger, \quad (s, s' = \pm 1), \quad (2) $$

with the constraint that the number of spinons must be conserved at any given site, $\sum_s c_{i,s}^\dagger c_{i,s} = 2S$. The index $s = \pm 1$ denotes up or down spins. In Eq. (2), $\sigma$ are the Pauli matrices, and $c_{i,s}^\dagger$ ($c_{i,s}$) denotes the creation (annihilation) operator for a spinon with spin $s$ at site $i$. The spin amplitude $S = 1/2$ is considered in this paper.

Substituting Eq. (2) into the spin Hamiltonian (1), we obtain

$$ H_{SB} = -2J_1 \sum_{(i,j)} \vec{A}_{ij} \cdot \vec{A}_{ij}^\dagger - \frac{D_2}{2} \sum_{(i,j)} \sum_s v_{ij} \mathcal{F}_{ij,s} c_{j,-s}^\dagger c_{i,s}^\dagger + \frac{\eta}{2} \sum_{(i,j)} \mathcal{F}_{ij,s} c_{i,s}^\dagger c_{j,-s}^\dagger - h_s \sum_i \mu_i \left( \sum_s c_{i,s}^\dagger c_{i,s}^\dagger - 2S \right), \quad (3) $$

where $\vec{A}_{ij} \equiv (c_{i+1,s} c_{j+1,s} - c_{i+1,s} c_{j+1,s}^\dagger)/2$ is the antiferromagnetic NN bond operator, and $\mathcal{F}_{ij,s} \equiv c_{i,s}^\dagger c_{j,s}$ is the second NN bond operator. $\mu_i$ is a Lagrange multiplier to impose the local constraint at the mean-field level. We note that $\vec{A}_{ij} = -\vec{A}_{ji}$ is antisymmetric.

$^1$While it is not easy to apply such a field externally, similar effects could arise when the SU(2) symmetry is broken by the single-ion anisotropy for $S > 1/2$ or the Ising-type anisotropy in the exchange coupling within a Schwinger boson (SB) mean-field approach [32]. This allows magnetic ordering in low-dimensional systems at finite temperature.

$^2$Note that our definition of the $\mathcal{I}$ operator has an additional matrix $\sigma_z$. It flips the sign of the spinon operator on the $B$ site, and is needed to make sure the NN bond term $\chi_{ij}$ transforms into itself under the $\mathcal{T} \mathcal{I}$ operation. The $\sigma_z$ matrix is allowed since there is an extra phase freedom in the spinon representation.
Imposing the $TI$ symmetry on the mean-field Hamiltonian (4) yields

$$A^\dagger_{ij,\ldots} = B_{-i,j,\ldots}. \quad (7)$$

We now assume that the bond order parameters and the chemical potential are spatially uniform. They are $A_{ij,s} = A^s, B_{ij,s} = B^s, \mu_i = \mu$, and $\mu_i = \mu$. Fourier transforming into the momentum space $\Psi_{ks} = \{a_{ks}, \bar{b}_{-k,-s}\} = (1/\sqrt{N}) \sum e^{ikR} [a_{ks}, \bar{b}_{-k,-s}]^T$, and using the condition (7), we obtain the mean-field Hamiltonian in the momentum space

$$H^{M}_{SB} = \sum_{k,s} \Psi_{ks}^\dagger \frac{1}{2} h^{\sigma}_{s} \sigma_{\mu} \Psi_{ks}, \quad (8)$$

where $\sigma_{\mu} = \{ I_{2\times 2}, \sigma_z \}$ and

$$h^0_s(k) = \mu - s - \frac{hs}{2} + \frac{D_s}{4} M_s \sigma_{s}(k), \quad (9a)$$

$$h^1_s(k) - ih^2_s(k) = -J_1 \chi_0 f(k), \quad (9b)$$

$$h^3_s(k) = \frac{D_s}{4} P_s \sigma_{s}(k), \quad (9c)$$

with $M_s^i \equiv A^s - B^s, \quad P_s \equiv A^s + B^s$. The structure factors are $g_{s\mu}(k) = -2 \sum \sin(k \cdot a_i) g_{s}(k) = 2 \sum \cos(k \cdot a_i)$, and $f(k) = \sum e^{i k \cdot a_i} \sigma_{s}(k)$ is an even function of $k$, and $g_{s}(k)$ and $|f(k)|$ are even functions of $k$.

**B. Schwinger boson mean-field solution**

The spinon Hamiltonian (8) contains six parameters that need to be determined self-consistently, namely, $\mu, \chi_0, \mu$, and $M_s, P_s$ with $s = \pm 1$. To diagonalize the Hamiltonian (8), we perform the Bogoliubov transformation $\Psi_{ks} = U^{-1}_s(k) \psi_{ks} = [a_{ks}, \bar{b}_{-k,-s}]^T$, where $U^{-1}_s(k)$ is a unitary matrix given by

$$U^{-1}_s(k) = \begin{bmatrix} \cos \frac{\theta_s(k)}{2} & \sin \frac{\theta_s(k)}{2} e^{-i \varphi_s(k)} \\ \sin \frac{\theta_s(k)}{2} e^{i \varphi_s(k)} & \cos \frac{\theta_s(k)}{2} \end{bmatrix}. \quad (10)$$

Here the Bogoliubov angles $\theta$ and $\varphi$ are defined by $h^s$ in Eq. (9): $h^s_0 = h^s \cos \theta_s$, $h^s_2 = h^s \sin \theta_s \cos \varphi_s$, and $h^s_0 = h^s \cos \varphi_s$.

$$U^{-1}_s(k) = \sqrt{h^2_0 - h^2_1 - h^2_2} = \sqrt{h^2_0 - h^2_1 - h^2_2} \quad (11)$$

The wave function of the $\alpha_{ks}$ ($\beta_{ks}$) quasiparticle is given in Appendix C.

This degeneracy originates from the combined $TI$ symmetry of our mean-field Hamiltonian. We note that the annihilation operator of a spinon $\alpha_{ks}$ transforms into $s\beta_{k,s}$ under the $\mathcal{T}$ operation defined in Eq. (5). From this, we find

$$E^\sigma_s(k) = E^\sigma_{-s}(k). \quad (12)$$

We call such a pair of degenerate modes as a $TI$ symmetry conjugate pair. This conjugate pair is crucial for the appearance of a pure transverse spin current as we discuss below.

We compute mean-field order parameters by solving a set of self-consistent equations detailed in Appendix B. The temperature dependence of order parameters at $D_s = 0.24 J_1$ with different $h_\perp$ are shown in Fig. 2, along with the spinon dispersion in Fig. 3. We first note that all order parameters vanish above $T_c \sim 0.826 J_1$. This is an artifact of the mean-field approach, and $T_c$ should be interpreted as a characteristic crossover temperature above which the system behaves as a paramagnet with local moments [33]. On the other hand, as the temperature approaches zero, the spinon gap at the $\Gamma$ point closes (Fig. 3), and the system undergoes a phase transition into the collinear AFM phase at the Néel temperature $T_N$ via the spinon condensation [34].

**FIG. 2.** The solution of order parameters with staggered field $h_\perp = 0$ for (a) and (c), and $h_\perp = 0.1 J_1$ for (b) and (d).

**FIG. 3.** The dispersions along high symmetry lines $\Gamma - M - K - \Gamma$ for (a) $h_\perp = 0$ and $T = 0.1 J_1$; (b) $h_\perp = 0.1 J_1$ and $T = 0.1 J_1$; (c) $h_\perp = 0$ and $T = 0.5 J_1$; and (d) $h_\perp = 0.1 J_1$ and $T = 0.5 J_1$. $\alpha_{\sigma (\beta)}$ denotes the mode $E^\sigma_{\sigma (\beta)}(k)$ with $s = \pm 1$ for spin $\uparrow (\downarrow)$. 
For the current two-dimensional model, $T_N$ is strictly zero because single-site spin anisotropy or anisotropic exchange coupling is absent. Spin ordering at finite $T$ is mimicked by the nonzero staggered field $h_{st}$.

### III. SPIN NERNST EFFECT OF SPINONS

#### A. Spin conservation and mirror symmetry

With a firm understanding of the spinon spectra, we now turn to the SNE. As a first step, we examine how many spins are carried by the spinon modes. In general, this is not a trivial question because, in the presence of the DMI, the spin angular momentum does not have to be conserved. Fortunately, our model also has the mirror symmetry $\mathcal{M}_z$ about the lattice plane, which leads to the conservation of the total spin $S_z$,

$$S_z = \frac{\hbar}{2} \sum_k s \bar{\psi}_k^{\dagger} \sigma_z \psi_k = \frac{\hbar}{2} \sum_k s \bar{\Phi}_k^{\dagger} \sigma_z \Phi_k.$$

We see that the $\alpha_k$ and $\beta_{k-x}$ modes have opposite angular momentum $(0|\alpha_k s \bar{\alpha}_k^{\dagger}|0) = hS/2$ and $(0|\beta_{k-x} s \bar{\beta}_{k-x}^{\dagger}|0) = -hS/2$, respectively. Here $|0\rangle$ is the vacuum state of spinons. The SNE is due to the opposite transverse motion of the two spin species driven by a longitudinal temperature gradient.

#### B. Spin Nernst effect coefficient in disordered state

Since spinons do not carry charge, they cannot be driven by an external electric field, but they can respond to a statistical force, such as the temperature gradient $\nabla T$. Due to the conservation of $S_z$, spin current can be written as $\mathbf{J}^{SN} = \sum_\lambda s \bar{\psi}^{\dagger} \hat{\nabla} s \psi$, where $\mathbf{J}^{\lambda}$ is the spin current of mode $\lambda$ and spin $s$. According to the authors of Refs. [16,17,20,25], the transverse $\mathbf{J}_{xy}^{\lambda}$ due to $\nabla T$ is given by

$$\mathbf{J}_{xy}^{\lambda} = \frac{\hbar}{2} \times \nabla T \int \frac{d\mathbf{k}}{(2\pi)^2} c_1(n^\lambda_s(k)) \Omega^\lambda_s(k).$$

where $c_1$ is the weight function $c_1(x) = x \ln x - (1 + x) \ln(1 + x)$, and $n^\lambda_s(k)$ and $\Omega^\lambda_s(k)$ are the Bose-Einstein distribution functions and the Berry curvature (defined below) for the mode $\mathbf{E}^\lambda_s(k)$, respectively.

We now analyze the symmetry properties of the Berry curvature, which for the mode $\mathbf{E}^\lambda_s(k)$ is expressed as

$$\Omega^\lambda_s(k) = i \partial_k u^\dagger_s(k) \times \sigma_3 \partial_k u^s_s(k)$$

$$\Omega^\lambda_s(k) = \frac{i}{2} \nabla_k \cosh \theta_s(k) \times \nabla_k \varphi_s(k),$$

where $u^\dagger_s(k)$ is the wave function of the $\alpha_k$ quasiparticle as presented in Appendix C. Under the $\mathcal{T}\mathcal{I}$ operation, $\alpha \rightarrow \beta$, $s \rightarrow -s$, and $k \rightarrow -k$. In addition, the Berry curvature should also flip sign due to the factor $i$ in its definition. As such, under the $\mathcal{T}\mathcal{I}$ operation, we have

$$\Omega^{\lambda -\lambda}_s(k) = -\Omega^\lambda_s(k).$$

Together with the energy dispersion relation $\mathbf{E}^\lambda_s(k) = \mathbf{E}^{\lambda -\lambda}_s(k)$ [see Eq. (12)], this relation indicates that $\mathbf{J}^{\lambda}_s$ and $\mathbf{J}^{\lambda -\lambda}_s$ are always opposite in sign, resulting in a pure transverse spin current.

Next we focus on a particular mode $\alpha$. For bosonic BdG equations, there is a general relation of the Berry curvature between the $\alpha$ and $\beta$ mode (see Appendix D)

$$\Omega^\beta_\alpha(k) = \Omega_{\alpha -\alpha}^{\beta}(k).$$

Combining this relation with Eq. (16), we have

$$\Omega^\alpha_\alpha(k) = -\Omega^\beta_\alpha(k).$$

This is clearly seen in Fig. 4(a). If the spinon dispersion is inversion symmetric, the transverse current $J^s_{xy}$ would vanish. However, as we can see from Eq. (11), the presence of the DMI breaks this symmetry, i.e., $E^\alpha_\alpha(k) \neq E^{\beta -\beta}_\beta(k)$ as illustrated in Fig. 4(b). After summing over all occupied states, there should be a net spin current. Therefore the second NN DMI is crucial for the appearance of the SNE.

We numerically calculate the spin Nernst coefficient given by [16,17,20,25]

$$\alpha_{\alpha xy} = \sum_s \int \frac{d\mathbf{k}}{(2\pi)^2} c_1(n^\lambda_s(k)) \Omega^\lambda_s(k).$$

where $\alpha_{\alpha xy}$ is defined by the relation $\mathbf{J}^{SN} = \alpha_{\alpha xy} \mathbf{z} \times \nabla T$. The temperature dependence of $\alpha_{\alpha xy}$ is calculated at different staggered field $h_{st}$ and DMI strength $D_2$ in Fig. 5. We find that $\alpha_{\alpha xy}$ will be zero at two ends of the temperature zone, i.e., $T = 0$ and $T = T_c$. When $T$ approaches zero, the fluctuating component of spinons is decreased. On the other hand, when the temperature approaches $T = T_c$, $\alpha_{\alpha xy}$ is reduced to zero. This will cause the SNE to vanish because the vanishing of $P_s$ effectively restores the inversion symmetry of the spinon dispersion.

In addition, the peak of the spin Nernst coefficient at a special temperature results from the competition between the enhancement of excited spinons engaging in transport and the reduction of the second-NN order parameter $P_s$ and $M_s$ as the temperature increases. The staggered field will weaken the spin Nernst coefficient in opposite to that of DMI because the staggered field supports a collinear configuration, but DMI favors a perpendicular one between two second-NN spin polarizations. In reality, $TN$ could be finite due to a variety of effects neglected here, and the temperature dependence of the spin Nernst coefficient is expected to depend on the competition between these effects and the DMI, especially near $T_N$. Nevertheless, the spin Nernst coefficient is shown to change continuously with increasing $h_{st}$. This implies that the spin Nernst coefficient changes continuously at the magnetic transition temperature as long as it is the second-order transition.

FIG. 4. The distributions of Berry curvature and spectrum for $\alpha_k$, spinon with $s = -1$ at temperature $T = 0.1 T_c$ without staggered fields: (a) The Berry curvature; (b) The spectrum.
In summary, we study the pure SNE in the PM state on an antiferromagnetic honeycomb lattice with a second-NN DMI, using the Schwinger boson mean-field method. We find that the pairs of the combined $TI$ conjugate modes of spinons support a transverse spin current without a transverse thermal current. Because of the competition between the short-range spin correlations, represented by the temperature-dependent mean-field order parameters, and spin fluctuations, represented by the thermal population of spinons, the spin Nernst coefficient shows a nontrivial temperature dependence for a rather simple model considered here. This might suggest that a paramagnetic insulator with AFM interaction of spins could serve as a spintronics device even above the magnetic transition temperature to generate or detect the spin current.

Before closing, we would like to discuss several issues left for future studies. Throughout this paper, we neglect the fluctuations from the mean-field solution. In fact, the Schwinger boson mean-field treatment is the result of the zeroth order of $O(1/N)$ in a large-$N$ expansion of a spin SU($N$) model [33]. Rigorously speaking, the low-energy part of fluctuations, i.e., the phase fluctuation of order parameters, could couple with the $U(1)$ gauge field, the dynamics of which may exhibit a confined or deconfined phase. Exploring these effects of fluctuations on spin transport will be an interesting problem in the future [35,36]. However, since our argument about the finite SNE in the PM state of the honeycomb AFM is based on the combined $TI$ symmetry, our conclusion would not be altered in a qualitative manner.

We do not use the full projected symmetry group method to analyze the spinon Hamiltonian. Such analyses would be necessary for spin liquid systems at low temperatures described by fermionic spinons. On the other hand, for investigating the pure SNE at high temperatures, it is sufficient to consider only the combined $TI$ symmetry based on the unprojected spinon wave function.

So far we only considered the so-called intrinsic contribution to the spinon SNE due to the Berry curvature of the spinon bands. Similar to the anomalous Hall effect [37], there should be extrinsic effect due to the scattering between spinons and other relevant physical degrees of freedom such as phonons. We note that there is an analogous effect of electrons [38,39] for which the impurity scattering has been discussed [40].

In real materials, such as transition-metal trichalcogenides, the situation is more complicated. In addition to the interactions described in Eq. (1), longer-range exchange interactions are present, stabilizing complex magnetic ordered states [41]. Furthermore, single-ion anisotropies and anisotropic exchange interactions could exist, making finite-temperature magnetic ordering possible even for the two-dimensional limit [42]. These effects not only require solving a set of self-consistent equations for many order parameters, but also require extending the current formalism as demonstrated in Ref. [32]. For $S > 1/2$ systems, $h_{\mu\nu}$ is related to the single-ion anisotropy $K_2$ as $h_{\mu\nu} \sim K_2 (S - 1/2) / S M_z$ with $M_z = \sum_{\sigma} \langle \sigma_{\mu\nu} S_{\mu\nu} \rangle$. For MnPS$_3$ as discussed in Ref. [25], $h_{\mu\nu} / J$ could become as large as 0.01 at low temperatures. This value is an order of magnitude smaller than the ones used in our analyses. Therefore, it is expected that the spin Nernst coefficient does
not change significantly across a magnetic transition temperature. Detailed material dependence of the SNE including these effects is left for future studies.

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APPENDIX A: SYMMETRY OPERATIONS

We discuss symmetry operations on the spinon Hamiltonian in the momentum space. These symmetry operations include inversion operation $I$, time-reversal operation $\mathcal{T}$, and mirror operation $\mathcal{M}_z$.

The spinon Hamiltonian matrix at each $k$ point is given by

$$H_s(k) = \sum_{\mu} h_{\mu}^s(k)\sigma_{\mu}.$$  \hfill (A1)

For the inversion operation, we follow the definition of Eq. (6) in the lattice space, which ensures that $I$ is an operation that changes the sign of all momenta. Accordingly, the Hamiltonian matrix $H_s(k)$ is transformed as

$$IH_s(k)I^{-1} = \sigma_z H_s^T(-k)\sigma_z,$$  \hfill (A2)

where $T$ stands for the matrix transposition.

The time-reversal operator $\mathcal{T}$ is defined in Eq. (5), and transforms $H_s(k)$ into

$$\mathcal{TH}_s(k)\mathcal{T}^{-1} = \sigma_3 H_s^*(-k)\sigma_3.$$  \hfill (A3)

Under the combined $\mathcal{TI}$ operation, $H_s(k)$ is thus transformed as

$$\mathcal{TI}H_s(k)(\mathcal{TI})^{-1} = \sigma_1 H_s(-k)\sigma_1.$$  \hfill (A4)

Therefore, if the system has the combined $\mathcal{TI}$ symmetry, then $H_s(k)$ should satisfy

$$\sigma_1 H_s(-k)\sigma_1 = H_s(k).$$  \hfill (A5)

The mirror symmetry operator $\mathcal{M}_z$ with respect to the lattice plane is defined as

$$\mathcal{M}_z c_{i,s} \mathcal{M}_z^{-1} = i (\sigma_3)_{i,s} c_{i,s},$$  \hfill (A6)

which leads to $\mathcal{M}_z S_i^z \mathcal{M}_z^{-1} = S_i^z$ and $\mathcal{M}_z S_i^{+,-} \mathcal{M}_z^{-1} = - S_i^{+,-}$. The Hamiltonian matrix is invariant under mirror operation $\mathcal{M}_z$

$$\mathcal{M} H_s(k) \mathcal{M}^{-1} = H_s(k).$$  \hfill (A7)

APPENDIX B: MEAN-FIELD SELF-CONSISTENT EQUATIONS

The mean-field order parameters and the Lagrange multiplier $\mu$ are determined by minimizing the free energy involving these parameters. By differentiating the free energy with respect to these parameters and equating to zero, one arrives at the following set of self-consistent equations:

$$1 + 2S = \frac{1}{2N} \sum_{k,s} \left[ \frac{h_0^s(k)}{h^s(i)} (n_{\alpha}^s + n_{\beta}^s - n_{\alpha}^s + n_{\beta}^s + 1) \right],$$  \hfill (B1a)

$$4\chi_0 = \frac{J_1\chi_0}{3N} \sum_{k,s} \left[ |f(k)|^2 \right] (n_{\alpha}^s + n_{\beta}^s - n_{\alpha}^s + n_{\beta}^s + 1),$$  \hfill (B1b)

$$M_s^z = \frac{1}{6N} \sum_k g_s(k) (n_{\alpha}^s - n_{\beta}^s - 1),$$  \hfill (B1c)

$$-P_s^z = \frac{1}{6N} \sum_k g_s(k) \frac{h_0^s(k)}{h^s(i)} (n_{\alpha}^s + n_{\beta}^s + 1),$$  \hfill (B1d)

where $n_{\alpha/\beta}^s = \exp(E_{\alpha/\beta}^s(k)/T) - 1$ is the Bose distribution function, and $N$ is the number of unit cells.

APPENDIX C: BDG EQUATION AND BERRY CURVATURE

In this section we present a detailed discussion of the bosonic BdG equation and the associated wave functions. Our starting point is the spinon mean-field Hamiltonian (8), reproduced here for convenience

$$H = \sum_{k,s} \Psi_{ks} H_s(k) \Psi_{ks}^\dagger,$$  \hfill (C1)

where $\Psi_{ks} = [a_{k,s}, b_{k,s}^\dagger]^T$ with $a_{k,s}$ and $b_{k,s}$ being the Fourier transform of the spinon operators on the A and B sublattices, respectively. Introducing the Bogoliubov transformation

$$\begin{pmatrix} a_{k,s} \\ b_{k,s}^\dagger \end{pmatrix} = U_s(k) \begin{pmatrix} \alpha_{k,s} \\ \beta_{k,s}^\dagger \end{pmatrix}.$$  \hfill (C2)

The boson commutation relation dictates that $U_s(k)$ is a parunitary matrix, i.e.,

$$U_s(k) \sigma_3 U_s^\dagger(k) = \sigma_3.$$  \hfill (C3)

By demanding that the Bogoliubov transformation diagonalizes the Hamiltonian, i.e., $H = \sum_{ks} \left[ E_{\alpha}^s(k) \alpha_{k,s} + E_{\beta}^s(k) \beta_{k,s}^\dagger \right]$, we obtain the BdG equation

$$H_s(k) U_s(k) = \sigma_3 U_s(k) \sigma_3 \Delta(k),$$  \hfill (C4)

where $\Delta(k) = \text{diag}(E_{\alpha}^s(k), E_{\beta}^s(-k))$ is the eigenvalue matrix. We note that both the excitation energies $E_{\alpha}^s(k)$ and $E_{\beta}^s(-k)$ must be positive. Otherwise the mean-field solution is unphysical. The explicit expression of $U_s(k)$ is given by Eq. (10).

For the purpose of calculating the Berry curvature, it is necessary to clarify the wave function of a spinon quasiparticle. Let us write $U_s(k) = [u_{\alpha}^s(k), u_{\beta}^s(-k)]$, where $u_{\alpha}^s(k)$ and $u_{\beta}^s(-k)$ are two-component column vectors. Inserting this expression into the BdG equation (C4), we have

$$H_s(k) u_{\alpha}^s(k) = E_{\alpha}^s(k) \sigma_3 u_{\alpha}^s(k),$$  \hfill (C5a)

$$H_s(k) u_{\beta}^s(-k) = -E_{\beta}^s(-k) \sigma_3 u_{\beta}^s(-k).$$  \hfill (C5b)

It is clear that $u_{\alpha}^s(k)$ and $u_{\beta}^s(-k)$ are the wave functions of the quasiparticle $\alpha_{k,s}$ with positive energy $E_{\alpha}^s(k)$ and the quasihole
The above discussion suggests that to find the quasiparticle eigenvalue problem given by Eq. (C5), the projector operator is defined by \[ P_n = \frac{|n\rangle\langle n|\sigma_3}{|n\rangle\langle n|\sigma_3} \].

For our disordered AFM described by the bosonic BdG Hamiltonian \( H_s(k) \), this leads to the formula
\[ \Omega^\pm(k) = \frac{i\delta k u^\dagger(k)}{u(k)\sigma_3 u(k)} \]
where \( u_s(k) \) is the wave function of \( \lambda \)-type quasiparticle or quasihole, and the normalization \( u^\dagger(k)\sigma_3 u_s(k) = \pm 1 \) for quasiparticle and quasiholes, respectively.

For a two-level system, it follows from Eq. (D1) that the Berry curvature has the property
\[ \Omega_n(k) = -\Omega_\beta(k), \]
where \( n \) and \( \bar{n} \) refers to the quasiparticle and quasihole bands, respectively. This property is a special case of \( \sum_n \Omega_n(k) = 0 \) with \( n \geq 2 \). Applying this relation to our Hamiltonian \( H_s(k) \), we have
\[ \Omega^\pm(k) = -\Omega^\mp(k). \]

Using Eq. (C9), one can deduce the relation
\[ \Omega^\pm(k) = -\Omega^\mp(k). \]

The result can be also applied to a general reduced BdG Hamiltonian.

In the presence of the \( T\overline{I} \) symmetry, the Berry curvatures for the two modes \( \alpha \) and \( \beta \) could be also related. Using Eq. (C11), we find
\[ \Omega^\pm(k) = -\Omega^\mp(k). \]